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TITLE LIE-POISSON BIFURCATIONS FOR THE MAXWELL-BLOCH EQUATIONS

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Lie-Poisson Bifurcations for the Maxwell-Bloch Equations

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Abstract

We present a study of the set of Maxwell-Bloch equations on $\mathbf{R}^3$ from the point of view of Hamiltonian dynamics. These equations are shown to be bi-Hamiltonian, on the one hand, and to possess several inequivalent Lie-Poisson structures, on the other hand, parametrized by the group SL(2,**R**). Each structure is characterized by a particular distinguished function. The level sets of this function provide two-dimensional surfaces onto which the motion takes various symplectic forms.

Keywords: Maxwell-Bloch equations, Lie-Poisson bifurcations, Bi-Hamiltonian system

Short title: Lie-Poisson structures of the Maxwell-Bloch equations

1. Introduction and reduction to the three-dimensional system.

In this note, we provide an example of a system that exhibits a multiplicity of Lie-Poisson structures. Under varying SL(2, **R**)-valued parameters, these structures undergo bifurcations. This property is certainly true of any bi-Hamiltonian system on $\mathbf{R}^3$ that can be put in the form

$$\dot{x} = \nabla H_1 \times \nabla H_2, \tag{1.1}$$

where the Hamiltonian functions H_1 and H_2 are quadratic in their arguments. The general case is presently under investigation by the author and will be presented elsewhere. Here, we show how this occurs for the travelling-wave three-dimensional Maxwell-Bloch system. This system is of use in the field of nonlinear optics [1] and arises from the following 2:1:1 resonant system on $\mathbf{C}^3$:

$$\begin{gathered} H: \mathbf{C}^3 \to \mathbf{R}: (u_1, u_2, u_3) \to (2\bar{u}_1 u_2 u_3 + 2u_1 \bar{u}_2 \bar{u}_3), \\ X_H = \frac{i}{2}\left(\frac{\partial H}{\partial u_\alpha}\frac{\partial}{\partial \bar{u}_\alpha} - \frac{\partial H}{\partial \bar{u}_\alpha}\frac{\partial}{\partial u_\alpha}\right), \\ \dot{u}_1 = -iu_2u_3, \qquad \dot{u}_2 = -iu_1\bar{u}_3, \qquad \dot{u}_3 = -iu_1\bar{u}_2. \end{gathered} \tag{1.2}$$

H is the Hamiltonian function for the system, X_H is the associated Hamiltonian vector field, and the equations of the motion are obtained, as usual, as $\dot{u}_\alpha = X_H(\alpha)$. The function H is invariant under the U(1) action $(u_1, u_2, u_3) \to (e^{2i\theta}u_1, e^{i\theta}u_2, e^{i\theta}u_3)$ and has the following two conserved quantities:

$$C = |u_1|^2 + |u_2|^3, \qquad K = |u_1|^2 + |u_3|^2. \tag{1.3}$$

Introduce a new set of coordinate functions on the dynamical space:

$$\begin{aligned} x_1 &= 2Re(u_3), & x_2 &= 2Im(u_1\bar{u}_2), & x_3 &= |u_1|^2 - |u_2|^2, \\ y_1 &= 2Im(u_3), & y_2 &= 2Re(u_1\bar{u}_2), & y_3 &= C. \end{aligned} \tag{1.4}$$

Then the system for the variables $\{x_1, x_2, x_3\}$ forms an invariant subsystem, with

$$H_1 = \frac{1}{2}(x_2^2 + x_3^2), \qquad H_2 = x_3 + \frac{1}{2}x_1^2, \tag{1.5}$$

$$\dot{x} = \nabla H_1 \times \nabla H_2, \tag{1.6}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1x_3, \qquad \dot{x}_3 = (-x_1x_2). \tag{1.7}$$

The travelling-wave maxwell-Bloch equations consist of equations (1.7). As can be seen from (1.6), the functions H_1 and H_2 provide two admissible Hamiltonian structures for the system; thus it is bi-Hamiltonian. Moreover, both these functions are quadratic in their arguments and therefore automatically induce an additional Lie-Poisson structure for the system. Geometrically, (1.7) implies that the dynamics takes place on intersections of level sets of the functions H_1 and H_2 in the space $\mathbf{R}^3$.

2. Classification of the Lie-Poisson structures

The three-dimensional Maxwell-Bloch system of equations possess a multiplicity of Lie-Poisson structures. In fact, we may characterize them by a three-parameter family of function pairs (H, C) where H is an admissible Hamiltonian function, and C an associated admissible distinguished (or Casimir) function. The set **P** of all such pairs forms a representation of the Lie group SL(2, **R**) and arises as the result of the invariance of the right-hand side of equation (1.7); the group homomorphism defining the representation is

$$SL(2, \mathbf{R}) \to P: \begin{bmatrix} \alpha & \beta \\ \mu & \nu \end{bmatrix} \to \begin{bmatrix} \alpha & \beta \\ \mu & \nu \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}. \tag{2.1}$$

Invariance of equation (1.7) yields the following

Proposition. Consider the two functions $H = H^{\alpha,\beta} = \alpha H_1 + \beta H_2$ and $C = H^{\mu,\nu} = \mu H_1 + \nu H_2$, with $\alpha\nu - \beta\mu = 1$. Then equation (1.7) is equivalent to $\dot{x} = \nabla H \times \nabla C = \nabla H^{\alpha,\beta} \times \nabla H^{\mu,\nu}$ For the proof, it suffices to compute

$$\begin{aligned} \dot{x} &= \nabla\ (\alpha H_1 + \beta H_2) \times \nabla\ (\mu H_1 + \nu H_2) \\ &= (\alpha\nu - \beta\mu) \nabla H_1 \times \nabla H_2 = \nabla H_1 \times \nabla H_2. \end{aligned} \tag{2.2}$$

The above proposition implies the invariance of the intersections of level surfaces of the functions H and C under the SL(2, **R**) action (2.1); in other words, it implies that the dynamics (the geometric loci of the solutions) remains unchanged in $\mathbf{R}^3$ under these group deformations.

Let us now examine the Lie-Poisson structure of our system. Consequent to the invariance of equation (1.1) under the SL(2, **R**) action, this structure will not be unique, and in fact presents bifurcations as we move along parametric curves in SL(2, **R**). We use Hamiltonian vector fields; any dynamical quantity Q thus evolves with time according to the equation

$$\dot{Q} = X_H Q, \tag{2.3}$$

and the correspondence with Poisson brackets is given through the following identity:

$$X_H F = -X_F H = \{F, H\}. \tag{2.4}$$

The equations governing the flow of X_H are the Hamilton equations for H. In view of (1.5-7) it is then clear that for all C^2 functions $G : \mathbf{R}^3 \to R$, exist associated vector fields

$$X_G = (\nabla G \times \nabla C) \cdot \nabla. \tag{2.5}$$

In component form, these are expressed as

$$\begin{aligned} X_G = (\nu + \mu x_3) \left(\frac{\partial G}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial G}{\partial x_1}\frac{\partial}{\partial x_2}\right) + \nu x_1 \left(\frac{\partial G}{\partial x_3}\frac{\partial}{\partial x_2} - \frac{\partial G}{\partial x_2}\frac{\partial}{\partial x_3}\right) \\ + \mu x_2 \left(\frac{\partial G}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial G}{\partial x_3}\frac{\partial}{\partial x_1}\right). \end{aligned} \tag{2.6}$$

We remark that expression (2.6) depends explicitly on the parameters μ and ν, through its dependence on the distinguished function C, as prescribed by the proposition. To determine the structure of the Lie algebra underlying the Poisson structure of the system, we first calculate the Lie bracket for the Hamiltonian vector fields associated to the coordinate functions x_i,

$$X_1 \equiv X_{x_1} = \mu x_2 \partial_3 - (\nu + \mu x_3)\,\partial_2,$$
$$X_2 \equiv X_{x_2} = (\nu + \mu x_3)\,\partial_1 - \nu x_1 \partial_3, \tag{2.7}$$
$$X_3 \equiv X_{x_3} = \nu x_1 \partial_2 - \mu x_2 \partial_1,$$

where $\partial_i = \partial/\partial x_i$. The non-vanishing commutators defining the Lie structure are

$$[X_1, X_2] = -\mu X_3,\ [X_2, X_3] = -\nu X_1,\ [X_3, X_1] = -\mu X_2. \tag{2.8}$$

Clearly, the Lie algebra spanned by the vector fields X_i also depends on the parameters μ and ν. This dependence is correlated with classes of orbits in the group SL(2, **R**) as follows.

Case 1. $\mu = 0, \nu \neq 0$. Define $Y_1 = -\nu X_1$, $Y_2 = X_2$, $Y_3 = X_3$. Then the structure of the algebra is

$$[Y_1, Y_2] = 0,\ [Y_2, Y_3] = Y_1,\ [Y_3, Y_1] = 0. \tag{2.9}$$

This algebra is the well known Heisenberg algebra.

Case 2. $\mu \neq 0, \nu = 0$. Define $Y_1 = -X_1/\mu$, $Y_2 = X_2$, $Y_3 = X_3$. Then the commutators become

$$[Y_1, Y_2] = Y_3,\ [Y_2, Y_3] = 0,\ [Y_3, Y_1] = Y_2. \tag{2.10}$$

This solvable algebra is isomorphic to the Euclidean algebra of the plane.

Case 3. $\mu \neq 0 = \varepsilon_\mu$, $\nu \neq 0 = \varepsilon_\nu$, with $\varepsilon_\sigma = \mathrm{Sign}(\sigma)$. We define $Y_1 = -\varepsilon_\nu X_1/|\mu|$, $Y_2 = X_2/(|\mu|\,|\nu|)^{1/2}$, $Y_3 = X_3/(|\mu|\,|\nu|)^{1/2}$. Then the structure of the algebra is

$$[Y_1, Y_2] = \varepsilon Y_3,\ [Y_2, Y_3] = Y_1,\ [Y_3, Y_1] = \varepsilon Y_2 \tag{2.11}$$

where $\varepsilon = \varepsilon_{\mu\nu} = \mathrm{Sign}(\mu\nu)$. Two subcases arise.

Subcase 3.1. $\varepsilon = 1$. This algebra is isomorphic to so(3).

Subcase 3.2. $\varepsilon = -1$. This algebra is isomorphic to so(2,1) and so(1,2).

Each of the above cases is associated with a particular family of distinguished functions C. We also point out that the groups corresponding to these algebras are compact if, and only if, the level sets of C are themselves compact sets. These functions are as follows.

Case 1. $\mu = 0, \nu \neq 0$. The level sets are parabolic cylinders along the x_2-axis,

$$C = \nu\left(x_3 + \frac{1}{2}x_1^2\right). \tag{2.12}$$

Case 2. $\mu \neq 0$, $\nu = 0$. The level sets are circular cylinders about the x_1-axis; they are defined whenever $C/\mu > 0$,

$$C = \frac{1}{2}\mu\,(x_2^2 + x_3^2)\,. \tag{2.13}$$

Case 3a. $\mu \neq 0$, $\nu \neq 0$, with $\mu\nu > 0$. The level sets are ellipsoids of revolution, with semi-major axis $r_1 = r$, $r_2 = r_3 = (\nu/\mu)^{1/2}r$, centered at $(0, 0, -\nu/\mu)$; they are defined whenever $4\mu C + \nu^2 > 0$.

$$x_1^2 + \frac{\mu}{\nu}\left[x_2^2 + \left(x_3 + \frac{\nu}{\mu}\right)^2\right] = \frac{2C}{\nu} + \frac{\nu}{2\mu} = r^2 \tag{2.14}$$

Case 3b. $\mu \neq 0$, $\nu \neq 0$, with $\mu\nu < 0$. Here, the level sets, here, are non-compact surfaces, namely two-sheeted hyperboloids of revolution if $4\mu C + \nu^2 < 0$, one-sheeted hyperboloids if $4\mu C + \nu^2 > 0$, or cones whenever $4\mu C + \nu^2 = 0$. The two varieties of hyperboloids correspond to the two choices of the algebra, either so(2,1) or so(1,2), respectively.

Note that for each case, the level sets provide foliations of the space $\mathbf{R}^3$ (in fact, these are symplectic foliations). Each of the above classes thus defines admissible pairs (H, C) prescribed by (2.1) where the appropriate SL(2,**R**) matrices given as follows:

$$\begin{aligned}
&\text{Case 1:}\quad g_1 = g_3|_{\mu=0} = \begin{bmatrix} 1/\nu & \beta \\ 0 & \nu \end{bmatrix}, \qquad H = \frac{H_1}{\nu} + \beta H_2, \\
&\text{Case 2:}\quad g_2 = g_3|_{\nu=0} = \begin{bmatrix} \alpha & -1/\mu \\ \mu & 0 \end{bmatrix}, \qquad H = \alpha H_1 - \frac{H_2}{\mu}, \\
&\text{Case 3:}\quad g_3 = \begin{bmatrix} \alpha & \beta \\ \mu & \nu \end{bmatrix}, \qquad H = \alpha H_1 + \beta H_2, \qquad \alpha\nu - \beta\mu = 1.
\end{aligned} \tag{2.15}$$

Clearly, the locus of the Hamiltonian function H depends on the parameters α or β and can bifurcate as we vary them. For instance, a change in sign may transform the topology of a level surface of energy from that of an ellipsoid to that of a hyperboloid. We note that the intersections of the level surfaces of C and H are unaffected by these bifurcations; in fact, they do not depend on α or β at all. However, the *representation* of the dynamics *does* depend on the choice of α or β, since the Hamiltonian function itself depends on these parameters, as will be made explicit in the next section where we show, for example, that $\beta = 0$ in Case 1 yields Duffing oscillator dynamics, while $\alpha = 0$ in Case 2 yields pendulum dynamics.

3. Reductions to the two-dimensional level sets of the distinguished functions

Each of the cases presented in Section 2 yields a distinct reduction of the initial Maxwell-Bloch system (1.5-6) to a symplectic system on a two-dimensional manifold specified by a level set of the distinguished functions C. These reductions however give different coordinate representations of the *same* solutions in $\mathbf{R}^3$. We now examine each reduction to a symplectic system and briefly describe the qualitative features. For more details about the nature of geometric reduction, the reader is referred to [2] and [3].

Case 1. $\mu = 0, \nu \neq 0$. The distinguished function C is given by (2.12) and the Hamiltonian function is, as prescribed by (2.1),

$$H = \frac{1}{2\nu}(x_2^2 + x_3^2) + \beta\left(x_3 + \frac{1}{2}x_1^2\right). \tag{3.1}$$

We introduce a new basis of coordinate functions (suggested by D. Holm)

$$x = x_1, \qquad y = x_2, \qquad z = \left(x_3 + \frac{1}{2}x_1\right), \tag{3.2}$$

In terms of these coordinates, X_H and the equations of the motion therefore reduce to

$$X_H = \nu\left(\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y}\right); \qquad \dot{x} = y, \qquad \dot{y} = \left(z - \frac{1}{2}x^2\right)x. \tag{3.3}$$

This system is a Duffing oscillator and it possesses the following three critical points: (x, y) = (0, 0), $(\pm\sqrt{2z}, 0)$. A linear stability analysis shows that the first one is unstable, and the two others are stable centers. The phase portrait is naturally the usual one, with the figure-eight pattern, but drawn on the parabolic cylinder in place of the flat plane.

Case 2. $\mu \neq 0, \nu = 0$. For this case, the distinguished function C is given by (2.13) and the Hamiltonian function is

$$H = \frac{\alpha}{2}(x_2^2 + x_3^2) - \frac{1}{\mu}\left(x_3 + \frac{1}{2}x_1^2\right). \tag{3.4}$$

We introduce a new basis of coordinate functions. Since the level sets of C are circular cylinders, it isappropriate to choose the usual cylindrical coordinates,

$$x_1 = z, \qquad x_2 = r\cos\theta, \qquad x_3 = r\sin\theta; \qquad r = \sqrt{2C/\mu}, \tag{3.5}$$

In terms of these coordinates, the distinguished and Hamiltonian functions become

$$C = \frac{\mu r^2}{2}, \qquad H = \frac{1}{\mu}\left(\alpha C - \frac{1}{2}z^2 - r\sin\theta\right). \tag{3.6}$$

The geometric locus of the level sets of the Hamiltonian is that of a parabolic cylinder along the x_2-axis. Thus the orbits of the motion are the intersection of such parabolic cylinders with a circular cylinder about the z-axis. These intersections are non-trivial only when $\mu H - \alpha C < r$. Therefore the orbits on the phase cylinder are periodic, except in the limit when one of the parabolic cylinders becomes tangent with the interior of the circular cylinder: when this occurs, a pair of homoclinic loops appears which partitions the phase cylinder into three distinct families of periodic orbits. The reduced vector field X_H and the reduced equations of motion are

$$X_H = \mu\left(\frac{\partial H}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial H}{\partial\theta}\frac{\partial}{\partial z}\right); \qquad \dot{\theta} = -z, \qquad \dot{z} = r\cos\theta. \tag{3.7}$$

These equations possess exactly two fixed points, $(\theta, z) = (\pm\pi/2, 0)$. The critical point $(\pi/2, 0)$ is stable whereas $(-\pi/2, 0)$ is unstable (i.e., is a saddle point). In these coordinates, the motion on the reduced phase cylinder is precisely the dynamics of a pendulum.

Case 3a. $\mu \neq 0$, $\nu \neq 0$, $\mu\nu > 0$, $4\mu C + \nu^2 > 0$. For this third case, the distinguished function C is given by expression (2.14). Introducing the constant

$$r = \sqrt{\frac{2C}{\nu} + \frac{\nu}{2\mu}}, \tag{3.8}$$

and keeping in mind that the level sets of the distinguished function are ellipsoids of revolution about the x_1-axis, it is natural to introduce a new basis of coordinate functions as follows

$$x_1 = r\cos\theta, \qquad x_2 = \sqrt{\nu/\mu}\, r\sin\theta\cos\varphi, \qquad x_3 = \sqrt{\nu/\mu}\, r\sin\theta\sin\varphi. \tag{3.9}$$

In terms of these coordinates, and using the fact that $\alpha\nu - \beta\mu = 1$, the distinguished and Hamiltonian functions become

$$C = \frac{2\mu^2 r^2 - \nu^2}{4\mu}, \qquad H = \frac{\nu(2-\alpha\nu) + \alpha\mu\nu r^2}{2\mu^2} - \frac{r^2\cos\theta^2}{2\mu} - \sqrt{\frac{\nu}{\mu}}\frac{r}{\mu}\sin\theta\sin\varphi. \tag{3.10}$$

Thus, a level set of C is a sphere of radius r. Notice that all the α and β dependence in the Hamiltonian is confined to the constant term (this actually occur for all cases). This implies that the equations of the motion, in contrast, will exhibit no dependence whatsoever on these two parameters. The geometric significance is that the orbits of the motion are invariant under SL(2, **R**) deformations of the functions C and H. Under the change of coordinate functions (3.9), the vector field X_H and the equations of the motion are

$$X_H = \frac{\mu}{r\sin\theta}\left(\frac{\partial H}{\partial\varphi}\frac{\partial}{\partial\theta} + \frac{\partial H}{\partial\theta}\frac{\partial}{\partial\varphi}\right); \qquad \dot\theta = -\sqrt{\frac{\nu}{\mu}}\cos\varphi, \qquad \dot\varphi = -\cot\theta\left(r\sin\theta - \sqrt{\frac{\nu}{\mu}}\sin\varphi\right). \tag{3.11}$$

These equations admit up to four distinct critical points, $(\theta, \varphi) = (\pi/2, \pm\pi/2)$, as well as the pair $(\sin\theta, \varphi) = [(\nu/\mu r^2)^{1/2}, \pi/2]$; whenever $r > \sqrt{\nu/\mu}$. Linear stability analysis gives the following information. The points $(\theta, \varphi) = (\pi/2, -\pi/2)$ and $(\sin\theta, \varphi) = [(\nu/\mu r^2)^{1/2}, \pi/2]$ are always stable. As for $(\theta, \varphi) = (\pi/2, \pi/2)$, this point is stable if $r < \sqrt{\nu/\mu}$ and is of saddle type whenever $r > \sqrt{\nu/\mu}$. Therefore, a Hamiltonian pitchfork bifurcation takes place at $r = \sqrt{\nu/\mu}$, i.e., when $4\mu C - \nu^2 = 0$. We mention that such a bifurcation did not occur for the previous case. Indeed, it is clear that the two homoclinic loops are not allowed to shrink to a point, due to the topology of the cylinder; this is a consequence of whether we reduce to compact level sets of the distinguished function or not.

Case 3b. $\mu \neq 0$, $\nu \neq 0$, $\mu\nu < 0$, $4\mu C + \nu^2 > 0$. For this last case, the distinguished function C is given by an hyperbolic quadric. Thus we consider subcases corresponding to the three possible geometries of the level sets of the distinguished function,

$$x_1^2 - (-\tfrac{\mu}{\nu})\,[x_2^2 + (x_3 + \nu/\mu)^2] = \frac{4\mu C + \nu^2}{2\mu\nu} \equiv R. \tag{3.12}$$

Subcase 3b.1. $4\mu C + \nu^2 < 0$. Then $R = r^2 > 0$. For this first subcase, the level sets of the distinguished function are two-sheeted hyperboloids. A natural set coordinate functions is given by

$$x_1 = r\cosh u, \qquad x_2 = \sqrt{-\nu/\mu}\, r\sinh u\cos\varphi, \qquad x_3 = -\frac{\nu}{\mu} + \sqrt{-\nu/\mu}\, r\sinh u\sin\varphi. \tag{3.13}$$

In terms of these, the distinguished function and the Hamiltonian function take the form

$$C = \frac{2\mu\nu r^2 - \nu^2}{4\mu}, \qquad H = \frac{\nu(2 - \alpha\nu) + \alpha\mu\nu r^2}{2\mu^2} - \frac{r^2\cosh u^2}{2\mu} - \sqrt{\frac{-\nu}{\mu}}\, r\sinh u\sin\varphi. \tag{3.14}$$

Again, only the constant term of the Hamiltonian function shows any dependence on the parameter α; the geometry of the solutions in the unreduced phase space $\mathbf{R}^3$ are therefore blind to this parameter. The equations of the motion on the reduced space are

$$X_H = \frac{\mu}{r\sinh u}\left(\frac{\partial H}{\partial u}\frac{\partial}{\partial\varphi} + \frac{\partial H}{\partial\varphi}\frac{\partial}{\partial u}\right);$$
$$\dot u = \sqrt{\frac{-\nu}{\mu}}\cos\varphi, \qquad \dot\varphi = -\coth u\left(r\sinh u + \sqrt{\frac{-\nu}{\mu}}\sin\varphi\right). \tag{3.15}$$

These equations possess two critical points, $(\varphi, \sinh u) = (\pi/2, -\sqrt{-\nu/\mu}/r)$ (on the bottom sheet of the hyperboloid) and $(\varphi, \sinh u) = (-\pi/2, \sqrt{-\nu/\mu}/r)$ on the top sheet. Linear stability analysis shows that both points are stable. Thus each sheet of the hyperboloidal reduced space is foliated by a family of periodic orbits.

Subcase 3b.2. $4\mu C + \nu^2 > 0$. Then $R = -r^2 > 0$. For this second subcase, level sets of the distinguished function C are one-sheeted hyperboloids. We choose new coordinate functions as

$$x_1 = r\sinh u, \qquad x_2 = \sqrt{-\nu/\mu}\, r\cosh u\cos\varphi, \qquad x_3 = -\frac{\nu}{\mu} + \sqrt{-\nu/\mu}\, r\cosh u\sin\varphi. \tag{3.16}$$

In terms of these, the distinguished function C and the Hamiltonian function H become

$$C = \frac{-(2\mu\nu r^2 + \nu^2)}{4\mu}, \qquad H = \frac{\nu(2-\alpha\nu) - \alpha\mu\nu r^2}{2\mu^2} - \frac{r^2\sinh u^2}{2\mu} - \sqrt{\frac{-\nu}{\mu}}\frac{r}{\mu}\cosh u\sin\varphi. \tag{3.17}$$

The equations of the motion are consequently

$$X_H = \frac{\mu}{r\cosh u}\left(\frac{\partial H}{\partial u}\frac{\partial}{\partial\varphi} + \frac{\partial H}{\partial\varphi}\frac{\partial}{\partial u}\right);$$
$$\dot{u} = \sqrt{\frac{-\nu}{\mu}}\cos\varphi, \qquad \dot{\varphi} = -\tanh u\left(r\cosh u + \sqrt{-\frac{\nu}{\mu}}\sin\varphi\right). \tag{3.18}$$

These equations admit up to four fixed points, $(u, \varphi) = (0, \pm\pi/2)$, as well as the pair $(\cosh u, \varphi) = [\pm(\nu/\mu r^2)^{1/2}, -\pi/2]$; the latter exist whenever $r < \sqrt{-\nu/\mu}$. Linear stability analysis provides us with the following information. The points $(u, \varphi) = (0, \pi/2)$ and $(\cosh u, \varphi) = [\pm(\nu/\mu r^2)^{1/2}, -\pi/2]$ are always stable. As for $(u, \varphi) = (0, -\pi/2)$, this point is stable if $r > \sqrt{-\nu/\mu}$ and is of saddle type whenever $r < \sqrt{-\nu/\mu}$.

Subcase 3b.3. $4\mu C + \nu^2 = 0$. Then $R = 0$. For this last subcase, the level sets of the distinguished function C are cones. We choose new coordinate functions as follows:

$$x_1 = z, \qquad x_2 = \sqrt{-\nu/\mu}\, z\cos\varphi, \qquad x_3 = -\frac{\nu}{\mu} + \sqrt{-\nu/\mu}\, z\sin\varphi. \tag{3.19}$$

In terms of these, the distinguished function C and the Hamiltonian function H become

$$C = \frac{-\nu^2}{4\mu}, \qquad H = \frac{\nu(2-\alpha\nu)}{2\mu^2} - \sqrt{\frac{-\nu}{\mu}}\frac{z}{\mu}\sin\varphi - \frac{z^2}{2\mu}. \tag{3.20}$$

Once more, we note that the α-dependence of the Hamiltonian function is only within the constant term, so that it does not affect the dynamics, as far as the nature of the solutions is concerned. The equations of the motion are

$$X_H = 2\mu\left(\frac{\partial H}{\partial z}\frac{\partial}{\partial\varphi}+\frac{\partial H}{\partial\varphi}\frac{\partial}{\partial z}\right); \qquad \dot{z} = 2\sqrt{\frac{-\nu}{\mu}}z\cos\varphi, \qquad \dot{\varphi} = -2\left(z+\sqrt{\frac{-\nu}{\mu}}\sin\varphi\right). \tag{3.21}$$

These equations have three fixed points. One is $(z, \varphi) = (0, 0) = (0, \pi)$, located at the vertex of the cone. The other two are $(z, \varphi) = (-\sqrt{-\nu/\mu}, \pi/2)$, $(\sqrt{-\nu/\mu}, -\pi/2)$. Linear stability analysis shows that the latter are always stable, while the point at the vertex is unstable. The nature of the phase portrait is as follows. Two homoclinic orbits are connected to $z = 0$, one on each half of the cone; these separate the halves into two regions foliated by a family of periodic orbits.

Remark. For cases 2 and 3, the phase portraits are characterized by a pair of homoclinic loops connected to an unstable point. Under time-dependent Hamiltonian perturbations, this configuration is expected to break. If the perturbation is periodic and preserves the distinguished level surfaces, the usual homoclinic tangle phenomenon yielding horseshoe chaos will occur on each level set of the preserved distinguished function. If the perturbation does not preserve the distinguished level surfaces, the dynamical space becomes again three-dimensional, and provided the unstable point persists as either a saddle-sink or a saddle-source, Hamiltonian Shilnikov chaos may take place for some classes of Hamiltonian perturbations.

4. Conclusion

We investigated an invariant three-dimensional subsystem of the Maxwell-Bloch set of equations. We have shown that the form of the equations of motion is group-invariant under SL(2, **R**). This allowed us to classify the various Lie-Poisson structures admitted by the system. Each structure is associated with a distinct Hamiltonian reduction on a quadratic surface defined as a level set of the second Hamiltonian function, embedded in $\mathbf{R}^3$ as a phase space for a one-degree-of-freedom system. On the reduced space, the symplectic motion is coordinate-dependent, e.g., the phase portrait can be made to correspond to the dynamics of either a pendulum, or a Duffing oscillator. An extension of this work is in progress to investigate the whole class of bi-Hamiltonian dynamical systems on $\mathbf{R}^3$ which can be written in the form (1.1) with general quadratic Hamiltonian functions. Clearly, the same SL(2, **R**) dynamical invariance holds for any such system (see, e.g., ref. [4]). Thus, one can also classify these systems according to admissible pairs of Hamiltonian and distinguished functions. A similar analysis can also be performed for multi-Hamiltonian dynamical systems defined on higher dimensional manifolds.

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