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COMPUTATIONAL STABILITY OF THE VON NEUMANN-RICHTMYER METHOD FOR THE CASE OF THE IDEAL GAS LAW

D. L. Hicks

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COMPUTATIONAL STABILITY OF THE VON NEUMANN-RICHTMYER METHOD FOR THE CASE OF THE IDEAL GAS LAW

D. L. Hicks
Computational Physics & Mechanics Division I - 5162
Sandia Laboratories, Albuquerque, New Mexico 87185

ABSTRACT

Two stability concepts are of interest for partial difference equations--one arises in theory--the other in practice. The theoretical kind, referred to here as asymptotic stability, is essentially just asymptotic (as Δt, $\Delta x \to 0$) boundedness of the discrete solution. The other kind, referred to here as computational stability, is stability for a fixed Δt and Δx--computational instability is indicated in practice by oscillatory behavior of the discrete approximation--in particular, oscillations of period $2\Delta x$. This report is concerned with computational stability.

Only approximate stability analyses of the von Neumann-Richtmyer scheme have been done for the case of the ideal gas law. Herein a more rigorous computational stability analysis is sought. The analysis leads to a recommendation for the improvement of the time step restriction in WONDY for the case of the ideal gas law.

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1. INTRODUCTION

From conversations with R. D. Richtmyer [1], G. N. White [2], and others, it appears that only approximate stability analyses of the vN-R (von Neumann-Richtmyer [3]) scheme have been done for the case of the ideal gas law. At this time, some approximations still appear to be unavoidable. However, herein an attempt is made to avoid as many of the simplifying assumptions and approximations as possible in order to arrive at a "more rigorous" computational stability condition than has heretofore been obtained.

2. NOTATION

Except for minor variations, the notation and nomenclature of vN-R (von Neumann-Richtmyer [3]) and/or R-M (Richtmyer-Morton [4]) is followed here. They write the equations of hydrodynamics in a form equivalent to the following: $\partial u/\partial t = -\partial(p + q)/\rho^o\partial x$, $\partial V/\partial t = \partial u/\rho^o\partial x$, and $\partial\mathcal{E}/\partial t = -(p + q)\partial V/\partial t$. Here $(V,u,\mathcal{E})$ are specific (volume, momentum, internal energy) with (x,t) being the (space, time) variables. The stress variables (p,q) are (pressure, viscosity). If the material law is the ideal gas law, then $p = \Gamma\mathcal{E}\rho$ where $\Gamma(= \gamma - 1)$ is a positive constant and $\rho = 1/V$ is the mass density with ρ^o being the initial value of ρ. The original vN-R artificial viscosity had the form $q = \rho\ell^2(\partial u/\partial x)^2 \cdot \mathrm{sign}(-\partial V/\partial t)$ where $\ell = c_2\Delta x$ and c_2 is a dimensionless constant ≈ 1.0.

Let $x_j = j\Delta x$ and $t^n = n\Delta t$. The primary evaluation points for the discrete approximations in the vN-R scheme are as follows: u at $(x_j,t^{n+1/2})$, q at $(x_{j+1/2},t^{n+1/2})$, and V, ρ, $\mathcal{E}$, p at $(x_{j+1/2},t^n)$. The evaluations of the approximation functions are denoted as follows: $u_j^{n+1/2} \approx u(x_j,t^{n+1/2})$, $q_{j+1/2}^{n+1/2} \approx q(x_{j+1/2},t^{n+1/2})$, and $f_{j+1/2}^n \approx f(x_{j+1/2},t^n)$ for $f = \mathcal{E},p,V$. Differences and averages taken between primary evaluation points are denoted by the delta-dot notation: $\Delta_{\cdot}u_{j+1/2}^{n+1/2} = u_{j+1}^{n+1/2} - u_j^{n+1/2}$, $\Delta^{\cdot}u_j^n = u_j^{n+1/2} - u_j^{n-1/2}$, $u_{\cdot j+1/2}^{n+1/2} = (u_{j+1}^{n+1/2} + u_j^{n+1/2})/2$, $u^{\cdot n}_j = (u_j^{n+1/2} + u_j^{n-1/2})/2$; and for $f = \mathcal{E},p,V$: $\Delta_{\cdot}f_j^n = f_{j+1/2}^n - f_{j-1/2}^n$, $\Delta^{\cdot}f_{j+1/2}^{n+1/2} = f_{j+1}^{n+1} - f_{j+1/2}^n$, $f_{\cdot j}^n = (f_{j+1/2}^n + f_{j-1/2}^n)/2$, $f^{\cdot n+1/2}_{j+1/2} = (f_{j+1/2}^{n+1} + f_{j+1/2}^n)/2$, etc. That is, $(\Delta_{\cdot},\Delta^{\cdot})$ indicates a (space, time) difference and $(f_{\cdot},f^{\cdot})$ indicates a (space, time) average--both taken between adjacent evaluations of the primary approximation functions.

Let $r = \Delta t/(\rho^o\Delta x)$, then the vN-R difference scheme is:

$$[\Delta^{\cdot}u + r\Delta_{\cdot}p]_j^n = -r\Delta_{\cdot}q_j^{n-1/2} ,$$

$$[\Delta^{\cdot}V - r\Delta_{\cdot}u]_{j+1/2}^{n+1/2} = 0 ,$$

$$[q + \ell^2|\Delta_{\cdot}u/\Delta x|(\Delta_{\cdot}u/\Delta x)/V^{\cdot}]_{j+1/2}^{n+1/2} = 0 ,$$

$$[\Delta^{\cdot}\mathcal{E} + (p^{\cdot} + q)\Delta^{\cdot}V]_{j+1/2}^{n+1/2} = 0 .$$

For the ideal gas law the vN-R equation is

$$[p - \Gamma\mathcal{E}/V]_{j+1/2}^n = 0 .$$

Following vN-R the first variation of f is denoted δf. The equations of first variation of the difference equations are:

$$[\Delta^{\cdot}\delta u + r\Delta^{\cdot}\delta p]_j^n = -r\Delta_{\cdot}\delta q_j^{n-1/2} ,$$

$$[\Delta^{\cdot}\delta V - r\Delta_{\cdot}\delta u]_{j+1/2}^{n+1/2} = 0 ,$$

$$[\delta q + \ell^2|\Delta_{\cdot}u/\Delta x|(2\Delta_{\cdot}\delta u/\Delta x - (\Delta_{\cdot}u/\Delta x)\delta V^{\cdot}/V^{\cdot})/V^{\cdot}]_{j+1/2}^{n+1/2} = 0 ,$$

$$[\Delta^{\cdot}\delta\mathcal{E} + (\delta p^{\cdot} + \delta q)\Delta^{\cdot}V + (p^{\cdot} + q)\Delta^{\cdot}\delta V]_{j+1/2}^{n+1/2} = 0 ,$$

and

$$[\delta p - \Gamma\delta\mathcal{E}/V + \Gamma\mathcal{E}\delta V/V^2]_{j+1/2}^n = 0 .$$

<u>Definition</u> (Local Equations of First Variation)

Let $\underset{\sim}{c}$ be the coefficients of the equations of first variation. Evaluate $\underset{\sim}{c}$ at (x_j, t^n). The resulting equations are called the local equations of first variation at (x_j, t^n).

Definition (von Neumann Stability or Stable in the Sense of von Neumann)

Suppose that the local equations of first variation at (x_j, t^n) are stable under a time step restriction of the form

$$f_j^n(\underset{\sim}{c}(x_j, t^n), \Delta t, \Delta x) \leq 1 \quad (*)_j^n$$

where f may depend on n, j, $\underset{\sim}{c}$ evaluated at (x_j, t^n), Δt, and Δx. If $(*)_j^n$ is enforced for all n and j, then the difference scheme is said to be stable in the sense of von Neumann or, for short, von Neumann stable.

The definition of von Neumann stability is not complete until we agree on the definition of stable used there. There are two different stability concepts which come to mind in this context. One kind, referred to here as asymptotic stability, is well-defined and the definition may be found on p. 45 of [10]. It is essentially asymptotic boundedness (i.e., boundedness as Δt, $\Delta x \to 0$) of the discrete approximations. The other kind, referred to here as computational stability, is not well-defined but is of greater practical interest as it relates to stability for fixed Δt and Δx. Computational instability shows up in practice as a high-frequency oscillatory behavior of the discrete solution--in particular oscillations of the highest possible frequency, i.e., those with period $2\Delta x$.

The utility of a von Neumann stability analysis lies in what it can reveal about computational stability--it can find constraints on the growth rates of small perturbations. On the other hand, a von Neumann stability analysis does not seem to be appropriate nor to have any utility in establishing asymptotic stability. The appropriate way to establish asymptotic stability of nonlinear hyperbolic systems seems to be with energy inequalities. The von Neumann stability analysis procedure assumes

the coefficients of the equations of first variation to be bounded. This is often tantamount to assuming that the discrete solution is bounded. But to show boundedness of the discrete solution is the purpose of the asymptotic stability analysis. Thus a von Neumann stability analysis with the purpose of establishing asymptotic stability often leads into a circular argument.

Therefore, it seems that a von Neumann stability analysis should concern itself solely with the important problem of analyzing computational stability. Unfortunately, we do not have, at this time, a rigorous definition of a concept which fits exactly the intuitive notion of computational stability for partial difference equations. For ordinary difference equations there do exist concepts of computational stability--for example, "A-stable". However, that concept doesn't seem to fit the intuitive notion of computational stability for hyperbolic partial difference equations. It is not clear, at this time, how computational stability should be defined. However, the following definition seems to approach the intuitive notion.

<u>Definition</u> (Computational Stability)

Let $\underset{\sim}{G}(k,\Delta t,\Delta x)$ be the amplification matrix of the local equations of first variation for a fixed spatial mesh. That is, for Δx a constant. If there exists a positive number τ such that for all Δt in $(0,\tau)$, $0 \le n\Delta t \le t_f$, and for all k $\underset{\sim}{G}(k,\Delta t,\Delta x)^n$ is uniformly bounded, then we say that the system of partial difference equations associated with $\underset{\sim}{G}(k,\Delta t,\Delta x)$ is computationally stable for $0 < \Delta t < \tau$ and for $n\Delta t \le t_f$.

Definition (von Neumann's Necessary Condition for Computational Stability)

Let $|\lambda(k,\Delta t,\Delta x)|_{max}$ be the spectral radius of $\underset{\sim}{G}(k,\Delta t,\Delta x)$ where Δx is a constant. If $|\lambda(k,\Delta t,\Delta x)|_{max} \leq 1 + O(\Delta t)$ for all k and for $0 < \Delta t < \tau$ and $n\Delta t < t_f$, then we say that the system of partial differential equations associated with $\underset{\sim}{G}(k,\Delta t,\Delta x)$ satisfies von Neumann's necessary condition for computational stability for $0 < \Delta t < \tau$ and $n\Delta t \leq t_f$.

Remarks: In the von Neumann stability analysis the growth of perturbations of the form

$$\delta \underset{\sim}{U}_j^n = \delta \underset{\sim}{U}^0(k)(\lambda(k))^n \exp ikx_j$$

is studied. All practical hydrocode problems on the computer are mixed *initial-boundary value problems--the x interval must be finite--denote it* $[x_o,x_r]$. Without loss of generality let $x_o = 0$. Then the values of k of interest for a computational stability analysis are $k = 2\pi k'/x_r$ for $k' = 0,1,2,\ldots,J/2$ where $J = x_r/\Delta x$--for simplicity, assume J is an even integer. The highest frequency of interest is with $k^* = J/2$ (or $2\Delta x k = 2\pi$)--that is, the frequency with period $2\Delta x$--the notorious noise frequency. In practice computational instability shows the symptom of oscillations with frequency $k' = k^*$. It seems that one of the weaknesses of the foregoing definition of computational stability is that it does not take this into account. Apparently, in order to compensate for this weakness, one should try to enforce

$$|\lambda(k',\Delta t,\Delta x)|_{max} \leq 1 + \Delta t F(k',\Delta t,\Delta x)$$

with $F(k',\Delta t,\Delta x)$ as small as possible for $k' = k^*$.

3. LEMMAS

In Lemmas 1A and 1B the roots of the quadratic $A\lambda^2 - 2B\lambda + C = 0$ are the objects of interest. The proofs may be found in [5].

Lemma 1A

Let A, B, and C be real numbers with $A > 0$, $D = (B/A)^2 - C/A$, $\lambda_\pm = B/A \pm D^{1/2}$, and $|\lambda|_{max} = \max|\lambda_\pm|$.

Case (a): If $D \geq 0$ and $B^2 > A^2$, then

$$2 < |\lambda|_{max} .$$

Case (b): If $D \geq 0$ and $B^2 \leq A^2$, then

$$[|\lambda|_{max} \leq 1 \text{ iff } 2|B| \leq A + C] .$$

Case (c): If $D < 0$, then

$$[|\lambda|_{max} \leq 1 \text{ iff } C \leq A] .$$

Lemma 1B

Under the same assumptions as Lemma 1A let $A = 1 - a$, $B = 1 - b$, and $C = 1 - c$.

Case (B_+): If $D \geq 0$, $B^2 \leq A^2$, and $B > 0$, then

$$[|\lambda|_{max} \leq 1 \text{ iff } a + c \leq 2b] .$$

Case (B_-): If $D \geq 0$, $B^2 \leq A^2$, and $B \leq 0$, then

$$[|\lambda|_{max} \leq 1 \text{ iff } a - 2b + c \leq 4] .$$

Case (C): If $D < 0$, then

$$[|\lambda|_{max} \leq 1 \text{ iff } a \leq c] .$$

The following is useful in reducing quadratic constraints to linear constraints.

Lemma 2

Assume: A is a real number; α and B are positive, real numbers. Let $D = B^2 + A$ and $\alpha' = \alpha(B + D^{1/2})$. Consider the following inequalities:

$$(**) \qquad A\alpha^2 + 2B\alpha \leq 1$$

and

$$(*) \qquad \alpha' \leq 1 .$$

Case (A): If $D \geq 0$, then (**) iff (*).

Case (B): If $D < 0$, then (**) holds for all α.

4. BACKGROUND

First we review the approximate stability analysis presented by vN-R in their 1950 [3] paper. They considered the following system of equations:

$$\partial u/\partial t = -\partial(p + q)/(\rho^{o}\partial x) \tag{4.1}$$

$$\partial V/\partial t = \partial u/(\rho^{o}\partial x) \tag{4.2}$$

$$q = c\ell^{2}(\partial u/\partial x)^{2}\cdot\mathrm{sign}(-\partial u/\partial x) \tag{4.3}$$

and

$$\partial p/\partial t = -c[\gamma p + \Gamma q]\partial V/\partial t \ . \tag{4.4}$$

Equation 4,* referred to as the $\partial p/\partial t$ equation, follows from eliminating $\mathcal{E}$ from the $\partial\mathcal{E}/\partial t$ equation by use of the ideal gas law. Note that $c[\gamma p + \Gamma q] = a^{2} + \Gamma\rho q$ where $a = c\rho$ is the acoustic impedance. The equations of first variation of equations 1-4 are:

$$\partial\delta u/\partial t = -\partial(\delta p + \delta q)/(\rho^{o}\partial x) \ , \tag{4.5}$$

$$\partial\delta V/\partial t = \partial\delta u/(\rho^{o}\partial x) \ , \tag{4.6}$$

$$\delta q = -\rho\ell^{2}|\partial u/\partial x|[2\partial\delta u/\partial x - \rho\partial u/\partial x\ \delta V] \ , \tag{4.7}$$

and

$$\delta V\partial p/\partial t + V\partial\delta p/\partial t = -[\gamma\delta p + \Gamma\delta q]\partial V/\partial t + [\gamma p - \Gamma q]\partial\delta V/\partial t \ . \tag{4.8}$$

Equations 5-8 are equivalent to Equations 37-40 of vN-R [3]. Let $\underset{\sim}{\delta U}^{T} = (\delta u, \delta V, \delta q, \delta p)$ then substitute $\underset{\sim}{\delta U} = \underset{\sim}{\delta U}_{o}\, e^{ikx+\alpha t}$ into (4.5-8) to get

$$\underset{\sim}{H}\ \underset{\sim}{\delta U}_{o} = 0$$

---

*Within Section N a reference to equation M refers to equation N.M.

where $\underset{\sim}{H}$ is given by Equation 9.

$$\underset{\sim}{H}(\alpha) = \begin{bmatrix} \alpha\rho^o & 0 & ik & ik \\ -ik & \alpha\rho^o & 0 & 0 \\ H_{31} & H_{32} & 1 & 0 \\ 0 & H_{42} & \Gamma\dot{V} & H_{44} \end{bmatrix} \tag{4.9}$$

where

$$H_{31} = 2ik\ell^2\rho^o|\dot{V}|/V$$

$$H_{32} = -(\ell\rho^o)^2|\dot{V}|\dot{V}/V^2$$

$$H_{44} = \alpha V + \gamma\dot{W}$$

with

$$\dot{V} = \partial V/\partial t$$

and

$$\dot{p} = \partial p/\partial t \ .$$

The zero positions in $\underset{\sim}{H}$ reduce the determinantal equation for $\underset{\sim}{H}$ to

$$\det \underset{\sim}{H} = H_{11}H_{22}H_{33}H_{44} + H_{14}H_{21}H_{33}H_{42} - K^{21}(H_{21}H_{32} - H_{22}H_{31}) \tag{4.10}$$

where

$$K^{21} = H_{14}H_{43} - H_{13}H_{44} \ .$$

Substituting the values of the H_{ij} into Equation 10 yields

$$\det \underset{\sim}{H} = (\alpha\rho^o)^2(\alpha V + \gamma\dot{W}) + k^2\left\{(\ell\rho^o)^2|\dot{V}|[2\alpha^2 + \alpha\dot{V}/V - (\dot{V}/V)^2] + H_{42}\right\} \ . \tag{4.11}$$

Remark: Equation (4.11) above is equivalent to Equation (42) of vN-R [3]. This becomes evident when all appearances of $\partial u/(\rho^o \partial x)$ are replaced by $\dot{V}$. Then their second and seventh terms combine and their Equation (42) reduces to (4.11) above.

In analyzing the roots of det $\underset{\sim}{H}(\alpha) = 0$, vN-R restricted attention to the higher powers of α and k. In shock regions (q large) their analysis yielded

$$\alpha \approx -2(k\ell)^2 \dot{V}/V \ . \tag{4.12}$$

In nonshock regions (q negligible) they got

$$\alpha^2 x - (ak/\rho^o)^2 \ . \tag{4.13}$$

Identifying the terms in the equations of first variation leading to (4.12) and (4.13) led to: for shock regions

$$\partial \delta u/\partial t = \sigma \partial^2 \delta u/\partial x^2 \tag{4.14}$$

where $\sigma = 2\ell^2|\dot{V}|/V$ and for nonshock regions

$$\partial^2 \delta u/\partial t^2 = (a\partial/c^o \partial x)^2 \delta u \ . \tag{4.15}$$

To do their approximate stability analysis, vN-R simply analyzed the stability of the discrete analogs of (4.14) and (4.15). For nonshock regions this leads to the famous CFL condition, i.e.,

$$ar \le 1 \ . \tag{4.16}$$

For shock regions their stability inequality is

$$4r\Lambda \le 1 \tag{4.17}$$

where Λ is the coefficient of $-\Delta_1 u$ in the artificial viscosity. That is,

$$q = -\Lambda \Delta_{\cdot} u \ . \tag{4.18}$$

The original vN-R viscosity has

$$\Lambda = c_2^2 \, \rho \, |\Delta_{\cdot} u| \tag{4.19}$$

with c_2 a dimensionless constant ≈ 1. The Rosenbluth (see R-M [4], p. 313) modification of q sets $\Lambda = 0$ in expansion ($\Delta_{\cdot} u > 0$). Then Landshoff [6] modified q by setting

$$\Lambda = c_1 a + c_2^2 \, \rho |\Delta_{\cdot} u| \tag{4.20}$$

where c_1 is a dimensionless constant ≈ 0.1.

There are two variants of the stability analysis procedure introduced by von Neumann and Richtmyer. The first, called the α-method, is exemplified in the analysis that led to the $\underset{\sim}{H}(\alpha)$ matrix, Equation (4.9).

The second, called the λ-method, involves the substitution of

$$\delta \underset{\sim}{U}_j^n = \delta \underset{\sim}{U}^n \, \xi^j$$

into the finite-difference equations given in Section 2, where $\xi = \exp(ik\Delta x)$. This leads to an equation of the form

$$\underset{\sim}{H}_1 \, \delta \underset{\sim}{U}^{n+1} = H_o \, \delta U^n$$

which in turn leads (letting $\underset{\sim}{G} = \underset{\sim}{H}_1^{-1} \underset{\sim}{H}_o$) to

$$\delta \underset{\sim}{U}^{n+1} = \underset{\sim}{G} \, \delta \underset{\sim}{U}^n \tag{4.21}$$

where $\underset{\sim}{G}$ is the amplification matrix. The next step in the standard version of the λ-method is to find the roots of

$$\det(\underset{\sim}{G} - \lambda \underset{\sim}{I}) = 0 \ .$$

Another version of the λ-method does not require the computation of $\underset{\sim}{H}_1^{-1}$ and $\underset{\sim}{H}_1^{-1} \underset{\sim}{H}_o$ and is closely analogous to the α-method. Replace $\delta \underset{\sim}{U}^{n+1}$

by $\lambda \delta \underset{\sim}{U}^n$ in $\underset{\sim}{H}_1 \delta \underset{\sim}{U}^{n+1} = \underset{\sim}{H}_o \delta \underset{\sim}{U}^n$ to get

$$\underset{\sim}{J}(\lambda)\delta \underset{\sim}{U}^n = 0 \tag{4.22}$$

where $\underset{\sim}{J}(\lambda) = \lambda \underset{\sim}{H}_1 - \underset{\sim}{H}_o$. Note that $\det \underset{\sim}{J}(\lambda) = 0$ is equivalent to $\det(\underset{\sim}{G} - \lambda \underset{\sim}{I}) = 0$ provided $\underset{\sim}{H}_1^{-1}$ exists. Observe that Equation (22) is analogous to $\underset{\sim}{H}(\alpha)\delta \underset{\sim}{U}_o = 0$ in a certain fashion.

Let $\underset{\sim}{H}(\alpha)$ result from an α-method analysis of the differential (not the difference) equations of some system of PDE's (e.g., $H(\alpha)$ for Equations 1-4 is given by Equation 9). Let $\underset{\sim}{J}(\lambda)$ result from a λ-method analysis of the difference analogs of the differential equations (e.g., the $\underset{\sim}{J}(\lambda)$ for the vN-R difference analogs of 1-4 is derived in Section 5). The correspondence between α and λ is $\lambda = e^{\alpha \Delta t}$. It might appear that there should be a simple correspondence between $\underset{\sim}{H}(\alpha)$ and $\underset{\sim}{J}(\lambda)$. For example, consider Equations 1-4: corresponding to the operation $\partial e^{\alpha t}/\partial t = \alpha e^{\alpha t}$ in the derivation of $\underset{\sim}{H}(\alpha)$ there is the operation $\Delta' e^{\alpha t}/\Delta t = e^{\alpha t}(\lambda - 1)/\Delta t$ in the derivation of $\underset{\sim}{J}(\lambda)$. Similarly, the operation $\partial e^{ikx}/\partial x = ik\, e^{ikx}$ in the derivation of $\underset{\sim}{H}(\alpha)$ is analogous to the following operation in the derivation of $\underset{\sim}{J}(\lambda)$: $\Delta_. e^{ikx}/\Delta x = e^{ikx}(2i \sin \theta)/\Delta x$ where $\theta = k\Delta x/2$. Thus if one replaced $(\alpha, k, \partial/\partial t, \partial/\partial x)$ in $H(\alpha)$ by $((\lambda - 1)/\Delta t, 2 \sin \theta/\Delta x, \Delta'/\Delta t, \Delta./\Delta x)$ to derive say "$\underset{\sim}{J}(\lambda)$", then one might conjecture that "$\underset{\sim}{J}$" $= \underset{\sim}{J}$. This conjecture is true when the time and space differencing possess a certain uniformity. However, the uniformity is lacking in the vN-R scheme. The reader can see explicitly what is meant by considering the vN-R difference analogs of the $\partial u/\partial t$ and $\partial v/\partial t$ equations.

The failure of the above conjecture might lead to the weaker conjecture that the stability analysis of "$\underset{\sim}{J}$" should produce stability inequalities

similar to those required for the stability of $\underset{\sim}{J}$. If the last conjecture were true, then it would provide a justification of sorts for the approximate stability analysis of vN-R [3].

In 1954 H. G. Kolsky reported (see p. 13 of [7]) G. N. White's stability inequality:

$$(ar)^2 + 4\Lambda r \leq 1 \tag{4.23}$$

with $\Lambda = c_2^2 \rho |\Delta u|$. Note that White's inequality combines the vN-R inequalities (16) and (17) in a certain way. White's approximate stability analysis was never published [2]. White's inequality was also reported on p. 350 of [4]--unfortunately with a typographical error--the 2 exponent of ar is missing there.

About 1957 Richtmyer (see [8], pp. 218-222) returned to the stability analysis to say: "It is found that two of the eigenvalues of $\underset{\sim}{G}(\Delta t,k)$ are $\lambda = 1$ and $\lambda = 0$, independent of Δt and of k. The other two eigenvalues satisfy the quadratic equation

$$(\lambda - 1)\left[\lambda - 1 - \frac{\lambda + 1}{2}(\gamma - 1)z\Delta t\right]$$

$$+ \left(\frac{4\Delta t}{\Delta x^2}\sin^2\beta\right)\left\{\left[2(\lambda - 1) + \frac{\lambda + 1}{2}z\Delta t\right]\ell^2 z\left[1 + \frac{\gamma - 1}{2}z\Delta t\right] + \mu\Delta t\right\} = 0."$$

This quadratic in λ, call it $P_R(\lambda)$, appears to be incorrect--more about this later. In the above quadratic the notation used is

$$z = -\left[\frac{1}{V^\circ}\frac{\Delta^\circ V}{\Delta t}\right]_{j+1/2}^{n+1/2}$$

and

$$\mu = (V^\circ)^2\left[(\gamma p^\circ + \Gamma q)/V^\circ\right]_{j+1/2}^{n+1/2} .$$

It is also convenient to introduce the notation $\zeta = z\Delta t/2$ for later use.

About 1966 R. . Thompson (see Appendix of [9]) did an approximate stability analysis of the vN-R scheme for the case of a mechanical equation of state. Thompson's stability inequality is

$$(\Delta r)^2 + 2\Lambda r \leq 1 \tag{4.24}$$

where Λ is a nonnegative constant. Reason for the 2-4 discrepancy in the Λ coefficient in Thompson's and White's inequalities: White was considering a viscosity quadratic in $\Delta_{\bullet}u$ that leads to an extra factor of 2 in the first variation--Thompson was considering a viscosity linear in $\Delta_{\bullet}u$.

About 1967 Richtmyer with Morton returned again to the stability analysis of the vN-R scheme (see [4], pp. 320-324). They point out that the quadratic $P_R(\lambda)$ in Richtmyer [8] is in error and say that it should be

$$P_{RM}(\lambda) = (\lambda - 1)\left\{\lambda - 1 - \frac{\lambda + 1}{2}(\gamma - 1)z\Delta t\right\}$$

$$+\left(\frac{4\Delta t}{\Delta x^2}\sin^2\theta\right)\left\{\left[2(\lambda - 1) + \frac{\lambda + 1}{2}z\Delta t\right]\ell^2 z\left[1 + \frac{\gamma - 1}{2}z\Delta t\right] + \lambda\mu\Delta t\right\} .$$

Note that the only difference between P_R and P_{RM} is that the $\mu\Delta t$ term changed to $\lambda\mu\Delta t$. In their approximate analysis they take $\Delta t/\Delta x^2$ = constant and let Δt, $\Delta x \to 0$ to get $\lambda = 1$ and $\lambda = 1 - M$ (where $M = 8(\ell/\Delta x)^2 z\Delta t$) as the asymptotic roots of $P_{RM}(\lambda)$. This leads to their conclusion that

$$4(\ell/\Delta x)^2 z\Delta t \leq 1 \tag{4.25}$$

is necessary for stability. They went on to say that they believe their argument can provide only a rough guide to stability conditions.

$P_{RM}(\lambda)$ is apparently incorrect--more about this later.

In 1976 a study of the stability of WONDY (based on the vN-R scheme, see [9]) for rate dependent material laws was begun (see [10], [11], [5], and [12]). An example of a rate dependent law is Maxwell's material law:

$$\partial\sigma/\partial t + a^2\partial V/\partial t + R = 0$$

where σ is the stress, a is the instantaneous acoustic impedance, and R is the relaxation function. In Maxwell's material law, a and R are allowed to depend on σ and V. The WONDY difference equation for this is

$$\Delta^{\cdot}\sigma^{n+1/2}_{j+1/2} + a^2(\sigma^n_{j+1/2}, V^{\cdot n+1/2}_{j+1/2})\Delta^{\cdot}V^{n+1/2}_{j+1/2} + \Delta t R(\sigma^n_{j+1/2}, V^{\cdot n+1/2}_{j+1/2}) = 0 \ .$$

<u>Remark</u>: The reader might wonder why a^2 and R are evaluated at $\sigma^n_{j+1/2}$. The reason is basically one of computational convenience. The a^2 and R may be nonlinear functions of σ. In which case if the σ arguments were centered at n+1/2 some sort of iteration procedure would be necessary to solve the implicit equations. Instead what is done in WONDY is a subcycling when Δt is near τ (a certain parameter called the relaxation time of the material). For a further discussion of this and its effect on stability see [11].

The first stability analysis for rate dependent materials was done on a simple case of the Maxwell material law called Malvern's material law:

$$\partial\sigma/\partial t + a^2\partial V/\partial t + (\sigma - \sigma_{eq})/\tau = 0$$

where a, σ_{eq}, and τ are constant with σ_{eq} called the equilibrium stress and τ the relaxation time. The system of equations can in this case be reduced to two:

$$\partial u/\partial t = -\partial\sigma/(\rho^o\partial x)$$

$$\partial\sigma/\partial t = -a^2\partial u/\rho^o\partial x - (\sigma - \sigma_{eq})/\tau \ .$$

The WONDY difference scheme for these two equations is

$$[\Delta^{\cdot}u + r\Delta_{\cdot}\sigma]_j^n = -r[\Delta_{\cdot}q]_j^{n-1/2}$$

and

$$[\Delta^{\cdot}\sigma + a^2 r\Delta_{\cdot}u]_{j+1/2}^{n+1/2} = -h(\sigma_{j+1/2}^n - \sigma_{eq})$$

where

$$h = \Delta t/\tau \ .$$

In [10] the stability analysis of the two preceding equations with

$$q = -c_1 a\ \Delta_{\cdot}u$$

led to the following result: *A necessary condition for stability is*

$$(ar)^2 + h(\tfrac{1}{2} - c_1 ar) + 2c_1 ar \le 1 \ . \qquad (4.26a)$$

Note that if $h \approx 0$, then (26) is just Thompson's inequality. Therefore, as a corollary of (26) we have the result that if the material law is Hooke's law in one dimension (i.e., $p = -a^2 V$ = constant) and if q is linear in $\Delta_{\cdot}u$ with a constant coefficient, then Thompson's inequality is *necessary for stability*. The stability results in this paragraph are for both asymptotic stability and computational stability.

In [12] it is shown that if

$$0 \le h < 1 \quad \text{and} \quad 0 \le c_1 < (1 - h)^{1/2} - ar$$

then

$$(ar)^2 + h(\tfrac{1}{2} - c_1 ar) + 2c_1 ar < 1 \qquad (4.26b)$$

is sufficient for stability. Note the corollary for Hooke's law which follows from (26b) when $h = 0$. Referring to

$$(ar)^2 + 2c_1 ar < 1$$

as Thompson's <u>strict</u> inequality the result is: If the material law is Hooke's law in one dimension and q is linear in $\Delta_x u$ with constant coefficient then the constraints

$$0 \leq c_1 < 1 - ar$$

are sufficient for stability of the vN-R scheme, and the sufficient constraints imply $(ar)^2 + 2c_1 ar + c_1^2 < 1$ which is only slightly stronger than Thompson's strict inequality since $c_1 \approx .1$ in practice. When $c_1 = 0$, then Thompson's strict inequality reduces to the CFL strict inequality which is necessary and sufficient for stability of the vN-R scheme in the Hooke's law case. This last result (for $c_1 = 0$) follows from p. 263 of [4]. The stability results stated in this paragraph hold for both asymptotic stability and computational stability.

In the course of a previous study of more complicated rate dependent material laws [5] it was noted that a stress of the form of an ideal gas law pressure plus a viscous stress q may be cast in the rate dependent form. Recall Equation 4

$$\partial p/\partial t = -(a^2 - \Gamma\rho q)\partial V/\partial t \ .$$

Now $a^2 + \Gamma\rho q$ may be interpreted as the instantaneous acoustic impedance. Let's call it $(a^+)^2$, i.e.,

$$(a^+)^2 = a^2 + \Gamma\rho q \ .$$

Then for $\sigma = p + q$ we have:

$$\partial\sigma/\partial t + (a^+)^2\partial V/\partial t - \partial q/\partial t = 0$$

and R becomes identified with $-\partial q/\partial t$.

In [5] a stability analysis is done for the case when a and $\Gamma\rho$ are constant and with $q = -[c_1 a + c_2^2\rho^o|\Delta_- u|]\Delta_- u$. Note that this is another way of doing an approximate stability analysis for the ideal gas law. The result found in [5] is the following necessary condition for stability.

$$(a_2 r)^2 + 2K_2 r \le 1 \tag{4.27}$$

where

$$a_2^2 = a^2 + \mathrm{sign}(-\Delta_- u)[\Gamma\rho(2c_1 a + 3c_2^2\rho^o|\Delta_- u|)\Delta_- u + 2c_2^2\rho^o\Delta_-^2\sigma]$$

and

$$K_2 = ac_1 + 2c_2^2\rho^o|\Delta_- u| \ .$$

The inequality in (27) is, of course, for the WONDY rate-dependent differencing which is slightly different from the original vN-R scheme.

In the case of the aforementioned stability analyses [5,10,11,12] certain lemmas were developed (see Section 3). It has been noticed that they can be applied to $P_{RM}(\lambda)$. Rearranging P_{RM} to a standard form

$$P_{RM}(\lambda) = A_2\lambda^2 - 2B_2\lambda + C_2$$

where

$$A_2 = 1 - \Gamma\zeta \ , \ 2B_2 = 2 - \xi_1[z\ell^2\xi_2(2 + \zeta) + \mu\Delta t]$$

$$C_2 = \xi_2[1 - z\ell^2\xi_1(2 - \zeta)] \ , \text{ with}$$

$$\xi_1 = 4\Delta t \sin^2 \theta/\Delta x^2 \ , \text{ and } \xi_2 = 1 + \Gamma\zeta \ .$$

and dividing by ξ_2 (letting $A_3 = A_2/\xi_2$, $B_3 = B_2/\xi_2$, $C_3 = C_2/\xi_2$) we obtain:

$$A_3 = 1 - 2\Gamma\zeta/\xi_2 ,$$

$$2B_3 = 2 - 2\Gamma\zeta/\xi_2 - \xi_1[\mu\Delta t/\xi_2 + z\ell^2(2 + \zeta)]$$

$$C_3 = 1 - z\ell^2\xi_1(2 - \zeta) .$$

The division by ξ_2 is done under the assumption that $\xi_2 > 0$, i.e., $-\zeta < 1/\Gamma$. Apply Lemma 1B with $A_3 = 1 - a_3$, $B_3 = 1 - b_3$, $C_3 = 1 - c_3$: Lemma 1B requires $A_3 > 0$. In order to insure this, let's assume that the time step restriction

$$|\zeta| < 1/\Gamma$$

is imposed. In case (B_-) the requirement

$$a_3 + 2b_3 + c_3 \le 4$$

leads to

$$(a^2 + \rho c)r^2/\xi_2 + z\Delta t\left\{\left(\frac{2\ell}{\Delta x}\right)^2 + \frac{\Gamma}{2\xi_2}\right\} \le 1 . \tag{4.28}$$

Recall that White's inequality is

$$(ar)^2 + z\Delta t\left(\frac{2\ell}{\Delta x}\right)^2 \le 1$$

and the Richtmyer-Morton inequality is

$$z\Delta t\left(\frac{2\ell}{\Delta x}\right)^2 \le 1$$

and compare these with (4.28). This result (4.28) appears to be very encouraging since it contains White's inequality and the Richtmyer-Morton inequality. Unfortunately, it appears that $P_{RM}(\lambda)$ is incorrect.

5. RESULTS

A disagreement was found with [4] on p. 321 on the equation for the eigenvalues of the amplification matrix. Therefore, the steps in it derivation are given in great detail here.

The vN-R difference equations are:

$$[\Delta^{\cdot}u + r\Delta_{\cdot}p]_j^n = -r[\Delta_{\cdot}q]_j^{n-1/2} , \tag{5.1}$$

$$[\Delta^{\cdot}V - r\Delta_{\cdot}u]_{j+1/2}^{n+1/2} = 0 , \tag{5.2}$$

$$[q + c_2^2\ \Delta_{\cdot}u|\Delta_{\cdot}u|/V^{\cdot}]_{j+1/2}^{n+1/2} = 0 , \text{ and} \tag{5.3}$$

$$[\Delta^{\cdot}\mathcal{E} + (p^{\cdot} + q)\Delta^{\cdot}V]_{j+1/2}^{n+1/2} = 0 . \tag{5.4}$$

For the ideal gas law

$$\Gamma\Delta^{\cdot}\mathcal{E} = p^{\cdot}\Delta^{\cdot}V + V^{\cdot}\Delta^{\cdot}p . \tag{5.5}$$

Eliminating $\mathcal{E}$ from (4) and (5) yields

$$[V^{\cdot}\Delta^{\cdot}P + (\gamma p^{\cdot} + \bar{\gamma}q)\Delta^{\cdot}V]_{j+1/2}^{n+1/2} = 0 . \tag{5.6}$$

Remarks: The reader may verify that Equations (1), (2), (3), and (6) are equivalent to Equations (48), (49), (50), and (51) of [3]. Also Equation (4) is equivalent to the $\mathcal{E}$-difference equation in the system of Equations (2.45) on p. 318 of [4]. The other difference equations on that page need to be compared with those in [3], i.e., those in [4] are the same, except for notation, as those in [3].

To solve (6) for p^{n+1} requires

$$\zeta < 1/\gamma \tag{5.7}$$

where

$$\zeta = \left(V^n_{j+1/2} - V^{n+1}_{j+1/2}\right)/\left(V^n_{j+1/2} + V^{n+1}_{j+1/2}\right) . \tag{5.8}$$

Henceforth, it is assumed that the time step restriction imposed by (7) is enforced.

The equations of first variation of (1)-(3) are:

$$[\Delta^{\cdot}\delta u + r\Delta_{\cdot}\delta p]^n_j = -r[\Delta_{\cdot}\delta q]^{n-1/2}_j , \tag{5.9}$$

$$[\Delta^{\cdot}\delta V - r\Delta_{\cdot}\delta u]^{n+1/2}_j = 0 \text{ , and} \tag{5.10}$$

$$[\delta q + c_2^2|\Delta_{\cdot}u|(2\Delta_{\cdot}\delta u - \delta V^{\cdot}\Delta_{\cdot}u/V^{\cdot})/V^{\cdot}]^{n+1/2}_{j+1/2} = 0 . \tag{5.11}$$

Following [4] and [3] in deriving (11), the perturbation δu is assumed small enough such that sign($\Delta_{\cdot}u$) is not altered--*see* [4], p. 321. The first variation of (6) is:

$$[(\gamma\delta p^{\cdot} + \Gamma\delta q)\Delta^{\cdot}V + (\gamma p^{\cdot} + \Gamma q)\Delta^{\cdot}\delta V + V^{\cdot}\Delta^{\cdot}\delta p + \delta V^{\cdot}\Delta^{\cdot}p]^{n+1/2} = 0 . \tag{5.12}$$

Check Equations (9)-(12) by comparing with Equations (37)-(40) of [3]. Following [4], p. 321, the notation

$$z = -[\Delta^{\cdot}V/(V^{\cdot}\Delta t)]^{n+1/2}_{j+1/2} \tag{5.13}$$

is introduced. Note that $\zeta \approx z\Delta t/2$. Using (13) in (11) leads to

$$\left[\delta q + (\ell\rho^{o})^2|z|\left(z\delta V^{\cdot} + 2\Delta_{\cdot}\delta u/(\rho^{o}\Delta x)\right)\right]^{n+1/2}_{j+1/2} = 0 , \tag{5.14}$$

where $\ell = c_2\Delta x$. Again following [4], p. 321, the notation

$$\mu = (V^{o})^2[(\gamma p^{\cdot} + \Gamma q)/V^{\cdot}]^{n+1/2}_{j+1/2} \tag{5.15}$$

is used. Note that

$$\mu = (V^o)^2[(a^*)^2 + \Gamma p^* q]_{j+1/2}^{n+1/2}$$

where $\rho^* = 1/V'$ and $(a^*)^2 = \gamma p' \rho^*$. From (6), (13), (15) follows

$$\Delta' p = 2\mu\zeta V'(\rho^o)^2 \ . \tag{5.16}$$

Then using (13), (15), (16), in (12) yields

$$[\mu(\rho^o)^2(\Delta'\delta V + 2\zeta\delta V') - 2\Gamma\zeta\delta q + \Delta'\delta p - 2\nu\zeta\delta p']_{j+1/2}^{n+1/2} = 0 \ . \tag{5.17}$$

Using the ζ-definition in (11) yields

$$\left[\Delta t\,\delta q + 2|z|(\ell\rho^o)^2(\zeta\delta V' + r\Delta_{\cdot}\delta u)\right]_{j+1/2}^{n+1/2} = 0 \ . \tag{5.18}$$

Now consider the system of equations (9), (10), (18), and (17). Let $\xi = \exp(ik\Delta x)$ and for $\nu = n, n+1$ let

$$\delta u_j^{\nu-1/2} = \delta u^\nu \xi^j \ , \quad \delta q_{j+1/2}^{\nu-1/2} = \delta q^\nu \xi^{j+1/2} \ , \quad \delta V_{j+1/2}^\nu = \delta V^\nu \xi^{j+1/2} \ , \text{ and}$$

$$\delta p_{j+1/2}^\nu = \delta p^\nu \xi^{j+1/2} \ .$$

Also let $\beta = k\Delta x/2$ and $b = 2r \sin \beta$. Then let $\delta \underset{\sim}{U}^\nu = (\delta u^\nu, \delta V^\nu, \delta q^\nu, \delta p^\nu)^T$ and replace $\delta \underset{\sim}{U}^{n+1}$ by $\lambda \delta \underset{\sim}{U}^n$ to get

$$\underset{\sim}{J}(\lambda)\delta \underset{\sim}{U}^n = 0$$

where $\underset{\sim}{J}(\lambda)$ is given by:

$$\underset{\sim}{J}(\lambda) = \begin{bmatrix} \lambda-1 & 0 & bi & bi \\ -bi\lambda & \lambda-1 & 0 & 0 \\ J_{31} & J_{32} & \lambda\Delta t & 0 \\ 0 & J_{42} & -2\Gamma\lambda\zeta & J_{44} \end{bmatrix} \tag{5.19}$$

where

$$J_{31} = 2bi\lambda|z|(\ell\rho^{o})^{2} ,$$

$$J_{32} = \zeta|z|(\lambda + 1)(\ell\rho^{o})^{2} ,$$

$$J_{42} = \mu(\rho^{o})^{2}[-1 + \zeta + \lambda(1 + \zeta)] ,$$

and

$$J_{44} = -1 - \gamma\zeta + \lambda(1 - \gamma\zeta) .$$

Observe that

$$\det \underset{\sim}{J}(\lambda) = \lambda\Delta t\, J_{44}(\lambda - 1)^{2}$$

$$+ \lambda b^{2}\left\{\lambda\Delta t\, J_{42} + (J_{44} + 2\Gamma\lambda\zeta)(J_{32} + 2|z|(\lambda - 1)(\ell\rho^{o})^{2})\right\} . \tag{5.20}$$

Let

$$P_{c}(\lambda) = \det \underset{\sim}{J}(\lambda)/(\lambda\Delta t)$$

and

$$M_{2} = (b\rho^{o})^{2}/\Delta t ,$$

to get

$$P_{c}(\lambda) = (\lambda - 1)^{2}[-1 - \gamma\zeta + \lambda(1 - \gamma\zeta)] + M_{2}\left\{\mu\lambda\Delta[-1 + \zeta + \lambda(1 + \zeta)]\right.$$

$$\left. + \ell^{2}|z|[-2 + \zeta + \lambda(2 + \zeta)][-1 - \gamma\zeta + \lambda(1 + \zeta(\Gamma - 1))]\right\} . \tag{5.21}$$

Let

$$M_3 = 8(\ell/\Delta x)^2 \sin^2 \beta \, \mathrm{sign}(\zeta)$$

and

$$M_4 = 16\Gamma(\ell/\Delta x)^2 \sin^2 \beta \, \mathrm{sign}(\zeta)$$

to get

$$P_c(\lambda) = (1 - \lambda)^2[-1 - \nu\zeta + \lambda(1 - \nu\zeta)] + \lambda[M_4\zeta^2 + (a^*b)^2][-1 + \zeta + \lambda(1 + \zeta)]$$

$$+ \zeta M_3[-2 + \zeta + \lambda(2 + \zeta)][-1 - \nu\zeta + \lambda(1 + \zeta(\Gamma - 1))] \, . \qquad (5.22)$$

Then expand $P_c(\lambda)$ into the standard form

$$P_c(\lambda) = A_o + \lambda A_1 + A_2\lambda^2 + A_3\lambda^3$$

where for $i = 0,1,2,3$

$$A_i = A_{io} + \zeta A_{i1} + \zeta^2 A_{i2} + \zeta^3 A_{i3}$$

with

$$A_{oo} = -1 \, , \; A_{o1} = 2M_3 - \gamma \, , \; A_{o2} = M_3(2\gamma - 1) \, , \; A_{o3} = -\gamma M_3 \, ,$$

$$A_{10} = 3 - (a^*b)^2 \, , \; A_{11} = \gamma - 4M_3 + (a^*b)^2 \, , \; A_{12} = -M_4 - 2\Gamma M_3 \, ,$$

$$A_{13} = M_4 - 2M_3 \, , \; A_{20} = -3 + (a^*b)^2 \, , \; A_{21} = \nu + 2M_3 + (a^*b)^2 \, ,$$

$$A_{22} = M_4 + M_3(2\Gamma - 1) \, , \; A_{23} = M_4 + M_3(\Gamma - 1) \, ,$$

$$A_{30} = 1 \, , \; A_{31} = -\gamma \, , \; A_{32} = 0 \, , \text{ and } A_{33} = 0 \, .$$

Let the roots of $P_c(\lambda)$ be denoted $\lambda_1(\zeta)$, $\lambda_2(\zeta)$, and $\lambda_3(\zeta)$. First, we determine the value of the roots at $\zeta = 0$. If $\zeta = 0$, then

$$P_c(\lambda)\Big|_{\zeta=0} = -1 + \lambda[3 - (a^*b)^2] + \lambda^2[-3 + (a^*b)^2] + \lambda^3 .$$

One root is unity so let $\lambda_1(0) = 1$. The other two roots satisfy the quadratic equation

$$1 - \lambda(2 - (a^*b)^2) + \lambda^2 = 0 .$$

If $a^*r \le 1$ (the CFL inequality) then the roots $\lambda_2(0)$ and $\lambda_3(0)$ are complex conjugates with magnitude unity. Henceforth, it is assumed that $a^*r \le 1$. Therefore let

$$\lambda_2(0) = \bar{\lambda}_3(0) = e^{i\theta} \text{ where } 2\cos\theta = 2 - (a^*b)^2 .$$

Next, we assume that

$$\lambda_i(\zeta) = a_{i0} + a_{i1}\zeta + O(\zeta^2)$$

where $a_{10} = 1$ and $a_{20} = \bar{a}_{30} = e^{i\theta}$. See Section 6 for a discussion of the existence of these expansions.

Consider

$$P_c(\lambda) = (1 - \gamma\zeta)(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= (1 - \gamma\zeta)[\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2)\lambda - \lambda_1\lambda_2\lambda_3]$$

and note that

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{10} + a_{20} + a_{30} + \zeta(a_{11} + a_{21} + a_{31}) + O(\zeta^2)$$

$$\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2 = a_{20}a_{30} + a_{10}a_{30} + a_{10}a_{20} + \zeta[a_{11}(a_{20} + a_{30})$$

$$+ a_{21}(a_{10} + a_{30}) + a_{31}(a_{10} + a_{20})] + O(\zeta^2)$$

and

$$\lambda_1\lambda_2\lambda_3 = a_{10}a_{20}a_{30} + \zeta(a_{11}a_{20}a_{30} + a_{10}a_{21}a_{30} + a_{10}a_{20}a_{31}) + O(\zeta^2)$$

Matching λ^2 coefficients yields:

$$(-1 + \gamma\zeta)[a_{10} + a_{20} + a_{30} + \zeta(a_{11} + a_{21} + a_{31})] = -3 + (a^*b)^2$$

$$+ \zeta[\gamma + 2M_3 + (a^*b)^2] + O(\zeta^2)$$

and this leads to

$$a_{11} + a_{21} + a_{31} = K_1 ,$$

where

$$K_1 = 2(\gamma - M_3) - (1 + \gamma)(a^*b)^2 .$$

Matching λ^1 coefficients yields:

$$(1 - \gamma\zeta)\Big\{a_{20}a_{30} + a_{10}a_{30} + a_{10}a_{20} + \zeta[a_{11}(a_{20} + a_{30}) + a_{21}(a_{10} + a_{30})$$

$$+ a_{31}(a_{10} + a_{20})]\Big\} = 3 - (a^*b)^2 + \zeta[\gamma - 4M_3 + (a^*b)^2] + O(\zeta^2) ,$$

and this leads to

$$a_{11}(a_{20} + a_{30}) + a_{21}(a_{10} + a_{30}) + a_{31}(a_{10} + a_{20}) = K_2 ,$$

where

$$K_2 = 4(\gamma - M_3) + (1 - \gamma)(a^*b)^2 .$$

Matching λ^0 coefficients yields:

$$(-1 + \nu\zeta)[a_{10}a_{20}a_{30} + \zeta(a_{11}a_{20}a_{30} + a_{10}a_{21}a_{30} + a_{10}a_{20}a_{31})]$$

$$= -1 + \zeta(-\nu + 2M_3) + O(\zeta^2)$$

which leads to

$$a_{11}(a_{20}a_{30}) + a_{21}(a_{10}a_{30}) + a_{31}(a_{10}a_{20}) = K_3$$

where

$$K_3 = 2(\nu - M_3) \ .$$

There are three equations for a_{11}, a_{21}, a_{31} and the determinant of that system of equations is seen to be

$$a_{20}(a_{10}^2 - a_{30}^2) + a_{30}(a_{20}^2 - a_{10}^2) + a_{10}(a_{30}^2 - a_{20}^2) \ .$$

Therefore if the roots coalesce, then the determinant goes to zero. This can happen when $(a*b)^2 \to 0$. This is discussed further in the appendix. For now let's proceed under the assumption that $(a*b)^2 > 0$. Solving for a_{11} yields $a_{11} = -2$. Therefore

$$\lambda_1 = 1 - 2\zeta + O(\zeta^2) \ .$$

Therefore in compression ($\zeta > 0$)

$$|\lambda_1| \le 1 + O(\zeta^2)$$

provided $\zeta \le 1$. However, in expansion ($\zeta < 0$)

$$|\lambda_1| = 1 + 2|\zeta| + O(\zeta^2)$$

and this suggests that a time step restriction of the form

$$|\zeta| \le \delta \tag{5.23}$$

with ζ fairly small should be imposed. That is, $\zeta \le \min(.25, 1/\gamma)$ or so. Next note that

$$(\lambda - a_{20} - a_{21}\zeta)(\lambda - a_{30} - a_{31}\zeta) = \lambda^2[a_{20} + a_{30} + \zeta(a_{21} + a_{31})]\lambda + (a_{20} + a_{21}\zeta)(a_{30} + a_{31}\zeta) ,$$

$$a_{20} + a_{30} + \zeta(a_{21} + a_{31}) = 2 - (a^*b)^2 + \zeta[2 + 2\gamma - 2M_3 - (1 + \gamma)(a^*b)^2] ,$$

and

$$(a_{20} + a_{21}\zeta)(a_{30} + a_{31}\zeta) = 1 + \zeta[2 + 2\gamma - 2M_3] + O(\zeta^2) .$$

Note that matching coefficients with $A_1\lambda^2 - 2B_1\lambda + C_1$ leads to

$$A_1 = 1 ,$$

$$2B_1 = 2 - (a^*b)^2 - \zeta[-2 - 2\gamma + 2M_3 + (1 + \gamma)(a^*b)^2] ,$$

and

$$C_1 = 1 - \zeta(-2 - 2\gamma + 2M_3) + O(\zeta^2) .$$

Applying Lemma 1B with $A_1 = 1 - a_1$, $B_1 = 1 - b_1$, $C_1 = 1 - c_1$ yields: In Case (B_+) the condition $a_1 + c_1 \le 2b_1$ leads to

$$O(\zeta^2) \le \zeta(1 + \gamma)(a^*b)^2 + (a^*b)^2 .$$

Now, $b = 2r \sin \theta$, and $\sin \theta = 1$ for the highest frequency, so this leads us to the constraint

$$O(\zeta^2) \le \zeta(1 + \gamma)4(a^*r)^2 + 4(a^*r)^2 ,$$

which leads to

$$O(\zeta^2) - \zeta(1 + \gamma) \le 1 .$$

This last inequality is always satisfied (modulo (ζ^2)) in compression ($\zeta > 0$), but in expansion ($\zeta < 0$) it imposes the constraint

$$O(\zeta^2) + |\zeta| \le 1/(1 + \gamma) . \tag{5.24}$$

In Case (c) $|\lambda_2|^2 = |\lambda_3|^2 = C_1 + O(\zeta^2)$ so requiring that $|\lambda_2|^2$ and $|\lambda_3|^2 \le 1$ leads to

$$C_1 + O(\zeta^2) \le 1$$

and this is equivalent to

$$\zeta(2 + 2\gamma) + O(\zeta^2) \le |\zeta| 8(\ell/\Delta x)^2 \sin^2 \beta .$$

Again since we are mainly interested in constraining the growth factors for the highest frequency component ($\sin \beta = 1$), we set $\sin \beta = 1$ to get the constraint

$$\zeta(2 + 2\gamma) + O(\zeta^2) \le |\zeta| 8(\ell/\Delta x)^2$$

which leads to

$$\mathrm{sign}(\zeta)(2 + 2\gamma) + O(\zeta) \le 8c_2^2 .$$

For expansion ($\zeta < 0$) this is always satisfied (modulo $O(\zeta)$). However, for compression ($\zeta > 0$) we have

$$2 + 2\gamma + O(\zeta) \le 8c_2^2$$

and thus a constraint on c_2^2 arises

$$O(\zeta) + \frac{1 + \gamma}{4} \le c_2^2 . \tag{5.25}$$

Note that for $c_2^2 = 1$ the previous inequality holds (modulo $O(\zeta)$) for $\gamma \le 3$. In Case (B_-) the condition

$$a_1 + 2b_1 + c_1 \le 4$$

leads to

$$(a^*r)^2[1 + \zeta(1 + \gamma)] + 8|\zeta|(\ell/\Delta x)^2 \le 1 + \zeta(1 + \gamma) \ . \qquad (5.26)$$

Compare this with White's inequality which is

$$(ar)^2 + 8\zeta(\ell/\Delta x)^2 \le 1 \ .$$

<u>Remark</u>: Note that if $a^* = a$ and $ar \le 1$, then White's inequality implies (26). In the ideal gas law case the difference between a^* and a at $t^{n+1/2}$ is as follows

$$(a^*)^{n+1/2} = \gamma \frac{(P^{n+1} + P^n)}{(V^{n+1} + V^n)}$$

and

$$a^{n+1/2} = \gamma\left(\frac{P^{n+1}}{V^{n+1}} + \frac{P^n}{V^n}\right) \ .$$

Therefore, if P^n is close to P^{n+1} and V^n is close to V^{n+1}, then a^* and a are close.

6. APPENDIX TO SECTION 5: ON THE EXISTENCE OF THE EXPANSIONS OF THE EIGENVALUES

This appendix presents proof of the existence of expansions of the eigenvalues in the form

$$\lambda_i(\zeta) = a_{io} + a_{i1}\zeta + O(\zeta^2) \ . \tag{6.1}$$

The proof is divided into two cases:

Case (1) has $a^*b = 0$; Case (2) has $a^*b \neq 0$.

Case (2) is divided into three subcases: Case (2a) has $\zeta = 0$; Case (2b) has $\zeta > 0$; Case (2c) has $\zeta < 0$.

Consider Case (1).

$$P_c(\lambda)\Big|_{a^*b=0} = (1 - \lambda)^2(-1 - \gamma\zeta + \lambda(1 - \gamma\zeta)) \ .$$

The roots are $\lambda = 1$ (twice) and

$$\lambda = (1 + \gamma\zeta)/(1 - \gamma\zeta) \ .$$

If $-1/\gamma \leq \zeta \leq 0$ ($\zeta < 0$ is expansion) then

$$|\lambda| \leq 1 \ .$$

If $0 < \zeta < 1/\gamma$ ($\zeta > 0$ is compression) then

$$|\lambda| \approx 1 + 2\gamma|\zeta| + O(\zeta^2) \ .$$

This suggests that for computational stability a time step restriction $|\zeta| \leq \delta$ should be enforced with δ small and certainly $\delta < 1/\gamma$.

Consider subcase (2a).

$$P_c(\lambda)\Big|_{\zeta=0} = -1 + \lambda[3 - (a^*b)^2] + \lambda^2[-3 + (a^*b)^2] + \lambda^3 .$$

One root is unity so let $\lambda_1(0) = 1$. The other two roots $(\lambda_2(0), \lambda_3(0))$ satisfy the quadratic equation

$$1 - \lambda[2 - (a^*b)^2] + \lambda^2 = 0 .$$

Recall that $b = 2r \sin \beta$. If $a^*r \le 1$, then the roots are complex conjugates with magnitude unity. Henceforth, assume that $a^*r \le 1$. Therefore let

$$\lambda_2(0) = e^{i\theta} = \overline{\lambda}_3(0)$$

where

$$2 \cos \theta = 2 - (a^*b)^2 .$$

Thus $a_{10} = 1$, $a_{20} = e^{i\theta}$, and $a_{30} = e^{-i\theta}$.

<u>Remark</u>: The reason for dividing into Cases (2b) and (2c) is that the coefficients of P_c depend on $|\zeta|$--the two-sided derivatives w.r.t. ζ at $\zeta = 0$ are nonexistent. But if restricted to the case $\zeta > 0$ (or $\zeta < 0$), then the coefficients are just polynomials in ζ. The case $\zeta < 0$ can be treated in a manner parallel to the treatment of $\zeta > 0$ so only the details of the $\zeta > 0$ case are presented here.

Consider subcase (2b). First verify that if $a_{10} = 1$, $a_{20} = e^{i\theta}$, $a_{30} = e^{-i\theta}$, $a_{11} = -2$, and if a_{21}, a_{31} satisfy

$$a_{21} + a_{31} = K_1 + 2$$

$$a_{21}\, e^{-i\theta} + a_{31}\, e^{i\theta} = K_3 + 2$$

where $K_3 = 2(\gamma - M_3)$ and $K_1 = K_3 - (1 + \gamma)(a^*b)^2$ then

$$P_c(\lambda) = (1 - \gamma\zeta)\prod_{i=1}^{3} (\lambda - a_{io} - a_{i1}\zeta) + O(\zeta^2) .$$

It follows that

$$P_c(a_{io} + a_{i1}\zeta) = O(\zeta^2) .$$

The next thing to show is that there exists $\Delta\lambda_i$ such that $P_c(a_{io} + a_{i1}\zeta + \Delta\lambda_i) = 0$ and $\Delta\lambda_i = O(\zeta^2)$. Use Newton's method. Let $\lambda^o = a_{io} + a_{i1}\zeta$ and $\lambda^{m+1} = \lambda^m - P_c(\lambda^m)/P_c'(\lambda^m)$. The basic idea of the proof is an induction on m as follows: If $P_c'(\lambda^m) \neq 0$, $P_c'(\lambda^m) = O(\zeta^o)$, $P_c(\lambda^m) = O(\zeta^2)$, and $\lambda^m = \lambda^o + O(\zeta^2)$ is true for m then it is true for m+1. One can see that this can be proven provided ζ is small enough, $P_c'(\lambda^o) \neq 0$, $P_c'(\lambda^o) = O(\zeta^o)$, and provided the Newton's iteration converges.

To prove the convergence of the Newton iteration one may use the following Theorem N (see p. 168 of [13]). Let $f(x^*) = 0$ and let f' be continuous and invertible in a neighborhood of x^o. Suppose $|f'(x^o)| > 1/\eta$ and $|f(x^o)| < |x^* - x^o|/(2\eta)$. Then Newton's method started at x^o converges to a number x^{**} such that $f(x^{**}) = 0$ and $|x^{**} - x^o| \leq |x^* - x^o|$.

$P_c(\lambda^o)$ can be made sufficiently small by choosing ζ small enough because $P_c(\lambda^o) = O(\zeta^2)$. To use Theorem N it must be established that $P_c'(\lambda)$ is invertible at $\lambda = \lambda^o$. To do this one may use the Inverse Function Theorem (see [14], p. 144) which says that if g in C^1 and $g'(x^o) \neq 0$ then there is a neighborhood of x^o such that g^{-1} exists and is C^1.

Thus what is required is that

$$P_c'(\lambda^o) \neq 0 \neq P_c''(\lambda^o) \quad \text{and} \quad P_c'(\lambda^o) = O(\zeta^o) .$$

Note that

$$P_c'(\lambda) = (1 - \gamma\zeta) \sum_{i=1}^{3} (\lambda - a_{io} - a_{i1}\zeta)(\lambda - a_{i+1,o} - a_{i+1,1}\zeta) + O(\zeta^2)$$

Therefore

$$P_c'(a_{io} + a_{i1}\zeta) = (a_{io} - a_{i+1,o})(a_{io} - a_{i+2,o}) + O(\zeta)$$

and since $a^*b \neq 0$ it follows that

$$P_c'(\lambda^o) \neq 0 \quad \text{and} \quad P_c'(\lambda^o) = O(\zeta^o) \ .$$

Also note that

$$P_c''(\lambda) = 2(1 - \gamma\zeta) \sum_{i=1}^{3} (\lambda - a_{io} - a_{i1}\zeta) + O(\zeta^2)$$

Therefore

$$P_c''(a_{io} + a_{i1}\zeta) = 2[2a_{io} - a_{i+1,o} - a_{i+2,o}] + O(\zeta)$$

and it follows that

$$P_c''(\lambda^o) \neq 0 \quad \text{and} \quad P_c''(\lambda^o) = O(\zeta^o)$$

since $a^*b \neq 0$. This completes the sketch of the proof that the eigenvalues possess expansions of the form (1) provided ζ is small enough.

7. SUMMARY

The eigenvalues of the amplification matrix of the local equations of first variation of the vN-R scheme for the case of the ideal gas law are of the form (for i = 1,2,3,4)

$$\lambda_i = a_{io} + a_{i1}\zeta + O(\zeta^2)$$

where

$$\zeta = \left(V^n_{j+1/2} - V^{n+1}_{j+1/2}\right)/\left(V^n_{j+1/2} + V^{n+1}_{j+1/2}\right) .$$

In order that the volume remain positive, the time step restriction $\zeta < 1$ is required--for the vN-R scheme to be solvable $\zeta < 1/\gamma$ is required--for the suppression of noise oscillations of period $2\Delta x$ the inequality $-1/(1 + \gamma) < \zeta$ should be enforced (note that this last inequality is a constraint only in expansion where $\zeta < 0$). In general, the computational stability analysis in Sections 5 and 6 suggests that

$$|\zeta| \leq \delta \tag{7.1}$$

should be enforced with δ small. How small may be determined by numerical experimentation--experience suggests $\delta \leq \min(1/4, 1/(1 + \gamma))$.

The computational stability analysis in Section 5 also leads to the following constraint

$$(a^*r)^2[1 + \zeta(1 + \gamma)] + 8|\zeta|(\ell/\Delta x)^2 \leq 1 + \zeta(1 + \gamma) \tag{7.2}$$

where

$$[a^*]^{n+1/2}_{j+1/2} = \gamma[P'/V']^{n+1/2}_{j+1/2}$$

is a certain approximation to the acoustic impedance at $t^{n+1/2}$ and $x_{j+1/2}$ and where $r = \Delta t/(\rho^{o}\Delta x)$. It was also found that

$$(1 + \gamma)/4 \leq (\ell/\Delta x)^2 \tag{7.3}$$

should be enforced to suppress noise oscillations of period $2\Delta x$ when in compression (when $\zeta > 0$).

The artificial viscosity in WONDY is of the form

$$q = -\Lambda\Delta_{\cdot}u$$

where

$$\Lambda = ac_1 + c_2^2|\Delta_{\cdot}u|/V^{\cdot} \tag{7.4}$$

with c_1 and c_2 dimensionless constants--$c_1 \approx 0.1$, $c_2 \approx 1.0$. The ℓ and c_2 are related by $\ell = c_2\Delta x$. The stability inequality in WONDY at this time is

$$(ar)^2 + 2\Lambda r \leq \theta < 1 \tag{7.5}$$

where $\theta \approx 0.9$. Equation (7.4) may be written $\Lambda = ac_1r + 2c_2^2|\zeta|$ therefore (7.5) may be written

$$(ar)^2 + 2ac_1r + 4c_2^2|\zeta| \leq \theta \ . \tag{7.6}$$

Consideration of (7.2) and the stability studies mentioned in Section 4 suggest that WONDY should be enforcing

$$(a^*r)^2[1 + \zeta(1 + \gamma)] + 2a^*c_1r + 8c_2^2|\zeta| \leq \theta[1 + \zeta(1 + \gamma)] \tag{7.7}$$

along with (7.1) and (7.3) instead of (7.6) for the case of the ideal gas law with artificial viscosity of the form

$$\Lambda = a^*c_1 + c_2^2|\Delta_{\cdot}u|/V^{\cdot} \ .$$

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