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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFUSION-LIKE PARTIAL DIFFERENTIAL EQUATIONS INVARIANT TO A FAMILY OF AFFINE GROUPS

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ABSTRACT

This report deals with the asymptotic behavior of certain solutions of partial differential equations in one dependent and two independent variables (call them c, z, and t, respectively). The partial differential equations are invariant to one-parameter families of one-parameter affine groups of the form

$$c' = \lambda^{\alpha} c \ ,$$

$$t' = \lambda^{\beta} t \ ,$$

$$z' = \lambda z \ ,$$

where λ is the group parameter that labels the individual transformations and α and β are parameters that label groups of the family. The parameters α and β are connected by a linear relation,

$$M\alpha + N\beta = L \ ,$$

where M, N, and L are numbers determined by the structure of the partial differential equation.

It is shown that when L/M and N/M are <0, certain solutions become asymptotic to $z^{L/M} t^{-N/M}$ for large z or small t. Some practical applications of this result are discussed.

INTRODUCTION

In two earlier publications[1,2] I studied the properties of certain partial differential equations with one dependent and two independent variables (call them c , z, and t, respectively). These partial differential equations are invariant to one-parameter families of one-parameter affine groups of the form

$$\left.\begin{aligned} c' &= \lambda^{\alpha} c \\ t' &= \lambda^{\beta} t \\ z' &= \lambda z \end{aligned}\right\} \qquad 0 < \lambda < \infty , \tag{1}$$

where λ is the group parameter that labels the individual transformations and α and β are parameters that label groups of the family. The parameters α and β are connected by a linear relation,

$$M\alpha + N\beta = L , \tag{2}$$

where M, N, and L are numbers determined by the structure of the partial differential equation. Because of relation (2), only one of the two parameters α and β may be chosen freely.

Similarity solutions are solutions of the partial differential equation that are invariant to one group of the family, say, that for which $\alpha = \alpha_0$ and $\beta = \beta_0$. Such solutions most generally have the form

$$c = t^{\alpha_0/\beta_0} y(z/t^{1/\beta_0}) , \tag{3}$$

where y is a function of the single variable $x = z/t^{1/\beta_0}$. When substituted into the partial differential equation, Eq. (3) yields an ordinary differential equation for y called the principal ordinary differential equation. The form of the principal ordinary differential equation depends on both the form of the partial differential equation and the values of α_0 and β_0.

Much attention has been paid in refs. 1 and 2 to diffusion-like partial differential equations such as $c_t = (c^m c_z)_z$ (the so-called porous medium equation), $cc_t = c_{zz}$ (which describes thermal expulsion of a compressible liquid from a long, slender, heated tube), and $c_t = (c_z^{1/3})_z$ [which describes heat transport in turbulent superfluid helium (He-II)]. Among the interesting solutions of these equations are those that obey the boundary and initial conditions

$$c(\infty,\ t) = 0 , \tag{4a}$$

$$c(z,\ 0) = 0 . \tag{4b}$$

To define a solution completely, an additional boundary condition is necessary. If it takes the form

$$c(0,t) = At^n , \tag{5}$$

where A and n are constants, then the solution is a similarity solution of the form (3). Equations (4a) and (4b) then collapse to the single condition $y(\infty) = 0$.

Many detailed calculations of similarity solutions of the three partial differential equations mentioned above for various α_0 and β_0 showed that quite often, when $L/M < 0$, the function y approaches zero as x approaches ∞ asymptotically to $ux^{L/M}$, where u is a constant. When substituted into Eq. (3), this gives the asymptotic form $uz^{L/M}t^{-N/M}$ for c, which would obey the conditions (4a) and (4b) if L/M and N/M were both < 0. Demonstrating that c has the asymptotic form $uz^{L/M}t^{-N/M}$ in the special cases studied was quite laborious, and it is the purpose of this report to outline broad conditions under which this asymptotic form can be verified if not at a glance, then at least with a minimum of computational effort.

FORM OF THE PARTIAL DIFFERENTIAL EQUATION

The partial derivative c_t transforms under transformation (1) according to $c'_{t'} = \lambda^{\alpha-\beta}c_t$; similarly, all other partial derivatives transform by multiplication by some power of λ. If a partial differential equation involving z, t, c, c_t, c_z, c_{zz}, ... is to be invariant to all groups of the family (1), then it can only contain λ-less combinations of z, t, c, c_t, c_z, c_{zz}, etc. Here the term "λ-less" is to be understood as the term "dimensionless" is understood in ordinary dimensional analysis. Thus the partial differential equation must have the form

$$F\left(\frac{c}{z^{L/M}t^{-N/M}}\,,\ \frac{tc_t}{c}\,,\ \frac{zc_z}{c}\,,\ \frac{t^2c_{tt}}{c}\,,\ \frac{z^2c_{zz}}{c}\,,\dots\right) = 0\,, \tag{6}$$

where F can be any function.

It is easy enough to see that Eq. (6) is invariant to Eq. (1): if we imagine Eq. (6) to be written in terms of the primed variables and substitute for them from Eq. (1), we obtain Eq. (6) again in the unprimed variables. It is proved in Appendix A that only forms composed of λ-less terms have this property.

Can this partial differential equation have solutions of the form

$$c = uz^a t^b \ ? \tag{7}$$

Direct substitution into Eq. (6) shows that Eq. (7) can only be a solution when $a = L/M$, $b = -N/M$, and u satisfies the equation

$$F\left[u\,,\ -\frac{N}{M}\,,\ \frac{L}{M}\,,\ \frac{N}{M}\left(\frac{N}{M}+1\right)\,,\ \frac{L}{M}\left(\frac{L}{M}-1\right)\,,\ \dots\right] = 0\,. \tag{8}$$

If Eq. (8) has real solutions for u, then the partial differential equation has real solutions of the form Eq. (7); if not, it has none. If $L/M < 0$ and $N/M < 0$, then Eq. (7) is capable of representing the asymptotic limit of solutions obeying the boundary and initial conditions (4a) and (4b). But under what conditions *must* Eq. (7) represent this limit?

ORDERED SOLUTIONS

For the superfluid diffusion equation and the thermal expulsion equation, it can be shown that for solutions $c_1(z,t)$ and $c_2(z,t)$ that obey the boundary and initial conditions (4a) and (4b), if $c_1(0,t) > c_2(0,t)$, then $c_1(z,t) \geq c_2(z,t)$ for all z. Such solutions are thus ordered according to their values at $z = 0$.

One consequence of such ordering is that for any similarity solutions, the function y obeys the inequalities

$$0 \leq y(x) \leq u x^{L/M} \tag{9}$$

as long as $y(0)$ is finite. Here u is the smallest real solution of Eq. (8). Let us now consider the functions $y(x)$ belonging to the values α_0 and β_0 of α and β. These functions are a one-parameter family ordered according to their intercepts $y(0)$ on the y-axis. In refs. 1 and 2, it is shown that the principal ordinary differential equation for the functions $y(x)$ is invariant to the affine group*

$$\left.\begin{aligned} y' &= \mu^{L/M} y \\ x' &= \mu x \end{aligned}\right\} \quad 0 < \mu < \infty \,. \tag{10}$$

So each of the curves $y(x)$ is the image of any other because $y'(0) = \mu^{L/M} y(0)$ can be given any value by the appropriate choice of μ while $y'(\infty) = \mu^{L/M} y(\infty)$ remains zero. Thus, the entire family of these similarity solutions $y(x)$ is transformed into itself.

Because of Eq. (10),

$$\frac{y'(x')}{u x'^{L/M}} = \frac{\mu^{L/M} y(x)}{u(\mu x)^{L/M}} = \frac{y(x)}{u x^{L/M}} \,, \tag{11}$$

so that the limits as x and x' approach infinity of the left- and right-hand sides are the same. Suppose this limit < 1. The infinitude of curves $y(x)$, being bounded from above, has an upper limit $y_\infty(x)$. This limit, too, is a solution of the principal ordinary differential equation. Furthermore, because the entire family is invariant to Eq. (10), its upper limit $y_\infty(x)$ must also be invariant to Eq. (10). Curves invariant to Eq. (10) must have the form $y = v x^{L/M}$, where v is a constant. Since this invariant curve also satisfies the principal ordinary differential equation, it must correspond to a solution $c = v z^{L/M} t^{-N/M}$ of the partial differential equation. But then v must be a root of Eq. (8). Since $v < u$, and u is the smallest root of Eq. (8), we have a contradiction. Therefore, the limit of both sides of Eq. (11) as x and x' approach infinity must be 1. This means that all the similarity solutions have the asymptotic form $u z^{L/M} t^{-N/M}$.

Because of the ordering of the solutions of the partial differential equations, this conclusion holds as well for any solution $c(z,t)$ that obeys the boundary and initial conditions (4a) and (4b) and whose value at the origin $c(0,t)$ is bounded above and below by powers of t.

THE SUPERFLUID DIFFUSION EQUATON

We now investigate the ordering of the solutions of the superfluid diffusion equation,

$$c_t = (c_z^{1/3})_z \,. \tag{12}$$

*A different proof from that given in refs. 1 and 2 can be found in Appendix B.

To do so, we consider the infinitesimal difference δc between two neighboring solutions. It obeys the linear partial differential equation obtained by taking the first variation of Eq. (12), namely,

$$(\delta c)_t = \left[\frac{1}{3}\, c_z^{-2/3}(\delta c)_z\right]_z \ , \tag{13a}$$

which can be rearranged as

$$3c_z^{2/3}u_t = -\frac{2}{3}\,\frac{c_{zz}}{c_z}u_z + u_{zz} \tag{13b}$$

after setting $\delta c = u$. The difference u obeys the boundary and initial conditions

$$u(0,t) > 0 \ , \tag{14a}$$

$$u(\infty,t) = 0 \ , \tag{14b}$$

$$u(z,0) = 0 \ . \tag{14c}$$

The conditions (14b) and (14c) follow from the fact that both neighboring solutions obey the boundary and initial conditions (4a) and (4b). To prove the ordering of the two solutions, we must prove that $u(z,t) \geq 0$ for all z and t.

To avoid difficulties created by semi-infinite domains, let us begin by replacing Eq. (14b) by

$$u(L,t) = 0 \tag{14d}$$

and restricting ourselves to the rectangle R of length L along the z-axis and length T along the t-axis. We propose to prove that $u \geq 0$ in R by proving that the smallest value of u must lie on one of the sides S_1: ($z = 0$, $0 \leq t \leq T$), S_2: ($t = 0$, $0 \leq z \leq L$), or S_3: ($z = L, 0 \leq t \leq T$). On S_1, S_2, and S_3, the smallest value of u is zero.

If Eq. (13b) were replaced by the strict differential inequality

$$3c_z^{2/3}u_t > -\frac{2}{3}\,\frac{c_{zz}}{c_z}\,u_z + u_{zz} \ , \tag{15}$$

we could easily prove that the smallest value of u could not be attained either in the interior of R or at an interior point of side S_4: ($t = T$, $0 \leq z \leq L$). Suppose, for example, that the smallest value of u were attained at a point P in the interior of R. The point P would then be a relative minimum at which $u_z(P) = u_t(P) = 0$ and $u_{zz}(P) \geq 0$. But these stipulations contradict Eq. (15). Suppose, instead, that the smallest value of u were attained at an interior point P of S_4. Then the point P would be a relative minimum along S_4 so that at P, $u_z(P) = 0$ and $u_{zz}(P) \geq 0$. Since $c_z^{2/3}$ is always positive no matter what the sign of c_z, Eq. (15) then implies that $u_t > 0$; this means that yet smaller values of u lie inside R directly under point P, again a contradiction. Since the smallest value of u cannot be attained either in R or on S_4, it must lie on S_1, S_2, or S_3.

We can convert Eq. (13b) into the strict inequality (15) by adding a small, positive source term ϵ to the right-hand side of Eq. (13b). If we assume that δc is a continuous function of ϵ, then $\lim_{\epsilon \to 0} \delta c \geq 0$ in R. It is worth noting that this limiting process dilutes the strict inequality $\delta c > 0$ in R, which is what we have actually proved when $\epsilon > 0$, to the weaker inequality $\delta c \geq 0$ in R, but the latter is sufficient for our purposes.

Finally, we let L approach infinity and so return from the boundary condition (14d) to the boundary condition (14b).

THE THERMAL EXPULSION EQUATION

The thermal expulsion equation

$$cc_t = c_{zz} \tag{16}$$

has as its first variation the following equation for $u = \delta c$:

$$uc_t + cu_t = u_{zz} \ , \tag{17}$$

which we again consider in the rectangle R with the boundary and initial conditions (14a), (14c), and (14d). As before, we can convert Eq. (17) to a strict inequality by adding an infinitesimal positive source term ϵ to its right-hand side. We cannot be sure of the sign of c_t, and the standard trick for dealing with such uncertainty is to set

$$u = ve^{\lambda t} \ , \tag{18}$$

in which case Eq. (17) becomes

$$(c_t + \lambda c)v + cv_t = v_{zz} + \epsilon e^{-\lambda t} \ , \tag{19}$$

where v, too, obeys the boundary and initial conditions (14a), (14c), and (14d). If $c \geq \delta > 0$ in R, we can choose λ large enough so that $c_t + \lambda c > 0$ in R. Then if the minimum of v were attained at a point P in R, P would be a relative minimum at which $v(P) \leq 0$, $v_t(P) = 0$, and $v_{zz}(P) \geq 0$. But these stipulations contradict Eq. (19). If the minimum of v were attained at an interior point P of S_4, then $v_{zz}(P) \geq 0$, $v(P) \leq 0$, so that from Eq. (19) we find that $v_t(P) > 0$. Thus, yet smaller values of v lie inside R, again a contradiction. Since the smallest value of v must then lie on S_1, S_2, or S_3, $v > 0$ in R. In the limit as ϵ approaches zero, this strict inequality weakens to $v \geq 0$ in R. In view of Eq. (18), this is equivalent to $u \geq 0$ in R, which was to be proved.

All of this depends on showing that $c \geq \delta$ in the interior of R and on S_4. This will be so if $c(0,t) \geq \delta$. As before, we add a small, positive source term ϵ to Eq. (16) and also replace the right-hand sides of Eqs. (14a), (14c), and (14d) written for c with a small, positive quantity δ. Later, we shall let ϵ and δ approach 0.

The smallest value of c cannot occur in the interior of R. If it did, say at a point P, then P would be a relative minimum and $c_t(P) = 0$ and $c_{zz}(P) \geq 0$. These two requirements contradict Eq. (16) augmented by the source term ϵ.

The smallest value of c cannot occur at an interior point of S_4 either. But now, owing to the factor c (of uncertain sign on S_4) on the left-hand side of Eq. (16), we cannot prove this with the argument of the previous section. But let us consider the rectangle R': $(0 \leq z \leq L,\ 0 \leq t \leq T')$, where $T' \gg T$, and let $c(0,t)$ take the fixed value $c(0,T)$ for $T \leq t \leq T'$. If T' is large enough, then on S_4'

$$c(z,T') = \frac{z}{L}\,\delta + \frac{L-z}{L}\,c(0,T)\ , \tag{20}$$

which is the steady solution that $c(z,t)$ approaches when $c(0,t)$ is constant. The smallest value of c cannot occur in R', as just shown, and surely does not occur along the interior of S_4', where c is given by Eq. (20). Hence, it must occur on S_1', S_2', or S_3', which means that $c > \delta$ in R' and thus in R. In the limit as ϵ approaches zero, we find $c \geq \delta$ in R.

By the argument previously given, $v \geq 0$ in R for any value of $\delta > 0$. In the limit as δ approaches zero, $v \geq 0$ in R. Thus $u \geq 0$ in R, which proves the ordering of the solutions. Finally, as before, we let L approach infinity.

THE POROUS MEDIUM EQUATION

Both of the two partial differential equations just discussed have L/M and $N/M < 0$ as required for the solution (7) to fulfill the boundary and initial conditions (4a) and (4b). For the porous medium equation

$$c_t = (c^m c_z)_z\ , \tag{21}$$

$L/M = 2/m$ and $N/M = 1/m$, so that these two ratios can only be negative if $m < 0$. In many applications, $m > 0$, so that solutions obeying the boundary and initial conditions (4a) and (4b) cannot have asymptotic limits of the form (7). At least some of the similarity solutions of Eq. (21) for $m > 0$ are known to vanish at and beyond certain finite, time-dependent values of z (Refs. 3–5).

DISCUSSION

The limiting processes used in demonstrating the ordering of the solutions of the superfluid diffusion equation and the thermal expulsion equation are based on unproven assumptions of continuity. For certain linear partial differential equations related to the ordinary heat diffusion equation, rigorous proofs exist that do not depend on such assumptions.[6] These proofs are quite lengthy and involved; what is more, they are not always easy to generalize for use with nonlinear partial differential equations. Therefore, I prefer the heuristic approach given here, even though it does not conform to a high standard of rigor.

The importance of the results proved here rests on the fact that the asymptotic form $c \sim uz^{L/M}t^{-N/M}$ is simple and independent of the boundary value $c(0,t)$. This form can therefore be used without paying an exorbitant cost in computation; furthermore, it can be relied on even when the boundary value $c(0,t)$ is uncertain. An excellent example of such an application is the protection of superconducting magnets wound with cable-in-conduit conductors.[7] A local normal zone in such a conductor induces flow in the helium, the velocity of which can be modeled by the thermal expulsion equation.[8] Since such conductors are typically very long compared with their hydraulic diameters ($L/D \sim 10^5$), the induced velocity in the helium at the end of a hydraulic path can be estimated using the asymptotic form given above. Two data of great practical utility that can be obtained in this way are the velocity of expulsion from the ends of the tube and the time at which thermal hydraulic quenchback[9] is complete. The expulsion velocity is large enough in many practical situations to be used as a nonelectrical means of quench detection.[10] The phenomenon of thermal hydraulic quenchback strongly affects the maximum quench pressure in the conductor as well as the hot-spot temperature. Both thermal expulsion and thermal hydraulic quenchback have been studied by means of detailed calculations based on simple models that give plausible values of $c(0,t)$ (refs. 10 and 11). From the conclusions reached here, we can see that those results of the detailed calculations that are founded on the asymptotic form $uz^{L/M}t^{-N/M}$ are model independent and could have been obtained after only a few lines of calculation.

A similar conclusion applies to the experimental temperature distributions in superfluid He-II measured by van Sciver[12] and by van Sciver and Lottin.[13] Although these two sets of measurements differ markedly in their boundary and initial conditions (van Sciver introduced a constant heat flux into a half-space; van Sciver and Lottin delivered an instantaneous heat pulse to an infinite medium), in both cases the temperature at short times and large distances is described by the asymptotic form $uz^{L/M}t^{-N/M}$, as detailed calculations show.[14,15] With the theorems of this paper, a partial but convincing comparison of theory and experiment can be carried out with a minimum of computation.

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APPENDIX A

MOST GENERAL FORM OF THE PARTIAL DIFFERENTIAL EQUATION

As we have seen in the main text, the partial derivative c_t transforms under the family of groups (1) by multiplication by $\lambda^{\alpha-\beta}$: $c'_{t'} = \lambda^{\alpha-\beta}c_t$. In view of the linear constraint (2), we can eliminate β from the exponent in favor of α; thus $\alpha - \beta = (M/N + 1)\alpha - L/N$. So c_t transforms by multiplication by a power of λ, the power being a linear function of α. The same is true of all other partial derivatives of c. Thus, in finding the most general form of a partial differential equation involving z, t, c, and its derivatives, we are led to consider functions F of N arguments x_i that transform according to the one-parameter family of groups

$$x'_i = \lambda^{a_i\alpha+b_i}x_i \,, \quad 0 < \lambda < \infty \,. \tag{A1}$$

Here a_i and b_i are constants [$(M/N+1)$ and $-L/N$ in the case of c_t].

If the function F is invariant to all the groups of the family (A1), then

$$F(x_1,\, x_2,\, \ldots,\, x_N) = F(x'_1,\, x'_2,\, \ldots,\, x'_N) \tag{A2}$$

or

$$F(x_1,\, x_2,\, \ldots,\, x_N) = F(\lambda^{a_1\alpha+b_1}x_1 \,,\, \lambda^{a_2\alpha+b_2}x_2 \,,\, \ldots,\, \lambda^{a_N\alpha+b_N}x_N) \,. \tag{A3}$$

Equation (A3) is an identity true for all values of α and λ. If we differentiate with respect to λ and then set $\lambda = 1$, we find

$$\sum_{i=1}^{N}(a_i\alpha + b_i)x_i\frac{\partial F}{\partial x_i} = 0 \,. \tag{A4}$$

Since Eq. (A4) is also an identity true for all values of α, it is equivalent to the pair of first-order, linear partial differential equations

$$\sum_{i=1}^{N} a_i x_i \frac{\partial F}{\partial x_i} = 0 \,, \tag{A5a}$$

$$\sum_{i=1}^{N} b_i x_i \frac{\partial F}{\partial x_i} = 0 \,. \tag{A5b}$$

The characteristic equations of these two partial differential equations are

$$\frac{dx_1}{a_1x_1} = \frac{dx_2}{a_2x_2} = \ldots = \frac{dx_N}{a_Nx_N} \,, \tag{A6a}$$

$$\frac{dx_1}{b_1x_1} = \frac{dx_2}{b_2x_2} = \ldots = \frac{dx_N}{b_Nx_N} \,. \tag{A6b}$$

The most general solution of Eq. (A5a) is an arbitrary function of the $N-1$ integrals of Eq. (A6a), and similarly the most general solution of Eq. (A5b) is an arbitrary function of the $N-1$ integrals of Eq. (A6b). The integrals of either set of characteristic equations are products of powers of the x_i, that is, functions $u(x_1, \ldots, x_N)$ of the form

$$u = \prod_{i=1}^{N} x_i^{c_i} \ . \tag{A7}$$

Since $du = 0$ in the direction given by Eq. (A6a), if we take the logarithm of Eq. (A7) and then differentiate in that direction we find

$$0 = \frac{dx_1}{a_1 x_1} \left(\sum_{i=1}^{N} c_i a_i \right) , \tag{A8}$$

so that the constants c_i must obey the constraint

$$\sum_{i=1}^{N} c_i a_i = 0 \ . \tag{A8a}$$

Similarly, the powers c_i in the integrals of Eq. (A6b) must obey the constraint

$$\sum_{i=1}^{N} c_i b_i = 0 \ . \tag{A8b}$$

If we look at Eqs. (A8a) and (A8b) as expressing the orthogonality of an N-dimensional vector c to two other vectors a and b, then we see at once that the admissible vectors span a subspace of dimension $N-2$. Hence there are $N-2$ independent mutual integrals of Eqs. (A6a) and (A6b). The most general simultaneous solution of Eqs. (A5a) and (A5b) is an arbitrary function of these $N-2$ independent mutual integrals.

Now let us consider invariants of the entire family of groups of the form

$$v = \prod_{i=1}^{N} x_i^{c_i} \ . \tag{A9}$$

Invariance means that

$$\prod_{i=1}^{N} x_i^{c_i} = \prod_{i=1}^{N} x_i'^{c_i} = \prod_{i=1}^{N} x_i^{c_i} \lambda^{(a_i\alpha + b_i)c_i} \tag{A10}$$

for any choice of x_i, α, and λ. Taking logarithms, we find that

$$\ln \lambda \sum_{i=1}^{N} (a_i \alpha + b_i) c_i = 0 \ , \tag{A11}$$

which leads as before to the constraints (A8a) and (A8b) on the c_i. Thus there are $N-2$ independent invariants v of the entire family of groups, and their exponent vectors c span the same subspace as the exponent vectors of the mutual integrals of Eqs. (A6a) and (A6b). So the most general function F is then an arbitrary function of the $N-2$ invariants v. Because of the first equality in Eq. (A10), these invariants are "λ-less" combinations of the x_i.

APPENDIX B

INVARIANCE OF THE PRINCIPAL ORDINARY DIFFERENTIAL EQUATION

We start by writing the "λ-less" equation (6) in the more convenient form

$$G\left[\frac{c}{z^{L/M}t^{-N/M}}\,,\ \frac{tc_t}{c}\,,\ \frac{zc_z}{c}\,,\ \frac{t(tc_t)_t}{c}\,,\ \frac{z(zc_z)_z}{c}\,,\ \ldots\right]=0\,. \tag{B1}$$

In view of the definition $x = zt^{-1/\beta}$, we can write the operator equations

$$t\frac{\partial}{\partial t} = -\frac{1}{\beta}x\frac{\partial}{\partial x} \qquad (z \text{ held fixed})\,, \tag{B2a}$$

$$z\frac{\partial}{\partial z} = x\frac{\partial}{\partial x} \qquad (t \text{ held fixed})\,. \tag{B2b}$$

Furthermore, we can write Eq. (3) in either of the following alternate forms:

$$c = t^{\alpha/\beta}y(x) = z^{\alpha}x^{-\alpha}y(x)\,. \tag{B3}$$

If we substitute Eq. (B3) into Eq. (B1), we find with the help of Eq. (B2) that Eq. (B1) takes the form

$$G\left\{\frac{y(x)}{x^{L/M}}\,,\ -\frac{1}{\beta}\,\frac{x\frac{d}{dx}(x^{-\alpha}y)}{x^{-\alpha}y}\,,\ \frac{x}{y}\frac{dy}{dx}\,,\ \frac{1}{\beta^2}\,\frac{x\frac{d}{dx}\left[x\frac{d}{dx}(x^{-\alpha}y)\right]}{x^{-\alpha}y}\,,\ \frac{x\frac{d}{dx}\left(x\frac{dy}{dx}\right)}{y}\,,\ \ldots\right\}=0\,, \tag{B4}$$

which one can immediately see is invariant to the group (10).