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ABSTRACT

We consider the non-relativistic N-body scattering problem for a system of particles in which some subsets of the particles are identical. We demonstrate how the particle identity can be included in a general class of linear integral equations for scattering operators or components of scattering operators. The Yakubovskii, Yakubovskii-Narodestkii, Rosenberg, and Bencze-Redish-Sloan equations are included in this class. Algebraic methods are used which rely on the properties of the symmetry group of the system. Operators depending only on physically distinguishable labels are introduced and linear integral equations for them are derived. This procedure maximally reduces the number of coupled equations while retaining the connectivity properties of the original equations.

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I. INTRODUCTION

Numerous methods for writing down mathematically well-behaved equations for the scattering operators in the non-relativistic N-body problem have been developed in the past few years.[1-9] In most of these methods the particles are initially considered to be distinguishable, and any effects of particle identity are put in only when specific problems are treated. Since the number of physical situations and equation types is large, we here present a general method for constructing reduced equations which include the effect of particle identity.

In the bound state case, the treatment of particle identity is generally considered a technical problem. One must solve the Schrödinger eigenvalue problem on the subspace of properly symmetrized[10] vectors of the Hilbert space. Such problems can be typically handled either by the elaborate mathematical machinery of the shell model[11] or with hyperspherical harmonics.[12] Both methods rely heavily on the theory of the symmetric group.

On the other hand the inclusion of exchange symmetry in a scattering problem is far from trivial. The reason is that when exchange scattering is present, the asymptotic form of the scattering wave function includes both incoming and scattered waves in different regions of the many-body configuration space. There is no longer a single channel Hamiltonian which describes the asymptotic waves so all the well-known problems of describing rearrangement channels arise. The conventional procedure is simply to solve the scattering problem for distinguishable particles and afterwards to sum the contributions of physically indistinguishable channels to the cross section.

In some of the exact formulations of N-particle scattering the exchange symmetry can be incorporated directly into the scattering equations with a resulting decrease in the number of coupled equations. Such a procedure has been carried out by Lovelace in the three-body problem[13] and by Kharchenko and Kuzmichev[14] for the four-body Faddeev-Yakubovskii equations. The first explicit treatment of the scattering of an arbitrary number of identical particles was carried out by the authors in Ref. 15 (henceforth referred to as I). In this paper abstract group theoretic methods were used to incorporate exchange symmetry into the Bencze-Redish-Sloan (BRS) N-particle scattering equations.[5,6,16,17]

The treatment presented in I made use of the specific properties of the BRS equations. In this paper we develop a general algebraic method of including exchange symmetry which can be applied to a large variety of N-particle scattering equations and allows the treatment of an arbitrary number of different kinds of identical particles, which may be bosons or fermions. Specifically we consider two important classes of N-particle equations, the channel coupling class of equations, which are written in terms of transition operators, and the chain coupling class which are written for components of the N to N transition operator. The former class includes the BRS equations, the set of equations described by Bencze and Tandy,[19] and the equations of Chandler and Gibson.[9] The latter include the Rosenberg,[1] Yakubovskii,[3] and Yakubovskii-Narodestkii[4] equations. For the sake of completeness, it should be mentioned that there also exist N-particle formalisms intermediate between the chain and channel coupling classes. In these formalisms[20-22] integral equations are written for operators labelled by chains, but these

operators are components of the physical transition operators which are labelled by two partitions. While it is straightforward to apply our algebraic method to these equations, for simplicity of discussion we restrict our considerations to equations of the chain and channel coupling classes.

The paper is organized as follows. Section II contains a discussion of the general form of the N-particle equations and the basic group theoretic results associated with the treatment of identical particles. The symmetrization of the equations for the case of arbitrary numbers of identical particles is carried out in Sect. III and the mathematical properties of the symmetrized kernel are discussed. Section IV contains some applications and a study of the associated combinatorial problems. The results are summarized and conclusions presented in Sect. V.

II. PERMUTATION SYMMETRY IN N-PARTICLE SYSTEMS

There exist numerous formulations of N-particle scattering theory. Although they are all exact in principle, the N-particle dynamics is handled in different ways in the different theories. A bommon property of these exact theories is that they obtain coupled equations for operators or wave functions which are labelled by partitions or by chains of partitions[3] of the N-particle system.

Equations of the channel coupling class are written for the transition operators

$$T^{ab} = V^a + V^a G V^b \tag{1}$$

where a and b label partitions of the system. Equations of the chain coupling class are written for the quantities T^A where T^A is a component of the N to N transition operator

$$T^{00} \equiv T = \sum_{A} T^{A}. \tag{2}$$

We have used the notation 0 to indicate the N-cluster partition. The index A may be either a single partition or a set of partitions satisfying certain internal conditions (a chain). We will use Greek letters α, β, γ,... to indicate either a partition or a chain of partitions and we write the set of possible labels as $\mathcal{L}$. The general equation then takes the form

$$T^{\alpha} = I^{\alpha} + \sum_{\beta \in \mathcal{L}} K^{\alpha\beta} T^{\beta}. \tag{3}$$

If Eqs. (3) refer to transition operators, the quantities T^{α} and I^{α} also carry a second label, γ, which indicates the initial state of the scattering process, i.e., $T^{\alpha} \to T^{\alpha\gamma}$ and similarly for I^{α}. For the sake of simplifying the notation we suppress this index except where it is relevant.

Let us now assume that the N-particle system contains some particles which are identical. In this case, the permutations of the identical particles form a finite symmetry group whose elements commute with the exact N-particle Hamiltonian. If all the N-particles are identical, the symmetry group would be the full symmetric group on N objects, $\mathfrak{S}_N$. In the case that there are k different kinds of identical particles with n_i particles of the i-th type $(N = n_1 + n_2 + \ldots + n_k)$, then the symmetry group of the system, S, will be isomorphic to the direct product of symmetric groups, S_{n_i}, viz.,

$$S \cong S_{n_1} \times S_{n_2} \ldots \times S_{n_k}. \tag{4}$$

If the particles of the system are permuted by some element of S, then the system labels will in general also be affected. The group S induces a transformation group of the label set, $\mathcal{L}$, into itself. Symbolically we write

$$P\alpha = \alpha' \qquad P \in S. \tag{5}$$

Depending on the system of labelling, certain elements of the symmetry group will leave a given label invariant. For example, if the label is a partition, the interchange of a pair of identical particles within a single cluster of the partition or the exchange of two identical clusters will produce a new partition which is identified as being identical to the original one. The set of elements $P \in S$ for which

$$P\alpha = \alpha \tag{6}$$

forms a subgroup of S which we denote as S_α.

The reduction of the integral equations (3) by the transformation group S is made possible by the transformation properties of the inhomogeneous term and the kernel under S. Specifically we assume that

$$P\, I^{\alpha}\, P^{-1} = I^{P\alpha} \tag{7}$$

and

$$P\, K^{\alpha\beta}\, P^{-1} = K^{P\alpha P\beta}. \tag{8}$$

In the case of the channel coupling equations, the suppressed index of the inhomogeneous term also is assumed to transform, viz.,

$$P\, I^{\alpha\beta}\, P^{-1} = I^{P\alpha P\beta}. \tag{9}$$

We describe any labelled quantity which transforms via a relation analogous to (7) or (8) as <u>label transforming</u>. Specifically, this means that the transform of the labelled quantity is that same quantity with only its labels transformed.

We now demonstrate that the quantities T^α are label transforming. In general, the kernel $K^{\alpha\beta}$ is a function of a complex parameter z which is taken to be equal to the scattering energy approached from above in the complex plane.

We further assume that the kernel is compact or pre-compact in the complex z-plane cut along the positive real axis. The analytic Fredholm theorem[23] then implies that Eq. (3) has a unique solution everywhere in the complex cut plane except possibly on a discrete set D. We then have the following:

Theorem I: If I^{α} and $K^{\alpha\beta}$ transform by Eq. (7) and (8) and $E \notin D$ then

$$P\, T^{\alpha}\, P^{-1} = T^{P\alpha}. \tag{10}$$

In the channel coupling case the second index of T transforms like (9).

Proof: Apply P to the left of (3) and P^{-1} to the right. Introduce $P^{-1}P$ between the K and the T. Using (7) and (8) gives

$$(P\, T^{\alpha}\, P^{-1}) = I^{P\alpha} + \sum_{\beta \in \mathcal{L}} K^{P\alpha P\beta} (P\, T^{\beta}\, P^{-1}).$$

For any $P \in S$ as β runs over $\mathcal{L}$, so does $P\beta$. Replacing the unknown vector of operators $P\, T^{\alpha}\, P^{-1}$ by the operators $T^{P\alpha}$ yields a solution by (3). Since $E \notin D$ the solution is unique. Q.E.D.

The situations in which these transformation properties are relevant can easily be seen. In the formalisms discussed above the dependence on a particular particle is entirely through a potential involving that particle. In that case it is easy to see that a renumbering of the particles produced by any permutation simply changes the resulting labels as required. For example, in Ref. 19 classes of equations are considered in which $I^{\alpha\gamma} = C_{\gamma} V^{\alpha}_{\gamma}$ and $I^{\alpha\gamma} = V^{\alpha} W_{\alpha\gamma}$. In the first case, the coefficients C_{γ} depend only on the number of clusters in each partition and are therefore invariant under a transformation ($C_{Pa} = C_a \; \forall \; P \in S$). The inhomogeniety is therefore label transforming. In the second case (Kouri-Levin[7] couplings) $W_{\alpha\gamma}$ is a numerical matrix whose elements sum to unity along each row and

column. The value of a matrix element in this case depends on specific partitions, so in general $W_{PaPb} \neq W_{ab}$. This coupling scheme therefore does not satisfy transformation property (9).

Consider any label α. Since the permutation of identical particles does not change any physical property of the system, labels related by $\alpha' = P\alpha$ are physically equivalent. We therefore introduce a binary relation

$$\alpha' \mathcal{R} \alpha \qquad \text{iff} \quad \exists \; P \in S \ni P\alpha = \alpha'.$$

Lemma: $\mathcal{R}$ is an equivalence relation on $\mathcal{L}$.

The proof is straightforward.

The relation $\mathcal{R}$ therefore splits the set $\mathcal{L}$ into disjoint equivalence classes. Since the quantities which depend on the label set are in some sense physically equivalent for all labels in a given equivalence class, we expect that class functions can be constructed carrying all the physically relevant information. These class functions will be constructed in Sect. III.

We now construct the projection operators on states of proper symmetry. In the physical description of many-particle systems containing identical particles, the wave function of the system must transform by a one-dimensional irreducible representation of the symmetry group, namely

$$P\psi = \pm\psi$$

the plus or minus being chosen according to whether an even or odd number of fermions is exchanged by the operator P. Since S is a product of permutation groups S_i, and since every element of S_i can be written as a product of transpositions, it follows that every element of S can be written as the product of transpositions of identical particles. We define f_P to be +1 if, when written as the product of transpositions of identical particles, P contains an even number of fermion transpositions, and -1 if it contains an

odd number of fermion transpositions. For convenience we write $\hat{P} = f_P P$ and $|S|$ for the order of the group S. The symmetrizer on states of appropriate symmetry is[24]

$$R = \frac{1}{|S|} \sum_{P \in S} \hat{P}. \tag{11}$$

It satisfies the following properties:

$$R^2 = R \tag{12}$$

and

$$\hat{Q} R = R \qquad \forall\, Q \in S. \tag{13}$$

One may obtain a useful factorization of the operator R, which generalizes the one given in I. Consider a particular element of the label set, $\alpha \in \mathcal{L}$. Those elements of S which leave α invariant form a subgroup of S which we label S_α. We then have

$$P\alpha = \alpha \qquad \forall\, P \in S_\alpha \subset S. \tag{14}$$

If we label the equivalence class by $a = [\alpha]$ and if N_a denotes the number of elements of $\mathcal{L}$ in a, then by Lagrange's theorem[24] we have

$$N_a = |S|/|S_\alpha|. \tag{15}$$

We then have:

Theorem: For any element α of the equivalence class a, if

$$R_\alpha = \frac{1}{|S_\alpha|} \sum_{P \in S_\alpha} \hat{P} \tag{16}$$

then

$$R = \frac{1}{N_a} \sum_{\alpha' \in a} \hat{P}_{\alpha'\alpha} R_\alpha \tag{17}$$

$$R = \frac{1}{N_a} \sum_{\alpha' \in a} R_\alpha \hat{P}_{\alpha\alpha'} \tag{18}$$

where $P_{\alpha'\alpha}$ is any permutation in S which maps α into α'.

Proof: For each $\alpha' \in a$, $P_{\alpha'\alpha} R_\alpha$ is a sum of elements of S (together with their phases) which exhausts a single coset of S_α. The summation over α' produces

a sum over cosets. Since the cosets are disjoint, when the sum in (17) is expanded using (16), the resulting sum exhausts the entire group. The correctness of the normalization follows from (15). Similar arguments yield Eq. (18). Q.E.D.

Finally, we observe the result.

Lemma: For any label transforming quantity, A^{α}

$$R_{\alpha} A^{\alpha} = A^{\alpha} R_{\alpha}. \tag{19}$$

Proof: By the label transforming property, we have

$$P A^{\alpha} = A^{\alpha} P \qquad \forall P \in S_{\alpha}.$$

The result follows upon inserting the explicit expression for R_{α}, Eq. (16).

III. INTEGRAL EQUATIONS FOR IDENTICAL PARTICLE SCATTERING

In this section we construct operators labelled by the equivalence classes of the label set. These operators carry the complete content of the permutation symmetry of the N-particle system. Integral equations for these quantities are derived, reducing the number of coupled equations and the number of quantities required for the description of real processes.

Our first task is to define appropriate operators labelled by equivalence classes of labels. We review briefly the results of I to clarify the procedure. There, the quantities considered were transition operators for a system of N identical particles. Their matrix elements between fully symmetrized channel states gave the transition probabilities. Class operators are constructed by defining

$$T^{ab} = \sqrt{\frac{N_b}{N_a}} \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0 \alpha} T^{\alpha \beta_0} \tag{20}$$

where N_b and N_a are constants associated with the normalization of the channel wave functions. Here α_0 and β_0 denote fixed, but arbitrary, representatives of the equivalence classes a and b, respectively. These are referred to as canonical labels. The transition probability for physical processes is given by the

on-shell matrix element

$$\mathcal{T}^{ab} = \langle \Phi_{\alpha_0} | T^{ab} | \Phi_{\beta_0} \rangle \tag{21}$$

where the states Φ_{α_0} and Φ_{β_0} are channel wave functions satisfying the symmetry internal to the bound clusters of the channel, i.e.,

$$\begin{aligned} R_{\alpha_0} \Phi_{\alpha_0} &= \Phi_{\alpha_0} \\ R_{\beta_0} \Phi_{\beta_0} &= \Phi_{\beta_0}. \end{aligned} \tag{22}$$

The class function in this case was constructed by defining a canonical initial label and symmetrizing on the left. This is sufficient due to the well-known property that one may symmetrize either the initial state or the final state in a many-body scattering matrix element.[25] It is not necessary to symmetrize on both sides. The internal symmetry of the wave functions used to calculate matrix elements was retained because of the fact that bound states of the proper symmetry are then required.

We will construct class operators for our two general classes of scattering equations in a similar way. We first demonstrate that Eq. (3) can be shown to give an equation for the class operator constructed for the case of a general symmetry group. Then we show how the physical matrix elements are related to those of the class operators in the two cases. Finally we demonstrate that the connectivity structure of the equation is not destroyed in the transition to an equation for the class operators.

The class function we construct is

$$T^a = \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0 \alpha} T^{\alpha}, \tag{23}$$

for the chain coupling class and

$$T^{ab} = \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0 \alpha} T^{\alpha \beta_0} \sqrt{\frac{N_b}{N_a}} \tag{24}$$

for the channel coupling class. In the channel coupling case we take the initial index to be β_0 and suppress it until the discussion of the matrix elements.

Multiplying Eq. (3) by $R_{\alpha_0} \hat{P}_{\alpha_0\alpha}$ and summing on the index α gives the equation

$$T^a = I^a + \sum_\beta X^{a\beta} T^\beta \tag{25}$$

where we have defined

$$I^a = \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0\alpha} I^\alpha \tag{26}$$

and

$$X^{a\beta} = \sum_{\alpha \in a}{}' R_{\alpha_0} \hat{P}_{\alpha_0\alpha} K^{\alpha\beta}. \tag{27}$$

The appropriate permutation operators must be extracted from $X^{a\beta}$ on the right. This is made possible by the following.

Proposition:

$$X^{a\beta} \hat{P}_{\beta\beta'} = X^{a\beta'}. \tag{28}$$

Proof: This property follows directly from the transformation property of K. From the definition of X and the transformation property of K, we have

$$X^{a\beta} \hat{P}_{\beta\beta'} = \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0\alpha} K^{\alpha\beta} P_{\beta\beta'}$$

$$= \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0\alpha} \hat{P}_{\beta\beta'} K^{Q\alpha Q\beta}$$

where for simplicity of notation we have written

$$Q = P^{-1}_{\beta\beta'}.$$

Now $P^{-1}_{\beta\beta'} = P_{\beta'\beta}$ so

$$Q\beta = P_{\beta'\beta}\,\beta = \beta'$$

and

$$\hat{P}_{\alpha_0\alpha} \hat{P}_{\beta\beta'} = \hat{P}^{-1}_{\alpha\alpha_0} \hat{Q}^{-1} = (\hat{Q} \hat{P}_{\alpha\alpha_0})^{-1}$$

$$= (\hat{P}_{Q\alpha\alpha_0})^{-1} = \hat{P}_{\alpha_0 Q\alpha}.$$

Therefore, we have

$$X^{a\beta}\hat{P}_{\beta\beta'} = \sum_{\alpha\in a} R_{\alpha_0}\hat{P}_{\alpha_0 Q\alpha}K^{Q\alpha\,\beta'}.$$

As α runs over all the elements of a, so does $Q\alpha$ for any Q. Therefore, the sum may be taken over $Q\alpha$ instead of over α; and relabelling the dummy index $Q\alpha$ as α gives the result. Q.E.D.

Corollary:

$$X^{a\beta}R_\beta = X^{a\beta}. \tag{29}$$

This result follows immediately from the proposition as all the terms $\hat{P}$ contained in the R_β sum leave β invariant.

These two propositions allow us to make the right index on the $X^{a\beta}$ in Eq. (27) canonical by pulling out the appropriate permutation operator. The resulting operator is a class operator and the sum over all β may be broken up into a sum over b and a sum over $\beta\in b$. Using the proposition, Eq. (25) becomes

$$T^a = I^a + \sum_\beta X^{a\beta_0}\hat{P}_{\beta_0\beta}T^\beta. \tag{30}$$

By the corollary the internal symmetrizer may be extracted to give

$$T^a = I^a + \sum_\beta X^{a\beta_0}R_{\beta_0}\hat{P}_{\beta_0\beta}T^\beta. \tag{31}$$

Finally, breaking up the sum gives the result

$$T^a = I^a + \sum_b \mathcal{K}^{ab}T^b \tag{32}$$

where we have written

$$\mathcal{K}^{ab} = X^{a\beta_0} = \sum_{\alpha\in a} R_{\alpha_0}\hat{P}_{\alpha_0\alpha}K^{\alpha\beta_0} \tag{33}$$

for the symmetrized kernel. Equation (32) is an integral equation for the class operators as desired.

We now consider the relation of the physical matrix elements to the matrix elements of the class operators. In the chain coupling case the physical matrix elements desired could be the matrix elements of the full T operator between some arbitrary initial state, ϕ; and a set of final non-interacting states of all the particles, ϕ_0. At least one of these states must have the correct symmetry. Because our equations (3) were chosen to have the T on the right, we have symmetrized from the left. We therefore require that the left wave function have the proper symmetry. The physical matrix element is therefore

$$\mathcal{T} = \langle \tilde{\phi}_0 | T | \phi' \rangle \tag{34}$$

where[25]

$$\langle \tilde{\phi}_0 | = \langle \phi_0 | R [n_1! n_2! \ldots n_k!]^{1/2}. \tag{35}$$

The state ϕ_0 may be a set of plane waves, for example, or a coordinate space state in which each of the particles is at a particular point. Using (2), the physical matrix elements may be written

$$\mathcal{T} = [n_1! n_2! \ldots n_k!]^{1/2} \langle \phi_0 | R \sum_\alpha T^\alpha | \phi' \rangle. \tag{36}$$

By properties (13) and (16), the required permutation operators may be introduced before the T^α. We may therefore write

$$\mathcal{T} = [n_1! n_2! \ldots n_k!]^{1/2} \langle \phi_0 | R \sum_\alpha R_{\alpha_0} \hat{P}_{\alpha_0 \alpha} T^\alpha | \phi' \rangle. \tag{37}$$

Decomposing the sum into a sum over classes, a, and a sum over the elements in those classes, $\alpha \in a$ gives

$$\mathcal{T} = [n_1! \ldots n_k]^{1/2} \langle \phi_0 | R \sum_a T^a | \phi' \rangle \tag{38}$$

so

$$\mathcal{T} = \langle \tilde{\phi}_0 | \sum_a T^a | \phi' \rangle. \tag{39}$$

For the channel coupling case the physical matrix elements are transition matrix elements. The indices of the T operators are partitions and one takes matrix elements between states of the appropriate partition Hamiltonians. Physical channels are labelled by class indices instead of partitions. The transition matrix element between a state of class b and one of class a will be

$$\mathcal{T}'_{ab} = \langle \tilde{\phi}_a | U | \tilde{\phi}_b \rangle \tag{40}$$

where U is the Ekstein transition operator (see Eq. (2.5) of I) and $|\tilde{\phi}_a\rangle$ and $|\tilde{\phi}_b\rangle$ are final and initial states of the proper symmetry. Following I, we express both final and initial states in terms of symmetrizations of canonical states. We therefore set

$$|\tilde{\phi}_b\rangle = \sqrt{N_a}\, R\, |\phi_{\beta_0}\rangle \tag{41}$$

and similarly for $|\tilde{\phi}_a\rangle$. The initial state consists of the bound clusters of the representative partition β_0 and plane waves for their relative motion. As in I we assume that the wave functions of the bound clusters in both the initial and final states have the proper symmetry for each cluster, but have not been symmetrized between members of different clusters. This allows the straightforward insertion of standard bound state theories (which have the correct symmetry) into the scattering equations.

The operator $\mathcal{T}'_{ab}$ may now be expressed in terms of matrix elements of the T operator (1) as follows. Using the fact that R commutes with U and $R^2 = R$, inserting (41) into (40) gives

$$\mathcal{T}'_{ab} = \sqrt{N_a N_b}\, \langle \phi_{\alpha_0} | R\, U | \phi_{\beta_0} \rangle . \tag{42}$$

Expanding R by Eq. (11) yields

$$\mathcal{T}'_{ab} = \sqrt{N_a N_b}\, \frac{1}{|S|} \sum_{P \in S} \langle \phi_{\alpha_0} | P^{-1} U | \phi_{\beta_0} \rangle$$

$$= \frac{\sqrt{N_a N_b}}{|S|} \sum_{P \in S} \langle \phi_{P\alpha_0} | U | \phi_{\beta_0} \rangle . \tag{43}$$

Since the matrix element of U agrees with that of T^{ab} on-shell, we may replace $\mathcal{T}'_{ab}$ by

$$\mathcal{T}_{ab} = \frac{\sqrt{N_a N_b}}{|S|} \sum_{P \in S} \langle \phi_{P\alpha_0} | T^{P\alpha_0\, \beta_0} | \phi_{\beta_0} \rangle . \tag{44}$$

Using Eq. (22), the definition of S_α and the fact that

$$\langle \phi_\alpha | = \langle \hat{P}_{\alpha\alpha_0} \phi_{\alpha_0} | = \langle \phi_{\alpha_0} | \hat{P}^\dagger_{\alpha\alpha_0} = \langle \phi_{\alpha_0} | \hat{P}_{\alpha_0\alpha} \tag{45}$$

we obtain

$$\mathcal{T}_{ab} = \frac{\sqrt{N_a N_b}}{|S|} |S_a| \sum_{\alpha \in a} \langle \phi_{\alpha_0} | R_{\alpha_0} \hat{P}_{\alpha_0\alpha} T^{P\alpha_0\, \beta_0} | \phi_{\beta_0} \rangle . \tag{46}$$

It is now only necessary to observe the following:

Lemma:

$$N_a = \frac{|S|}{|S_a|} . \tag{47}$$

This follows very directly from the arguments of Ref. 25. Assuming the asymptotic states $|\tilde{\phi}_a\rangle$ and $|\tilde{\phi}_b\rangle$ correspond to normalizable wave packets we must have

$$\langle \tilde{\phi}_a | \tilde{\phi}_a \rangle = 1$$

so

$$N_a = (\langle \phi_{\alpha_0} | R | \phi_{\alpha_0} \rangle)^{-1} .$$

Expanding out R by (11) gives

$$\langle \phi_{\alpha_0} | R | \phi_{\alpha_0} \rangle = \frac{1}{|S|} \sum_{P \in S} \langle \phi_{\alpha_0} | \hat{P} | \phi_{\alpha_0} \rangle .$$

If we choose wave pakcets such that all the clusters are well separated, interchanging between different clusters gives 0 and within the same cluster gives 1 (assuming $\langle \phi_{\alpha_0} | \phi_{\alpha_0} \rangle = 1$). Therefore we have

$$\langle \phi_{\alpha_0} | R | \phi_{\alpha_0} \rangle = \frac{|S_a|}{|S|} = \frac{1}{N_a} .$$

We therefore have

$$\mathcal{T}_{ab} = \sqrt{\frac{N_b}{N_a}} \sum_{\alpha \in a} \langle \phi_\alpha | T^{\alpha \beta_0} | \phi_{\beta_0} \rangle . \tag{48}$$

Introducing the class operator given by (24) the physical matrix element is given by

$$\mathcal{T}_{ab} = \langle \phi_{\alpha_0} | T^{ab} | \phi_{\beta_0} \rangle . \tag{49}$$

We observe that all the complex counting considered in paper I (including the δ term of Eq. (2.35)) is obtained from the straightforward group theoretic considerations of Eq. (47).

We now consider the connectivity structure of the resulting equations for the class operators, Eq. (32). We have the following:

Theorem: If the n-th power of the operator $K^{\alpha\beta}$ is completely connected, then so is the n-th power of the operator $\mathcal{K}^{ab}$.

Proof: Let us consider the m-th power of the operator $\mathcal{K}$. The theorem follows immediately from the following.

Lemma:

$$(\mathcal{K}^m)^{ab} = \sum_{\alpha \in a} R_{\alpha_0} \hat{P}_{\alpha_0 \alpha} (K^m)^{\alpha \beta_0} . \tag{50}$$

To prove the Lemma we observe that

$$\begin{aligned} (\mathcal{K}^m)^{ab} &= \sum_c (\mathcal{K}^{m-1})^{ac} \mathcal{K}^{cb} \\ &= \sum_c (\mathcal{K}^{m-1})^{ac} \sum_{\gamma \in c} R_{\gamma_0} \hat{P}_{\gamma_0 \gamma} K^{\gamma \beta_0} \end{aligned} \tag{51}$$

by the definition of matrix multiplication and of $\mathcal{K}$ (Eq. (33)). We now note that if we have (where $j = m - k$)

$$(\mathcal{K}^m)^{ab} = \sum_c (\mathcal{K}^j)^{ac} \sum_{\gamma \in c} R_{\gamma_0} \hat{P}_{\gamma_0 \gamma} (K^k)^{\gamma \beta_0} \tag{52}$$

with $j > 1$, we can write

$$\begin{aligned} (\mathcal{K}^m)^{ab} &= \sum_{d,c,\gamma \in c} (\mathcal{K}^{j-1})^{ad} \mathcal{K}^{dc} R_{\gamma_0} \hat{P}_{\gamma_0 \gamma} (K^k)^{\gamma \beta_0} \\ &= \sum_{d,\gamma} (\mathcal{K}^{j-1})^{ad} X^{d\gamma_0} R_{\gamma_0} \hat{P}_{\gamma_0 \gamma} (K^k)^{\gamma \beta_0} . \end{aligned}$$

Using the proposition and corollary (Eqs. (28) and (29)) gives

$$= \sum_{d,\gamma} (\mathcal{K}^{j-1})^{ad} X^{d\gamma} (K^k)^{\gamma\beta}{}_0$$

which by the definition of the operator X (Eq. (27)) becomes

$$= \sum_{d,\gamma} (\mathcal{K}^{j-1})^{ad} \sum_{\delta \in d} R_{\delta_0} \hat{P}_{\delta_0 \delta} K^{\delta\gamma} (K^k)^{\gamma\beta}{}_0$$

$$= \sum_{d} (\mathcal{K}^{j-1})^{ad} \sum_{\delta \in d} R_{\delta_0} \hat{P}_{\delta_0 \delta} (K^{k+1})^{\delta\beta}{}_0 . \qquad (53)$$

This equation has the same structure as Eq. (52) with k incremented by one. We may therefore induce on k beginning with k = 1 (Eq. (51)) and bring the operator over to the left until only a single power of $\mathcal{K}$ remains. The final step follows in a similar manner. Q.E.D.

We have therefore demonstrated that the symmetrization (23)-(24) defines class functions for both the chain and channel coupling cases, that equations for these class operators can be defined (Eq. (32)) with all the symmetry carried in the structure of the inhomogeneity (26) and the kernel (33), that the physical matrix elements are given simply in terms of matrix elements of the class operators (Eqs. (39) and (49)), and finally that the connectivity structure of the unsymmetrized equation is maintained by the symmetrization.

We conclude this section with a few comments about the KLT equations.[7,8] As we remarked above, the KLT method uses a numerical matrix in the kernel of their equations so although the KLT equations have the structure of (3), neither the kernel nor the resulting T operator are label transforming. There are many T operators corresponding to different choices of the W matrix. These T operators have different off-shell continuations, although the exact operators agree on-shell. The permutation transformation $T^{\alpha\beta} \to P T^{\alpha\beta} P^{-1}$ maps one T operator into a T operator corresponding to a different coupling scheme. This prevents the application of our procedure to this case. An attempt to symmetrize these equations was made by Tobocman.[18] In this work the equations are symmetrized before coupling in contrast to our method which symmetrizes after the coupled equations have been constructed. Tobocman's method does not produce an equation with connected kernel.

IV. APPLICATIONS AND ASSOCIATED COMBINATORIAL PROBLEMS

The results presented in the previous section make it straightforward to construct properly symmetrized N-particle integral equations once the symmetry gorup of the system is given. Only the basic combinatorial problems[26] of ennumerating equivalence classes needs to be solved to make the equations ready for practical applications.

In the following we present two specific applications in order to show precisely what problems arise and how they are dealt with. We consider first the nuclear physics example of N identical protons and Z identical neutrons described by channel coupling class equations employing minimal (two-cluster) coupling. As a second example we consider the problem of a homonuclear diatomic molecule with $2Z$ electrons and two identical spin zero (boson) nuclei, treated by the Yakubovskii equations,[3] a chain coupling case.

For the first example, N identical neutrons and Z identical protons, the permutation group of the system is isomorphic to

$$S \cong S_N \times S_Z$$

so the order of the group is $|S| = N!Z!$. We consider a formulation in which the indices are two-cluster partitions. An example of a formulation of this type is that of BRS which has a kernel whose first iterate is completely connected.

To construct the physical quanitites needed one must determine the equivalence classes of the labels and the number of elements in each equivalence class. The number of distinct labels for N + Z distinguishable particles is known[6] to be $2^{N+Z-1} - 1$. For the system with identical particles, all the distinct physical two-cluster partitions may be characterized by a

pair of positive integers, (n,z), which give the number of neutrons and protons in the smaller fragment. Since the number of neutrons and protons are both fixed, the second fragment must contain (N - n,Z - z) neutrons and protons, respectively.

A pair of integers, (N,Z), where N and Z cannot both vanish simultaneously, is known in the theory of combinatorics as a bipartite number.[27] The equivalence classes of the two-cluster partitions correspond precisely to all the partitions of a bipartite number into two bipartite numbers. The number of such partitions was given by Macmahon[27] in closed form as

$$R_{NZ} = \left[\frac{(N+1)(Z+1)-1}{2}\right] \tag{54}$$

This is therefore the number of equivalence classes and therefore the number of coupled equations in Eq. (32) for this example.

We now must determine the number of elements in each equivalence class in order to construct the operators R_{α_0}. Given a particular two-cluster partition of the form, (n,z) - (N - n,Z - z), how many distinct partitions are there in the same equivalence class? If one applies all the permutations in S one would obtain N!Z! partitions. Not all of them are distinct, however, as exchainging protons and/or neutrons in a single cluster doesn't lead to a new partition. We have to divide by the number of ways of permuting the protons and neutrons in each of the two clusters. This reduces the number of elements to N!Z!/(n!(N - n)!z!(Z - z)!)or $\binom{N}{n}\binom{Z}{z}$. If the two clusters of the partition are identical, then the complete interchange of the two clusters does not produce a new partition. This requires reducing the number of distinct partitions produced by an addition factor of two. We therefore obtain

$$N_a = N_{(n,z)} = \binom{N}{n}\binom{Z}{z}\frac{1}{1+\delta_{N/2\,n}\,\delta_{Z/2\,z}}. \tag{55}$$

These numbers fix the normalization of the operators R_{α_0} which will be used in constructing the inhomogeneous term by Eq. (26) and the kernel by Eq. (33). This is also the number of terms in the $\alpha \in a$ summations (Eqs. (23) and (24)).

Let us now investigate how the Yakubovskii equations have to be symmetrized for the case of a diatomic homonuclear molecule. The labels for these equations are chains of partitions, $a_1 \supset a_2 \supset a_3 \supset \cdots a_{N-1} \supset a_N$ where a_j represents a partition of the N-particles into j-clusters and the relation $a_j \supset a_{j+1}$ means the partition a_{j+1} can be obtained by breaking a single one of the clusters of a_j. The initial and final partitions, a_1 and a_N, being unique are usually omitted.

For distinguishable particles, the number of complete (maximal) chains can be ennumerated by elementary methods. This can be done because the set of partitions forms a semimodular, relatively complemented lattice.[28] When the symmetry group of the system is applied, the set partitions and therefore the set of labels is split up into disjoint equivalence classes. The equivalence classes of the partitions can still be regarded as a partially ordered set, since the ordering of the partitions is preserved by the mapping of each partition into its equivalence class. However, the lattice property of the set of partitions is in general not preserved. This hinders one considerably in trying to find a closed form expression for the number of equivalence classes of maximal chains. The problem is equivalent to a currently unsolved problem in graph theory; however, an algorithm can be given for ennumerating the number of equivalence classes of chains without explicitly symmetrizing the original equations. The explicit symmetrization procedure has been used in all previously considered examples, viz., the works of Kharchenko and Kuzmichev[14] and that of Roy-Choudhury, et al.[29]

Under the symmetrization group, each partition will map into a multipartite number. Therefore each chain will map into a chain of multipartite numbers. As a specific example we consider the system D_2 consisting of two deuterons and two electrons, labelled 1 = d, 2 = d, 3 = e, and 4 = e. All the possible distinct partitions are indicated in Fig. 1 with the possible inclusions shown by arrows. One can easily read off from this diagram that the number of maximal chains is 7. The number of coupled Yakubovskii equations for this system will therefore be 7. The kernels and inhomogeneous terms for the equations coupling the operators labelled by these chains can easily be constructed once the subgroups for each chain are determined. The number of Yakubovskii and BRS equations for the various cases of different numbers of identical particles in the three-, four-, and five-body problem are given in Table I.

To be specific, the full symmetry group is generated by the operators P_{12} (interchange of the deuterons) and P_{34} (interchange of the electrons). Taking into account the fact that the deuterons are bosons and the electrons are fermions yields the full symmetrizer

$$R = \frac{1}{4}(1 + P_{12} - P_{34} - P_{12}P_{34}).$$

Consider the chain a = (dd)(ee) $\supset$ (d)(d)(ee). A canonical chain for this equivalence class is the chain α_0 = (12)(34) $\supset$ (1)(2)(34). The full symmetry group leaves this chain invariant. Therefore, we construct the operator

$$R_{\alpha_0} = R.$$

There is only a single term in the α sum in equations (23), (26), and (33). For the chain b = (d)(dee) $\supset$ (d)(d)(ee) a canonical chain is β_0 = (1)(234) $\supset$ (1)(2)(34). Only the group elements 1 and P_{34} leave this chain invariant. Therefore we have

$$R_{\beta_0} = \frac{1}{2}(1 - P_{34}).$$

There are two chains in this equivalence class, β_0, and the chain $(2)(134) \supset (1)(2)(34)$. There will therefore be two elements in the sums $\beta \in b$.

In general, each chain can be associated with a tree. The trees for the chains α_0 and β_0 in the above example are shown in Fig. 2. The invariance subgroup of a particular chain is determined by the number of indistinguishable particles which are connected directly together rather than being joined to a cluster. The internal symmetry operator for each chain must be determined by considering the structure of the specific chain.

Upon the construction of these operators, the equations coupling the symmetrized operators, (32), may be written down directly for any case without the need of beginning with the original equations (3).

V. CONCLUSIONS

The treatment of identical particles in N-particle scattering gives rise to nontrivial mathematical problems. In this paper we demonstrate that a general algebraic treatment can be developed for a large class of N-particle scattering theories. This treatment relies on the properties of the permutation group of the system. These theories include channel coupling equations such as those of BRS or Chandler and Gibson (but explicitly excluding theories of KLT type) and chain coupling equations such as those of Rosenberg, Yakubovskii and Narodestkii-Yakubovskii.

There are two features of basic importance which make the general algebraic treatment possible. First, in all the N-particle theories the quantities to be determined are labelled by partitions or (incomplete or complete) chains of partitions of the N-particle system. Consequently, the

number of coupled equations is determined by the way of labelling rather than the treatment of N-body dynamics. Second, the permutation group of the system generates an equivalence relation on the set of labels and accordingly splits it into disjoint equivalence classes. Only the equivalence classes and operator valued functions of these classes have physical meaning. This statement is just an abstract formulation of the property that due to the indistinguishability of some of the particles the amplitudes of the physical processes can be expressed as coherent sums of "direct" and "exchange" processes. In our general algebraic treatment we construct operators which are sums of permutation operators acting on the scattering operators of the relevant theory. These class operators given by Eqs. (23) and (24) are labelled by the equivalence classes of the set of original labels. If the inhomogeneous term and the kernel of the relevant set of N-particle equations satisfy certain general conditions, i.e., if they are label transforming, it follows that the N-particle equations can be reformulated in terms of the class operator, with a simultaneous reduction in the number of coupled equations. Furthermore, the physical matrix elements are expressed in terms of matrix elements of the class operators. The entire burden of the symmetry is carried by the symmetrization of the inhomogeneous term and the kernel. We also demonstrate that the connectivity properties of the kernel are passed on to the symmetrized equation.

The most remarkable fact exhibited by the considerations of this paper is that the treatment of identical particles, i.e., exchange effects, is actually independent of the N-particle dynamics for a large class of N-particle scattering theories. This is the consequence of the nondynamical nature of the permutation symmetry of the system.

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TABLE CAPTION

Table I.: Number of coupled integral equations for N=3, 4, and 5 particles in the Yakubovskii (Y) and Bencze-Redish-Sloan (BRS) methods. The labels a, b, c, ... indicate distinct species of particles.

TABLE I

	N = 3		
	aaa	aab	abc
Y	1	2	3
BRS	1	2	3

	N = 4				
	aaaa	aaab	aabb	aabc	abcd
Y	2	5	7	11	18
BRS	2	3	4	5	7

	N = 5						
	aaaaa	aaaab	aaabb	aaabc	aabbc	aabcd	abcde
Y	4	15	26	45	61	105	180
BRS	2	4	5	7	8	11	15

FIGURE CAPTIONS

Fig. 1: The partition set for a four-body system consisting of two pairs of identical particles. Arrows indicate inclusion. A complete maximal chain is a path leading from the top element (ddee) to the bottom (d)(d)(e)(e). There are seven such distinct chains.

Fig. 2: Tree representations of the canonical chains in the four-body problem ddee. The chains are read off from the tree by following the successive connections down from the top of the tree. Identical particles are marked by curved brackets. The dotted boxes indicate parts of the tree which are invariant under an element of the symmetry group.

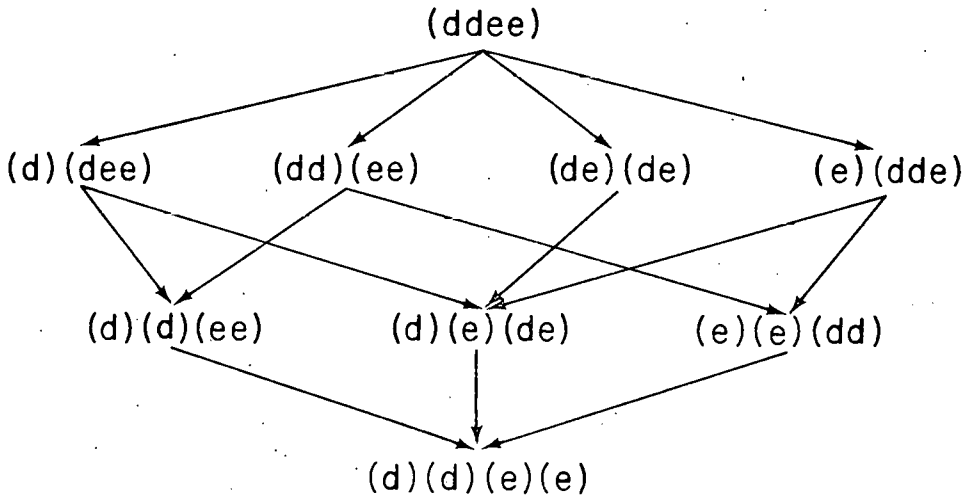

Fig. 1

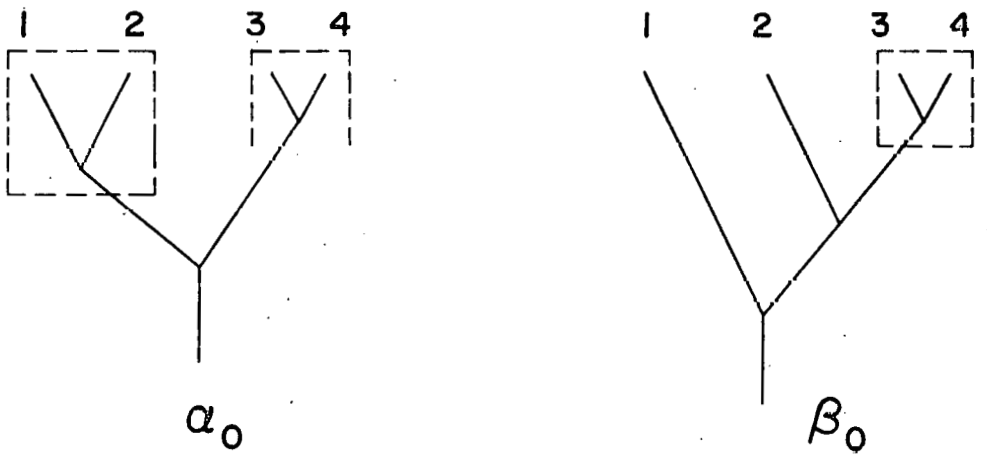

Fig. 2