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THE COMPLEX MODIFIED KORTEWEG-DEVRIES EQUATION, A NON-INTEGRABLE EVOLUTION EQUATION

BY

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Abstract

The two-dimensional steady-state propagation of electrostatic waves is governed by $\partial v/\partial\tau + \partial^3 v/\partial\xi^3 + \partial(|v|^2 v)/\partial\xi = 0$, the Complex Modified Korteweg-DeVries equation. The properties of this equation are studied.

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THE COMPLEX MODIFIED KORTEWEG-DEVRIES EQUATION,
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1 Derivation of Equation

Some anisotropic media, e.g. a magnetized plasma, can support propagating electrostatic waves. We wish to study the dispersion and nonlinear self-modulation of such waves when the system is two-dimensional (x and y, x being the principal axis) and has reached a steady state [all field quantities $\sim \exp(-i\omega t)$, $\omega =$ constant]. We will take the medium to be homogeneous and non-dissipative, and will assume that the dielectric tensor depends on the electric field amplitude squared. The equation for the complex electric potential, ϕ, is

$$\nabla \cdot \overline{\overline{K}}(\nabla, |\nabla\phi|^2) \cdot \nabla\phi = 0 \ . \tag{1}$$

We expand $\overline{\overline{K}}$ about $\nabla = 0$, $|\nabla\phi|^2 = 0$ (the long-wavelength, linear limit)

$$\overline{\overline{K}}(\nabla, |\nabla\phi|^2) = \overline{\overline{K}} + \varepsilon(1/2)(\partial^2\overline{\overline{K}}/\partial\nabla\partial\nabla):\nabla\nabla + \varepsilon(\partial\overline{\overline{K}}/\partial|\nabla\phi|^2)|\nabla\phi|^2 \ , \tag{2}$$

where ε is a formal expansion parameter. Assuming $K_{yy} > 0 > K_{xx}$ then to order ε^0 (1) becomes the wave equation $-|K_{xx}|\partial^2\phi/\partial x^2 + K_{yy}\partial^2\phi/\partial y^2 = 0$. Expanding ϕ about the right-going solution we let $\phi = \phi(\tau, \xi)$, where $\tau \propto y$ and $\xi \propto x - (|K_{xx}|/K_{yy})^{1/2} y$ and $\partial\phi/\partial\xi = O(1)$, $\partial\phi/\partial\tau = O(\varepsilon)$. To order ε we obtain

$$v_\tau + v_{\xi\xi\xi} + (|v|^2 v)_\xi = 0 \ , \tag{3}$$

where $v \propto \partial\phi/\partial\xi$, and subscripts denote differentiation. We call (3) the Complex Modified Korteweg-deVries equation [1].

2. Constants of the Motion

Only four constants of the motion are known. This is in contrast to equations soluble by the inverse scattering method which have an infinite number of constants of the motion. The four constants are:

$$I_1 = \int_{-\infty}^{\infty} v \, d\xi \ , \tag{4}$$

$$I_2 = \int_{-\infty}^{\infty} |v|^2 \, d\xi \ , \tag{5}$$

$$I_3 = \int_{-\infty}^{\infty} |v|^4/2 - |v_\xi|^2 \, d\xi \ , \tag{6}$$

$$I_4 = \int_{-\infty}^{\infty} |v|^2/k \, dk \ . \tag{7}$$

In (7) k is the Fourier-transform variable conjugate to ξ. Three of these constants have physical interpretations. I_1 = const. states that the electric field is derivable from a potential. I_2 = const. and I_4 = const. give the conservation of momentum and energy, i.e. the force and power balances.

3. Soluble Limits

Although (3) is not analytically soluble, it is closely related to the modified Korteweg-deVries equation,

$$v_\tau + v_{\xi\xi\xi} + \kappa|v|^2 v_\xi = 0 \quad , \tag{8}$$

which is soluble by the inverse scattering method [2]. To see this relation we rewrite (3) in two ways,

$$v_\tau + v_{\xi\xi\xi} + 3|v|^2 v_\xi = 2i|v|^2 v\theta_\xi \quad , \tag{9}$$

$$v_\tau + v_{\xi\xi\xi} + |v|^2 v_\xi = -v|v^2|_\xi \quad , \tag{10}$$

where $\theta = \arg(v)$. In the limits of slow and rapid phase variation, the right hand sides of (9) and (10) respectively are negligible and in these limits (3) reduces to (8), although the strength of the nonlinear term, κ, is different. When neither limit applies, we must solve (3) numerically.

4. Numerical Solution

We choose initial conditions of the form,

$$v(\tau = 0, \xi) = A\,\mathrm{sech}(\xi)\exp(ik_0\xi) \quad . \tag{11}$$

Figs. 1 and 2 show two examples of the evolution. We see that there are two types of solitary pulses produced; one had a constant phase (Fig. 1), while the other is an envelope pulse (Fig. 2).

5. Constant Phase Pulses

The constant phase pulses are a special case of the solitons of (8). Their form is

$$v = \sqrt{2}\,a\,\mathrm{sech}[a(\xi - \xi_0 - a^2\tau)]\exp(i\theta_0) \quad . \tag{12}$$

The area of these pulses is $\sqrt{2}\pi$. However, these pulses do not behave as solitons in (3). Fig. 3 shows the collision of two of these pulses which have different phases, θ_0. We see that after the collision the phase and amplitude of the pulses have changed and some "radiation" is produced.

6. Envelope Pulses

The general form of the envelope pulses is $v(\tau, \xi) = V(\zeta)\exp(ik_0\xi - i\omega_0\tau)$, where V is complex, $\zeta = \xi - c\tau$, $c = a^2 - 3k_0^2$, and $\omega_0 = k_0(3a^2 - k_0^2)$. Here <u>a</u> is the decay rate of V as $|\zeta| \to \infty$. V satisfies

$$(V_{\zeta\zeta} + |V|^2 V - a^2 V)_\zeta + ik_0(3V_{\zeta\zeta} + |V|^2 V - 3a^2 V) = 0 \quad . \tag{13}$$

For $k_0 = 0$, we recover the constant phase pulses. Numerically integrating (13), we find that for $0 < |k_0| \lesssim 0.5a$, V does not form a pulse, while for $|k_0| \gtrsim 0.5a$ we do obtain a pulse. The asymptotic form of the pulse for k_0 large is

$$V = \sqrt{6}\, a\, \mathrm{sech}(a\zeta)\{1 + i\varepsilon \tanh(a\zeta) + \varepsilon^2[\tanh^2(a\zeta) - 1/2]/3 + O(\varepsilon^3)\} \quad , \qquad (14)$$

where $\varepsilon = a/k_0$. In the limit $\varepsilon \to 0$ we recover the soliton of (6) (with $\kappa = 3$). The "absolute" area of V is (cf. the area of the constant phase pulses)

$$\int_{-\infty}^{\infty} |V|\, d\zeta = \sqrt{6}\,\pi[1 + \varepsilon^2/4 + O(\varepsilon^4)] \quad . \qquad (15)$$

7. Transition From Envelope to Constant Phase Pulses

Since solitary pulses do not exist for the full range of k_0/a it is clear that there cannot be a continuous transition from envelope to constant phase pulses. If we take an envelope pulse and alter the initial conditions in such a way that $|k_0| < 0.5a$, we see from (15) that it has sufficient area to break up into about 3 constant phase pulses. For initial conditions of the form of (11) this happens when $A \approx 2k_0$ (see Fig. 4). Fig. 5 shows schematically what pulses are produced for different initial conditions. The equivalent figure for (6) would consist of a single set of lines at $A(\kappa/6)^{1/2} = N - 1/2$.

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References

1. C. F. F. Karney, A. Sen, F. Y. F. Chu: Plasma Physics Laboratory Report PPPL-1452, Princeton University (1978)
2. M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur: Stud. Appl. Math. 53, 249-315 (1974)

Figure Captions

Fig. 1 Evolution of (3) with initial conditions given by (11) with $A = 3$, $k_o = 1$. The solid line, long dashes, and short dashes denote $|v|$, $Re(v)$, and $Im(v)$ respectively

Fig. 2 Same as Fig. 1, except $A = 5$, $k_o = 3$

Fig. 3 Collision of two constant phase pulses

Fig. 4 Same as Fig. 2, except $A = 6$

Fig. 5 Schematic showing numbers and types of pulses produced with initial conditions given by (11)

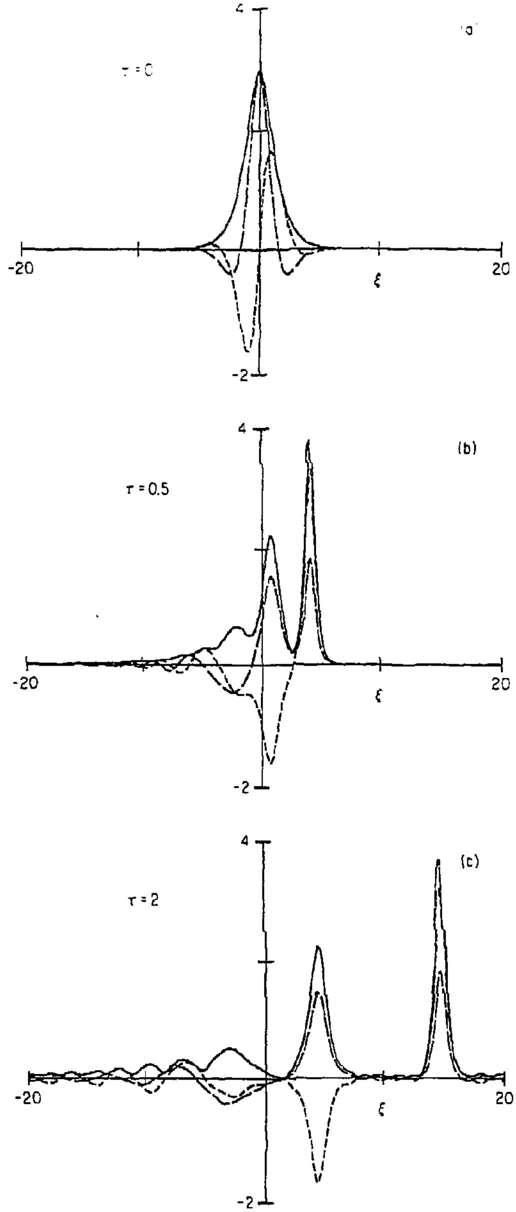

Fig. 1. 782165

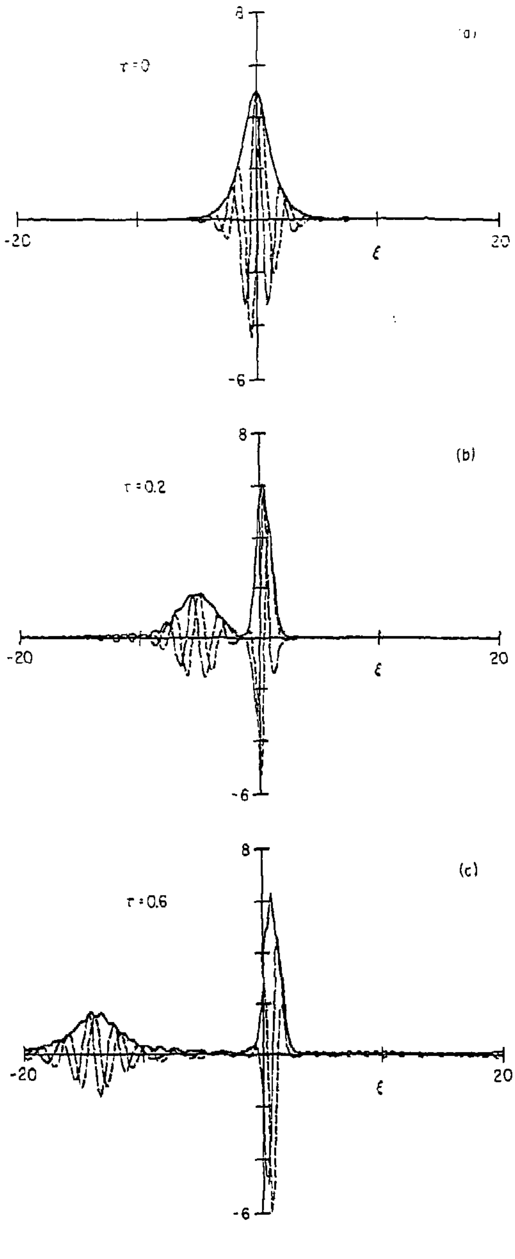

Fig. 2. 782166

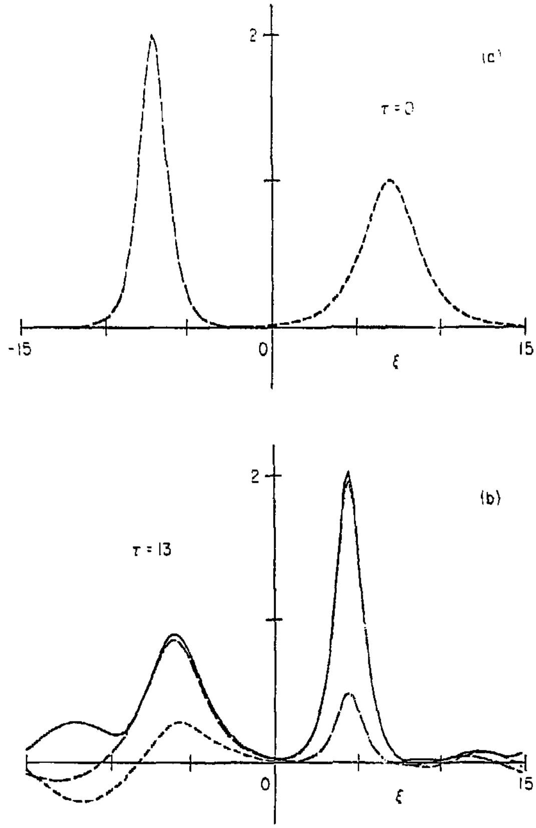

Fig. 3. 782164

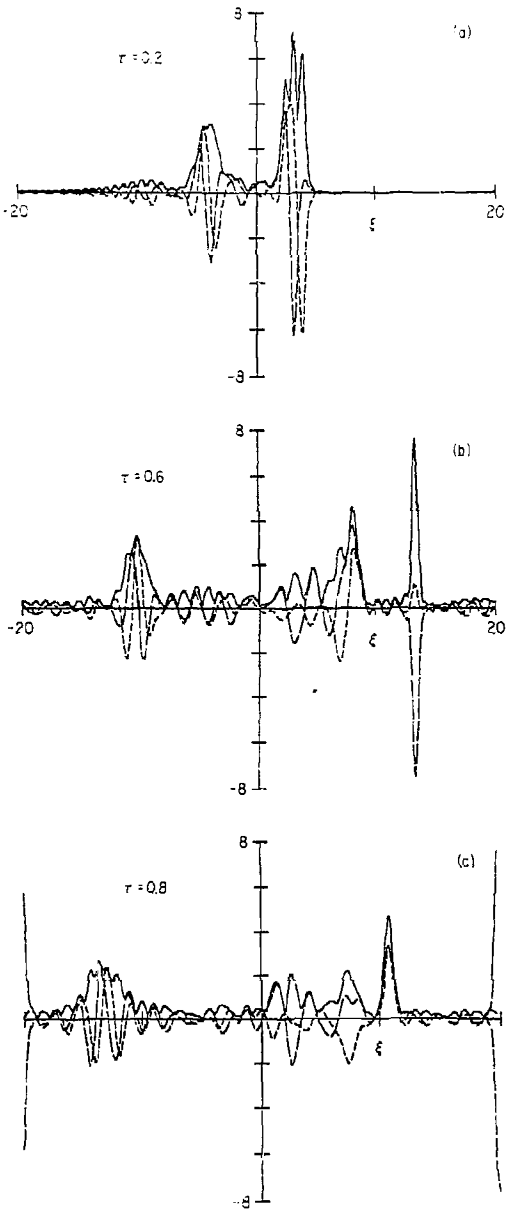

Fig. 4. 782163

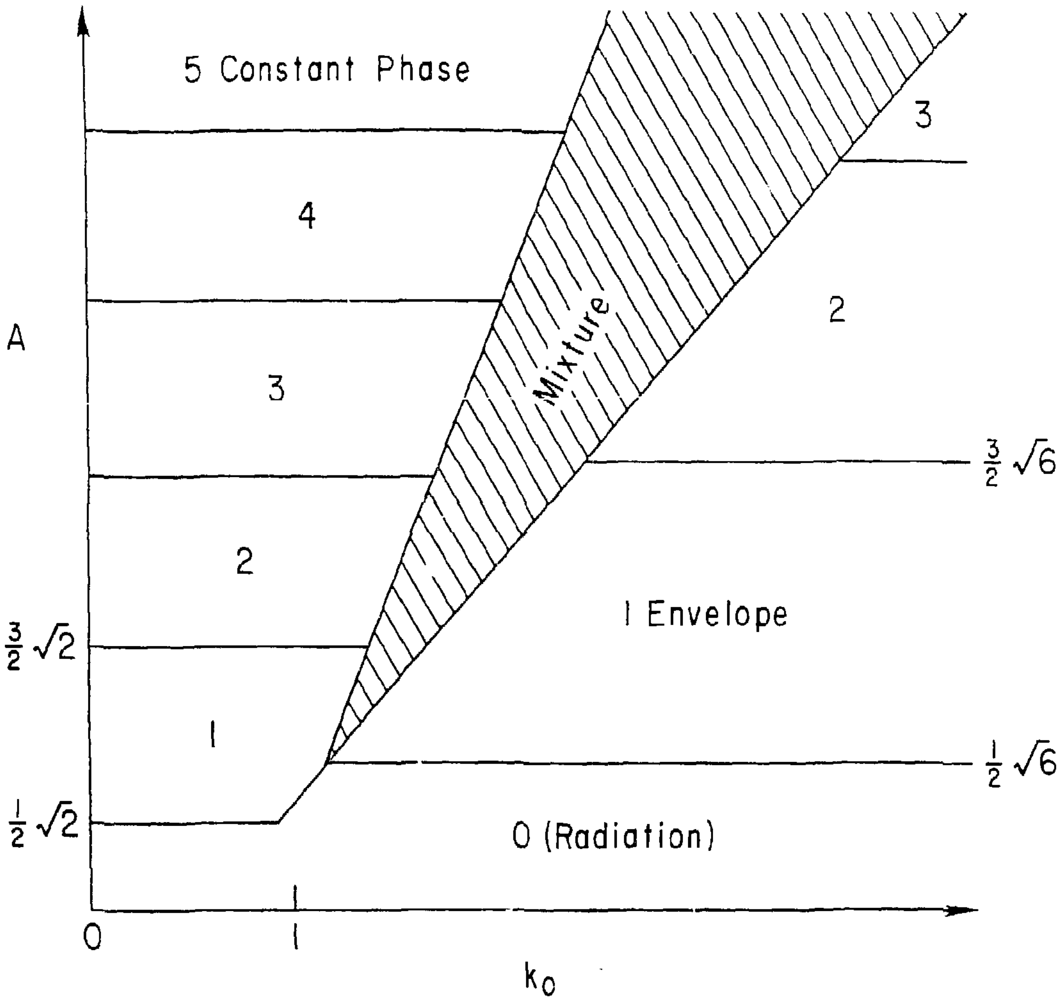
5 Constant Phase
4
3
2
1
A
$\frac{3}{2}\sqrt{2}$
$\frac{1}{2}\sqrt{2}$
Mixture
3
2
$\frac{3}{2}\sqrt{6}$
1 Envelope
$\frac{1}{2}\sqrt{6}$
0 (Radiation)
0
1
k_0

Fig. 5. 782159