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OCEANOGRAPHIC FLOW

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TOWARD AN EXTENDED-GEOSTROPHIC EULER–POINCARÉ MODEL FOR MESOSCALE OCEANOGRAPHIC FLOW

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Abstract

We consider the motion of a rotating, continuously stratified fluid governed by the hydrostatic primitive equations (PE). An approximate Hamiltonian (L1) model for small Rossby number ϵ is derived for application to mesoscale oceanographic flow problems. Numerical experiments involving a baroclinically unstable oceanic jet are utilized to assess the accuracy of the L1 model compared to the PE and to other approximate models, such as the quasigeostrophic (QG) and the geostrophic momentum (GM) equations. The results of the numerical experiments for moderate Rossby number flow show that the L1 model gives accurate solutions with errors substantially smaller than QG or GM.

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1. Introduction

We continue the study of intermediate models (McWilliams and Gent, 1980), derived under the assumption that the Rossby number ϵ is small, for possible application to mesoscale oceanographic flow fields. Previous work has involved approximate models for flows of homogeneous fluids governed by the f-plane shallow water equations (SWE) (Allen et al., 1990a,b; Barth et al., 1990; Allen and Holm, 1996) and for flows of continuously stratified fluids governed by the hydrostatic primitive equations (PE) (Allen, 1991; Allen, 1993; Allen and Newberger, 1993; Holm, 1996).

We use a traditional modeling approach of making approximations in Hamilton's principle. (For example, the finite-element method is based on such an approach.) This approach was developed for GFD and applied in Salmon (1983, 1985, 1996) to construct balanced GFD equations, by substituting leading order balance relations and asymptotic expansions into Hamilton's principle before taking variations. See also Allen and Holm, 1996, and Holm, 1996. In the present paper, we again use this approach in accomplishing our first objective – to derive an approximate model for mesoscale oceanographic flow. For this, we work in the framework of the Euler–Poincaré theorem for ideal continua with advected parameters (Holm, Marsden and Ratiu, 1997). Euler–Poincaré systems are the Lagrangian analogue of Lie-Poisson Hamiltonian systems (Holm, Marsden, Ratiu, and Weinstein, 1985, and references therein). In this framework, the resulting Eulerian approximate GFD equations possess a Kelvin-Noether circulation theorem, conserve potential vorticity on fluid particles and conserve volume integrated energy. A further objective is to assess the accuracy of the resulting model equations through numerical experiments.

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Motivation for this study is provided by the seemingly great potential usefulness of approximate models derived from Hamilton's principle and the apparent soundness of this approach. On the other hand, results of numerical experiments assessing the accuracy of different intermediate models applied to the SWE (Allen et al., 1990a,b; Barth et al., 1990) demonstrate clearly that, at moderate values of ϵ, Salmon's (1983) HP model and the geostrophic momentum (GM) approximation (Hoskins, 1975) provide disappointingly inaccurate solutions to the SWE, compared e.g., to those obtained from the balance equations (BE) (Gent and McWilliams, 1983). This is in spite of the fact that the HP and GM models have Hamiltonian structure, whereas the BE for the SWE do not conserve energy. Thus, possession of Hamiltonian structure is not sufficient in itself to ensure an accurate approximate model. The question arises of how more accurate approximate models can be derived from Hamilton's principle. Allen and Holm (1996) formulated an approximate model by extending Salmon's (1983) approach and utilizing higher order approximations in Hamilton's principle. Allen and Holm (1996) contended that this extended-geostrophic model should give more accurate solutions than the HP or GM models.

Here we take an initial step toward applying the expansion procedure of Allen and Holm (1996) to derive an extended-geostrophic model for continuously stratified flows governed by the PE. The initial step involves derivation of an approximate model following a strategy similar to that of Salmon (1983, 1985, 1996), but utilizing the methods of Holm, Marsden, and Ratiu (1997) for deriving the Euler-Poincaré equations for fluids. We refer to the resulting approximate equations as the L1 model.

The derivation is followed by numerical experiments to assess the L1 model accuracy compared to the PE. The idealized, mesoscale oceanographic problems utilized in Allen and Newberger (1993) to quantify the accuracy of different intermediate models are repeated. Thus, we find information not only on the absolute accuracy of the L1 model compared to PE solutions, but also on the relative accuracy compared to other intermediate models.

The outline of this paper is as follows. The Euler-Poincaré equations for fluids are summarized briefly in section 2 and the derivation of the L1 model equations is given in section 3. The solution procedure for the L1 model and the numerical experiments are presented in sections 4 and 5, respectively, with details of the numerical methods explained in the appendix. Brief summary comments are given in section 6.

2. The Euler–Poincaré equations for fluids

The Euler–Poincaré equations for fluids with advected parameters are derived in Holm, Marsden and Ratiu (1997). In abstract form, these equations are

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \mathbf{u}} = -\operatorname{ad}^*_{\mathbf{u}}\frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta a}\diamond a = -\mathcal{L}_{\mathbf{u}}\frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta a}\diamond a\,, \qquad \frac{\partial}{\partial t}a = -\mathcal{L}_{\mathbf{u}}\,a\,, \qquad (2.1a)$$

where l is the Lagrangian in the variational principle in Eulerian coordinates

$$\delta \int_{t_1}^{t_2} l\,(\mathbf{u}, a)\;dt = 0\,, \qquad (2.1b)$$

which holds on variables $(\mathbf{u}, a) \in \mathcal{X}(\mathcal{D}) \times V^*$, using *constrained* variations of the form

$$\delta \mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{u}, \mathbf{w}] = \frac{\partial \mathbf{w}}{\partial t} - \operatorname{ad}_{\mathbf{u}}\mathbf{w}\,, \qquad \delta a = -\mathcal{L}_{\mathbf{w}}\,a\,, \qquad (2.2)$$

in which $[\mathbf{u}, \mathbf{w}] = -\operatorname{ad}_{\mathbf{u}}\mathbf{w}$ is the Lie bracket between vector fields $\mathbf{u}, \mathbf{w} \in \mathcal{X}(\mathcal{D})$ defined on the domain $\mathcal{D}$, and $\mathbf{w}$ vanishes at the endpoints in time.

In equation (2.1a), $\mathcal{L}_{\mathbf{u}}$ denotes Lie-derivative with respect to the vector field $\mathbf{u}$ (the Eulerian fluid velocity), the variational derivative $\delta l/\delta \mathbf{u} \in \mathcal{X}(\mathcal{D})^*$ is a one form density, and the tensor field $a \in V^*$ is advected by $\mathbf{u}$. The operation $\diamond$ between elements of V and its dual space V^* producing an element of $\mathcal{X}(\mathcal{D})^*$ is defined in terms of the Lie derivative as

$$\langle v \diamond a, \mathbf{w}\rangle = -\int_{\mathcal{D}} v \bullet \mathcal{L}_{\mathbf{w}}\,a\,, \qquad (2.3)$$

where $v \bullet \mathcal{L}_{\mathbf{w}}\,a$ denotes the contraction between elements of V and elements of V^*. On a general manifold, tensors of a given type have natural duals. (For example, k–forms are naturally dual to $(n-k)$–forms, the pairing being given by taking the integral of their wedge

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product.) The $\diamond$ operation needs to be determined on a case by case basis, depending on the nature of the tensor field a.

Among other cases, Holm, Marsden and Ratiu, 1997, compute explicit formulas for the variations δa when the set of tensor fields a consists of scalar functions and densities in a Euclidean basis on $\mathcal{R}^3$, namely,

$$a \in \{\rho,\, D\, d^3 x\}\,. \tag{2.4}$$

Invariance of the advected quantities in the set a in the Lagrangian picture under the dynamics of $\mathbf{u}$ implies in the Eulerian picture that $\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) a = 0$, where $\mathcal{L}_{\mathbf{u}}$ denotes Lie derivative with respect to the velocity vector field $\mathbf{u}$. Hence, for a fluid dynamical action $L = \int dt\, l(\mathbf{u}; \rho, D)$, the advected variables $\{\rho, D\}$ satisfy the following Lie-derivative relations,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \rho = 0, \quad \text{or} \quad \frac{\partial \rho}{\partial t} = -\,\mathbf{u} \cdot \nabla \rho\,, \tag{2.5a}$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) D\, d^3 x = 0, \quad \text{or} \quad \frac{\partial D}{\partial t} = -\,\nabla \cdot (D\mathbf{u})\,. \tag{2.5b}$$

In oceanographic applications, the advected Eulerian variables ρ and D represent the buoyancy ρ and volume element D, respectively. According to equation (2.1c), the variations of the tensor functions a at fixed $\mathbf{x}$ and t are also given by Lie derivatives, namely $\delta a = -\mathcal{L}_{\mathbf{w}}\, a$, or

$$\delta \rho = -\mathcal{L}_{\mathbf{w}}\, \rho = -\mathbf{w} \cdot \nabla \rho\,,$$

$$\delta D\, d^3 x = -\mathcal{L}_{\mathbf{w}}\, (D\, d^3 x) = -\nabla \cdot (D\mathbf{w})\, d^3 x\,. \tag{2.6}$$

Hence, Hamilton's principle (2.1b) with this dependence yields

$$
\begin{aligned}
0 = \delta \int dt \; l(\mathbf{u}; \rho, D) \\
= \int dt \left[\frac{\delta l}{\delta \mathbf{u}} \cdot \delta \mathbf{u} + \frac{\delta l}{\delta \rho} \, \delta \rho + \frac{\delta l}{\delta D} \, \delta D \right] \\
= \int dt \left[\frac{\delta l}{\delta \mathbf{u}} \cdot \left(\frac{\partial \mathbf{w}}{\partial t} - \mathrm{ad}_{\mathbf{w}} \, \mathbf{u} \right) - \frac{\delta l}{\delta \rho} \, \mathbf{w} \cdot \nabla \rho - \frac{\delta l}{\delta D} \left(\nabla \cdot (D\mathbf{w}) \right) \right] \\
= \int dt \; \mathbf{w} \cdot \left[- \frac{\partial}{\partial t} \frac{\delta l}{\delta \mathbf{u}} - \mathrm{ad}_{\mathbf{u}}^* \frac{\delta l}{\delta \mathbf{u}} - \frac{\delta l}{\delta \rho} \, \nabla \rho + D \, \nabla \frac{\delta l}{\delta D} \right] \\
= - \int dt \; \mathbf{w} \cdot \left[\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) \frac{\delta l}{\delta \mathbf{u}} + \frac{\delta l}{\delta \rho} \, \nabla \rho - D \, \nabla \frac{\delta l}{\delta D} \right],
\end{aligned}
\tag{2.7}
$$

where we have consistently dropped boundary terms arising from integrations by parts, by invoking natural boundary conditions. Specifically, we impose $\hat{\mathbf{n}} \cdot \mathbf{u} = 0$ on the boundary, where $\hat{\mathbf{n}}$ is the boundary's outward unit normal vector.

The Euler-Poincaré equations for continua (2.1a) may now be summarized for advected Eulerian variables a in the set (2.4). We adopt the notational convention that a one form density can be made into a one form (no longer a density) by dividing it by the volume element D and using the Lie-derivative relation for the continuity equation $(\partial/\partial t + \mathcal{L}_{\mathbf{u}})Dd^3x = 0$. Then, the Euclidean components of the Euler-Poincaré equations for continua in equation (2.1) are expressed in the following Kelvin theorem form with a slight abuse of notation as

$$
\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) \left(\frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) + \frac{1}{D} \frac{\delta l}{\delta \rho} \nabla \rho \cdot d\mathbf{x} - \nabla \left(\frac{\delta l}{\delta D} \right) \cdot d\mathbf{x} = 0,
\tag{2.8}
$$

where the variational derivatives of the Lagrangian l are to be computed according to the usual physical conventions, i.e., as Frechét derivatives. Formula (2.8) is the Kelvin-Noether form of the equation of motion for ideal continua. Hence, we have the explicit

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Kelvin theorem expression,

$$\frac{d}{dt} \oint_{\gamma_t} \frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} = - \oint_{\gamma_t} \frac{1}{D} \frac{\delta l}{\delta \rho} \nabla \rho \cdot d\mathbf{x} \,, \tag{2.9}$$

where the curve γ_t moves with the fluid velocity $\mathbf{u}$. Then, by Stokes' theorem, the Euler equations generate circulation of $(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}})$ whenever $\nabla \rho$ and $\nabla(\frac{1}{D}\frac{\delta l}{\delta \rho})$ are not collinear.

In vector notation, equation (2.8) is

$$\frac{d}{dt} \frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} + \frac{1}{D} \frac{\delta l}{\delta u^j} \nabla u^j + \frac{1}{D} \frac{\delta l}{\delta \rho} \nabla \rho - \nabla \frac{\delta l}{\delta D} = 0, \tag{2.10}$$

or, in three dimensions,

$$\frac{\partial}{\partial t} \left(\frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \right) - \mathbf{u} \times \mathrm{curl} \left(\frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \right) + \nabla \left(\mathbf{u} \cdot \frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} - \frac{\delta l}{\delta D} \right) + \frac{1}{D} \frac{\delta l}{\delta \rho} \nabla \rho = 0 \,. \tag{2.11}$$

In writing the last equation, we have used the fundamental vector identity of fluid dynamics,

$$(\mathbf{b} \cdot \nabla)\mathbf{a} + a_j \nabla b^j = -\mathbf{b} \times (\nabla \times \mathbf{a}) + \nabla(\mathbf{a} \cdot \mathbf{b}) \,, \tag{2.12}$$

for any three dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ with, in this case, $\mathbf{a} = (\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}})$.

Using advection of buoyancy ρ, one finds conservation of potential vorticity Q on fluid parcels:

$$\frac{\partial Q}{\partial t} + \mathbf{u} \cdot \nabla Q = 0 \,, \quad \text{where} \quad Q = \frac{1}{D} \nabla \rho \cdot \mathrm{curl} \left(\frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \right) \,. \tag{2.13}$$

A **partial** Legendre transformation (in the variable $\mathbf{u}$ only) is defined by writing

$$\mu = \frac{\delta l}{\delta \mathbf{u}} \,, \quad h(\mu, a) = \langle \mu, \mathbf{u} \rangle - l(\mathbf{u}, a), \tag{2.14}$$

where $\langle \mu, \mathbf{u} \rangle$ denotes the contraction between elements of $\mathcal{X}(\mathcal{D})^*$ and elements of $\mathcal{X}(\mathcal{D})$. (This is the natural integral pairing between one form densities and vector fields.) Holm,

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Marsden and Ratiu (1997) show that $h(\mu, a)$ is the Hamiltonian for the Lie-Poisson Hamiltonian formulation of ideal fluid dynamics.

The Euclidean component formulae (2.10) and (2.11) are especially convenient for direct calculations of motion equations in geophysical fluid dynamics, to which we turn our attention next.

3. Derivation of the L1 model equations

We consider the motion of a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, adiabatic, primitive equations and derive an approximate model for small Rossby number through the use of Hamilton's principle. One important objective is to assess the accuracy of the resulting model equations by obtaining numerical solutions to the same idealized, mesoscale oceanographic problems utilized in Allen and Newberger (1993) to assess the accuracy of several different intermediate models. Consequently, for simplicity we restrict the derivation here to the idealized conditions utilized for the problems in that study, i.e., to an f-plane with a rigid lid and flat bottom and with the domain periodic in the horizontal (x, y) directions.

The primitive equations (PE) in dimensionless variables are

$$\nabla_3 \cdot \mathbf{u}_3 = 0\,, \tag{3.1a}$$

$$\epsilon \frac{D\mathbf{u}}{Dt} + \hat{\mathbf{z}} \times \mathbf{u} = -\nabla p, \tag{3.1b}$$

$$0 = -p_z - \rho, \tag{3.1c}$$

$$\frac{D\rho}{Dt} = 0, \tag{3.1d}$$

where

$$\mathbf{x} = (x, y, z)\,, \tag{3.2a}$$

$$\mathbf{u}_3 = (u, v, \epsilon w)\,, \qquad \mathbf{u} = (u, v, 0)\,, \tag{3.2b,c}$$

$$\nabla_3 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)\,, \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)\,, \tag{3.2d,e}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}_3 \cdot \nabla_3 \right)\,, \tag{3.2f}$$

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and $\hat{\mathbf{z}}$ is the unit vector in the vertical z direction.

Dimensionless variables are formed using the characteristic values (L, H, U_0, f_0) for, respectively, a horizontal length scale, vertical depth scale, horizontal velocity, and Coriolis parameter. The Rossby number

$$\epsilon = U_0/f_0 L\,. \tag{3.3}$$

With dimensional variables denoted by primes, we have

$$(x, y) = (x', y')/L\,, \qquad z = z'/H\,, \tag{3.4a, b}$$

$$(u, v) = (u', v')/U_0\,, \qquad \epsilon w = w'L/(U_0 H)\,, \tag{3.4c, d}$$

$$t = t'U_0/L\,, \qquad f = f'/f_0 = 1\,, \tag{3.4e, f}$$

so that $(u, v, \epsilon w)$ are the dimensionless velocity components in the (x, y, z) directions, and t is time.

The total dimensional density is given by

$$\rho'_T = \rho_0 + \overline{\rho}'(z') - \theta'(\mathbf{x}', t')\,, \tag{3.5}$$

where ρ_0 is a constant reference density, $\overline{\rho}'(z')$ the basic undisturbed z'-dependent field, and θ' the negative of the density fluctuation. We define

$$\overline{\rho}(z) = \overline{\rho}'(z')/\rho_C\,, \qquad \theta = \theta'/\rho_C\,, \tag{3.6a, b}$$

and

$$\rho = \overline{\rho}(z) - \theta\,, \tag{3.7}$$

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where $\rho_C = p_C/(Hg)$, $p_C = \rho_0 U_0 f L$, and g is the acceleration of gravity. In addition, we write

$$\overline{p}_z = -S(z)/\epsilon\,, \qquad S(z) = N^2(z)H^2/(f_0^2 L^2)\,, \qquad (3.8a,b)$$

where

$$N^2(z) = -g\overline{\rho}'_{z'}/\rho_0\,, \qquad (3.8c)$$

is the square of the basic Brunt-Väisälä frequency. Subscripts (x, y, z, t) denote partial differentiation. Pressure variables $p, \overline{p}$ and $\tilde{p}$ are defined by nondimensionalizing with p_C such that

$$p = \overline{p}(z) + \tilde{p}\,, \qquad (3.9)$$

where

$$\overline{p}_z = -\overline{\rho}, \qquad \nabla p = \nabla \tilde{p}\,. \qquad (3.10a,b)$$

We are interested in the limit of small Rossby number,

$$\epsilon \ll 1\,, \qquad (3.11)$$

with $S = O(1)$.

Theorem (Holm, 1996). The PE are Euler–Poincaré equations, with action $L = \int dt\, l$ given in dimensionless variables by

$$L = \int dt\, dx\, dy\, dz \left\{ D\left[\mathbf{u} \cdot (\mathbf{R} + \epsilon\mathbf{u}) - \frac{\epsilon}{2}|\mathbf{u}|^2 - \rho z\right] - p(D - 1) \right\}\,, \qquad (3.12)$$

where $\mathbf{R} = \mathbf{R}(x, y), \mathbf{R} \cdot \hat{\mathbf{z}} = 0$, and

$$\mathrm{curl}_3\, \mathbf{R} = \hat{\mathbf{z}}\,. \qquad (3.13)$$

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Proof. By direct substitution into equation (2.11).

We derive approximate equations for $\epsilon \ll 1$ using the Euler–Poincaré framework, by following a procedure similar to that applied by Salmon (1983, 1985, 1996). Thus, we define

$$\mathbf{u}_1 = \hat{\mathbf{z}} \times \nabla \tilde{\phi}, \qquad (3.14a)$$

with

$$\tilde{\phi}(\mathbf{x}, t) = \phi_S(x, y, t) + \int_z^0 dz'\, \rho, \qquad \tilde{\phi}_z = -\rho, \qquad (3.14b,c)$$

where $\phi_S(x, y, t)$ is a function to be determined, and utilize the following order $O(\epsilon)$ approximation for the action L,

$$L_1 = \int dt\, dx\, dy\, dz \left\{ D\left[\mathbf{u} \cdot (\mathbf{R} + \epsilon \mathbf{u}_1) - \frac{\epsilon}{2}|\mathbf{u}_1|^2 - \rho z \right] - p(D-1) \right\}. \qquad (3.15)$$

The action L_1 has variational derivatives given by

$$\delta L_1 = \int dt\, dx\, dy\, dz \left\{ D(\mathbf{R} + \epsilon \mathbf{u}_1) \cdot \delta \mathbf{u} + \left[\mathbf{u} \cdot (\mathbf{R} + \epsilon \mathbf{u}_1) - \frac{\epsilon}{2}|\mathbf{u}_1|^2 - \rho z - p \right] \delta D \right.$$
$$\left. + \epsilon D(\mathbf{u} - \mathbf{u}_1) \cdot \delta \mathbf{u}_1 - Dz\delta\rho - (D-1)\delta p \right\}. \qquad (3.16)$$

Using the relations

$$\mathbf{a} \cdot \delta \mathbf{u}_1 = -\left(\delta\phi_S + \int_z^0 dz'\, \delta\rho\right)\hat{\mathbf{z}} \cdot \operatorname{curl} \mathbf{a} + \operatorname{div}\left[\mathbf{a} \times \hat{\mathbf{z}}\left(\delta\phi_S + \int_z^0 dz'\, \delta\rho\right)\right], \qquad (3.17)$$

and

$$\int_{-1}^0 dz\, a(z) \left(\int_z^0 dz'\, b(z') \right) = \int_{-1}^0 dz\, b(z) \left(\int_{-1}^z dz'\, a(z') \right), \qquad (3.18)$$

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in (3.16) we obtain

$$\delta L_1 = \int dt\, dx\, dy\, dz \Big\{ D(\mathbf{R} + \epsilon\mathbf{u}_1)\cdot\delta\mathbf{u} + \Big[\mathbf{u}\cdot(\mathbf{R}+\epsilon\mathbf{u}_1) - \frac{\epsilon}{2}|\mathbf{u}_1|^2 - \rho z - p\Big]\delta D$$

$$- (D-1)\delta p - \delta\rho\Big[Dz + \int_{-1}^{z} dz'\,\hat{\mathbf{z}}\cdot\operatorname{curl}\epsilon D(\mathbf{u}-\mathbf{u}_1)\Big]\Big\}$$

$$+ \int dt\, dx\, dy\, \delta\phi_S \int_{-1}^{0} dz\,\hat{\mathbf{z}}\cdot\operatorname{curl}\epsilon D(\mathbf{u}-\mathbf{u}_1)$$

$$+ \int dt\, dz \oint ds \left(\delta\phi_S + \int_{z}^{0}\delta\rho\, dz'\right)\epsilon D(\mathbf{u}-\mathbf{u}_1)\cdot\hat{\mathbf{s}}, \qquad (3.19)$$

where $\hat{\mathbf{s}} = \hat{\mathbf{z}} \times \hat{\mathbf{n}}$ is the unit tangent vector on the boundary and $\hat{\mathbf{n}}$ is the unit outward normal vector. For the periodic domain in (x, y) assumed here, the boundary integral in (3.19) vanishes.

The δp and $\delta\phi_S$ variations in (3.19) yield, respectively,

$$D = 1, \qquad (3.20)$$

and

$$\int_{-1}^{0} dz\,\hat{\mathbf{z}}\cdot\operatorname{curl}(\mathbf{u}-\mathbf{u}_1) = 0. \qquad (3.21)$$

In this nondimensional notation, the Euler–Poincaré equation for the fluid motion generated by L_1 is, cf. (2.11),

$$\frac{\partial}{\partial t}\left(\frac{1}{D}\frac{\delta L_1}{\delta\mathbf{u}}\right) - \mathbf{u}_3 \times \operatorname{curl}_3\left(\frac{1}{D}\frac{\delta L_1}{\delta\mathbf{u}}\right)$$

$$- \nabla_3\left(\frac{\delta L_1}{\delta D} - \frac{1}{D}\frac{\delta L_1}{\delta\mathbf{u}}\cdot\mathbf{u}\right) + \frac{1}{D}\frac{\delta L_1}{\delta\rho}\nabla_3\rho = 0. \qquad (3.22)$$

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The resulting equation, from (3.19) and (3.22), is

$$\epsilon\frac{\partial \mathbf{u}_1}{\partial t} - \mathbf{u}_3 \times \mathrm{curl}_3\,(\mathbf{R} + \epsilon\mathbf{u}_1) + \nabla(p + \frac{\epsilon}{2}|\mathbf{u}_1|^2)$$

$$+ \rho\hat{\mathbf{z}} - \left[\epsilon \int\limits_{-1}^{z} dz'\,\hat{\mathbf{z}} \cdot \mathrm{curl}\,(\mathbf{u} - \mathbf{u}_1)\right]\nabla_3\,\rho = 0\,. \qquad (3.23)$$

The full set of approximate equations are given by (3.23), the constraint (3.21), the equation of continuity,

$$\nabla_3 \cdot \mathbf{u}_3 = 0, \qquad (3.24)$$

obtained from the equation (2.5b) for D and preservation of the constraint (3.20), and the equation for the density,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{u}_3 \cdot \nabla_3\,\rho = 0\,. \qquad (3.25)$$

We will refer to (3.23), (3.24), (3.25) and (3.21) as the L1 model equations.

By equation (2.9) for Euler–Poincaré systems the L1 model possesses the following Kelvin-Noether circulation theorem,

$$\frac{d}{dt} \oint_{\gamma_t(\mathbf{u}_3)} (\mathbf{R} + \epsilon\mathbf{u}_1) \cdot d\mathbf{x}_3 = - \oint_{\gamma_t(\mathbf{u}_3)} (z + \epsilon^2 I)d\rho\,, \qquad (3.26)$$

where

$$I = \epsilon^{-1} \int\limits_{-1}^{z} dz'\,\hat{\mathbf{z}} \cdot \mathrm{curl}\,D(\mathbf{u} - \mathbf{u}_1)\,, \qquad (3.27)$$

and $\gamma_t(\mathbf{u}_3)$ moves with the fluid velocity $\mathbf{u}_3$. Then, by Stokes' theorem, the L1 equations generate circulation of $(\mathbf{R} + \epsilon\mathbf{u}_1)$ whenever $\nabla\rho$ and $\nabla(z + \epsilon^2 I)$ are not collinear. For a loop γ_t on an isopycnal surface, the right hand side of (3.26) vanishes and Stokes' theorem implies the quantity $Q_\rho = \nabla_\rho \times (\mathbf{R} + \epsilon\mathbf{u}_1(\mathbf{x}, z(\mathbf{x}, \rho)))$ is conserved along fluid parcels, where

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∇_ρ is the horizontal gradient at constant buoyancy, ρ. This is the basis for an isopycnal L1 model.

Returning to z coordinates, the curl of equation (3.23) gives conservation of the following potential vorticity Q_1 on fluid particles, cf. (2.13),

$$\frac{DQ_1}{Dt} = 0 \,, \tag{3.28}$$

where

$$Q_1 = \text{curl}_3(\mathbf{R} + \epsilon \mathbf{u_1}) \cdot \nabla_3 \rho \,, \tag{3.29a}$$

$$= (\hat{\mathbf{z}} + \epsilon \, \text{curl}_3 \, \mathbf{u_1}) \cdot \nabla_3 \rho \,. \tag{3.29b}$$

The L1 model equations also conserve the volume integrated energy E_1, given in the general theory by (2.14), i.e.,

$$\frac{dE_1}{dt} = 0 \,, \tag{3.30}$$

$$E_1 = \int dx \, dy \, dx \, \left[\frac{1}{2} |\mathbf{u_1}|^2 + \rho z \right] \,. \tag{3.31}$$

4. Solution Procedure

We write the L1 model equations (3.23), (3.24) and (3.25) in the form:

$$\nabla_3 \cdot \mathbf{u}_3 = 0, \qquad (4.1a)$$

$$\epsilon \frac{\partial \mathbf{u}_1}{\partial t} - \mathbf{u}_1 \times (1 + \epsilon\zeta_1)\hat{\mathbf{z}} + \epsilon^2 w \mathbf{u}_{1z} + \nabla(\tilde{p} + \frac{\epsilon}{2} |\mathbf{u}_1|^2) + \epsilon^2 I \nabla \phi_z = -\epsilon\nu\nabla^4 \mathbf{u}_1, \qquad (4.1b)$$

$$\tilde{p}_z - \phi_z - \epsilon(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{u}_{1z} + \epsilon I(S + \epsilon\phi_{zz}) = 0, \qquad (4.1c)$$

$$\frac{\partial}{\partial t}\phi_z + \nabla_3 \cdot (\mathbf{u}_3 \phi_z) + Sw = 0, \qquad (4.1d)$$

with constraint (3.21), imposed by ϕ_S,

$$\int_{-1}^{0} dz\, \hat{\mathbf{z}} \cdot \text{curl}\,(\mathbf{u} - \mathbf{u}_1) = \langle \hat{\mathbf{z}} \cdot \text{curl}\,(\mathbf{u} - \mathbf{u}_1) \rangle = 0, \qquad (4.1e)$$

where

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi + \epsilon\nabla\chi, \qquad (4.2a)$$

and where the density and pressure fields have been decomposed as in (3.7) and (3.9),

$$\rho = \bar{\rho}(z) - \theta, \quad p = \bar{p}(z) + \tilde{p}, \qquad (4.2b, c)$$

so that

$$S(z) = -\epsilon\bar{\rho}_z. \qquad (4.2d)$$

With (4.2b), $\mathbf{u}_1$ may be written

$$\mathbf{u}_1 = \hat{\mathbf{z}} \times \nabla\phi, \qquad (4.3a)$$

where

$$\phi = \phi_S(x, y, t) - \int_{z}^{0} dz'\, \theta, \qquad \phi_z = \theta, \qquad (4.3b, c)$$

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and

$$\zeta_1 = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}_1 = \nabla^2 \phi. \tag{4.3d}$$

The constraint (4.1e) implies

$$\nabla^2 \langle \phi \rangle = \nabla^2 \langle \psi \rangle. \tag{4.4}$$

It will turn out to be convenient to define

$$\psi = \phi + \epsilon \hat{\psi}, \tag{4.5a}$$

where, for the satisfaction of (4.4) we require

$$\langle \hat{\psi} \rangle = 0. \tag{4.5b}$$

We also note that

$$I = \epsilon^{-1} \int_{-1}^{z} dz' \, \hat{\mathbf{z}} \cdot \mathrm{curl} \, (\mathbf{u} - \mathbf{u}_1) = \int_{-1}^{z} dz' \, \nabla^2 \hat{\psi}. \tag{4.6}$$

We include biharmonic momentum diffusion in the horizontal momentum equations (4.1b) so that it may be used in the numerical finite-difference solutions to provide dissipation at high wave numbers in otherwise nearly inviscid flows.

The horizontal momentum equations (4.1b) are replaced by vorticity and divergence equations formed, respectively, by the operations $\hat{\mathbf{z}} \cdot \nabla \times (4.1b)$ and $\nabla \cdot (4.1b)$:

$$\frac{\partial \zeta_1}{\partial t} + J(\psi, \zeta_1) + \nabla^2 \chi + \epsilon \nabla \cdot [w \nabla \phi_z + \zeta_1 \nabla \chi] + \epsilon J(I, \phi_z) + \nu \nabla^4 \zeta_1 = 0, \tag{4.7}$$

$$\nabla^2 \psi = \nabla^2 \tilde{p} - \epsilon 2 J(\phi_x, \phi_y)$$
$$- \epsilon^2 \left[J(\zeta_1, \chi) + J(w, \phi_z) + \nabla \cdot (\zeta_1 \nabla \hat{\psi}) - \nabla \cdot (I \nabla \phi_z) \right]. \tag{4.8}$$

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We rewrite (4.1d) as

$$\frac{\partial \phi_z}{\partial t} + J(\psi, \phi_z) + Sw + \epsilon\left[\nabla \cdot (\phi_z \nabla \chi) + (w\phi_z)_z\right] = 0\,, \tag{4.9}$$

and (4.1a) as

$$\nabla^2 \chi + w_z = 0\,. \tag{4.10}$$

The L1 model equations now consist of (4.7), (4.8), (4.1c), (4.9), (4.10) and (4.5). The variables are $\psi, \chi, w, \tilde{p}, \phi$ and $\hat{\psi}$. Note that if we set $\epsilon = 0$ in these equations, they reduce to the quasigeostrophic (QG) approximation (Pedlosky, 1987). To obtain numerical solutions, we follow a procedure similar to that developed and applied in Allen and Newberger (1993) for other intermediate models. The procedure is based on the assumption that $\epsilon \ll 1$ and essentially uses the solution of the QG approximation as the starting point for an iteration scheme. Accordingly, we form an equation for a linear approximation to the potential vorticity by eliminating $w_z = -\nabla^2 \chi$ between (4.7) and (4.9):

$$\frac{\partial}{\partial t}[\nabla^2 \phi + (S^{-1}\phi_z)_z] = -J(\psi, \zeta_1) - \epsilon \nabla \cdot [w\nabla\psi_z + \zeta_1 \nabla \chi]$$
$$- \epsilon J(I, \psi_z) - \nu\nabla^6\phi - \left\{S^{-1}\left[J(\psi, \phi_z) + \epsilon\left[\nabla \cdot (\phi_z \nabla \chi) + (w\phi_z)_z\right]\right]\right\}_z \,. \tag{4.11}$$

We also write (4.7) as

$$\nabla^2 \chi = -\frac{\partial \nabla^2 \phi}{\partial t} - J(\psi, \zeta_1) - \epsilon\nabla \cdot [w\nabla\phi_z + \zeta_1 \nabla\chi] - \epsilon J(I, \phi_z) - \nu\nabla^6\phi\,. \tag{4.12}$$

We eliminate $\tilde{p}$ by taking the z derivative of (4.8) and substituting for $\tilde{p}_z$ from (4.1c). The resulting equation is

$$\nabla^2 G = -2J_z(\phi_x, \phi_y) - \epsilon\left\{J_z(\zeta_1, \chi) + J_z(w, \phi_z)\right.$$
$$+ \nabla \cdot (\zeta_1 \nabla\hat{\psi})_z - \nabla \cdot (I\nabla\phi_z)_z - \nabla^2\left[\nabla\hat{\psi} \cdot \nabla\phi_z + J(\phi_z, \chi) - \phi_{zz}I\right]\right\}, \tag{4.13a}$$

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where

$$G = \hat{\psi}_z + SI \, . \tag{4.13b}$$

If, for $\epsilon \ll 1$, G is obtained from the solution of (4.13a), then $\hat{\psi}$ may be found from the solution to

$$\nabla^2 \hat{\psi} + [S^{-1}\hat{\psi}_z]_z = [S^{-1}G]_z, \tag{4.13c}$$

where (4.5b) implies

$$\hat{\psi}_z = G \quad \text{at} \quad z = 0, -1 \, . \tag{4.13d}$$

In a domain periodic in $(x, y,)$, (4.13a) determines G up to an arbitrary function of z. That function has no effect on the horizontal derivatives of $\hat{\psi}$ (or ψ) needed in the other equations. Thus, to obtain a unique solution for G we require

$$\int dx \, dy \, G = 0. \tag{4.13e}$$

The L1 model equations we solve are finally (4.10), (4.11), (4.12), (4.13) and (4.5). The variables are ψ, χ, w, ϕ, and $\hat{\psi}$. The numerical finite difference method used to solve this equation set is described in the appendix. The boundary conditions are

$$w(z = 0) = w(z = -1) = 0, \tag{4.14}$$

with periodicity of all variables over the domain in x and y.

5. Numerical Experiments

Numerical solutions to finite-difference approximations to the L1 model are obtained for the problems utilized to investigate accuracy of different intermediate models in Allen and Newberger (1993). The numerical experiments involve initial-value problems for the time-dependent development of an unstable baroclinic jet on an f-plane. Initial conditions involve a uniform, vertically-sheared jet with small perturbations. The jet is unstable and develops finite amplitude meanders that grow in time and eventually pinch off to form detached eddies. The weak jet and basic case numerical experiments from Allen and Newberger (1993) are repeated here for the L1 model. The accuracy is assessed by comparison with solutions of the primitive equations (PE).

The domain is periodic in the horizontal directions (x, y) and is of constant depth in the vertical direction (z). The finite difference methods are discussed in the appendix and in Allen and Newberger (1993). All models use the same variables on the same grid.

Dimensional variables are used for the numerical experiments. The Coriolis parameter $f = 9.20 \times 10^{-5}$ s^{-1}. The total depth $H_T = 3172$ m. The number of vertical grid cells is 6. The horizontal domain is

$$0 \le x \le L^{(x)}, \qquad 0 \le y \le L^{(y)}, \qquad (5.1a,b)$$

where the initial jet flow is parallel to the x axis. For the weak jet experiment, $L^{(x)} = 250$ km, $L^{(y)} = 640$ km. For the basic case experiment, $L^{(x)} = 250$ km, $L^{(y)} = 810$ m. The horizontal grid spacing is $\Delta x = \Delta y = 5$ km. The horizontal bi-harmonic diffusion coefficient $\nu = 8 \times 10^8$ m^4 s^{-1} is chosen to be small so that dissipative processes play a nearly negligible role in the time-dependent dynamics.

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The initial stratification and jet structure are based on observed oceanographic values from the Coastal Transition Zone (CTZ) off Northern California (Kosro et al., 1991; Pierce et al., 1991; Walstad et al., 1991). The Rossby radius for the first baroclinic vertical mode calculated from the initial stratification is $\delta_{R1} = 24.6$ km. The initial jet has a half-width of $L_J = 30$ km, comparable in magnitude to δ_{R1}. The experiments are characterized by different velocity magnitudes in the initial basic jet profiles. For the weak jet and basic case experiments, the maximum initial jet velocities are 0.52 m s^{-1} and 0.90 m s^{-1}, respectively. The vertical shear is such that the corresponding maximum initial jet velocities at 500 m depth are 0.23 m s^{-1} and 0.36 m s^{-1}.

The function $|\zeta(x, y, z, t)|/f$, where $\zeta = v_x - u_y$, indicates the magnitude of the local, flow-determined Rossby number. For the weak jet and basic case experiments, the maximum initial values of $|\zeta|/f$ are 0.174 and 0.287, respectively. The maximum values reached during the experiment are larger, 0.264 and 0.555, respectively. Thus, the two experiments cover a range of flow regimes characterized by maximum local Rossby numbers $|\zeta|/f$ that might be described as moderately small (weak jet) and moderate (basic case).

Quantitative measures of the errors of the L1 model solutions, compared to PE, are found as a function of time by calculating normalized rms differences between the corresponding variables from the L1 model and from the PE solutions as described in Allen and Newberger (1993). Prior to comparison with L1 model results and calculation of errors, the PE solutions are averaged over an inertial period to eliminate high frequency variability.

The time-dependent development of the unstable jet flow field in the basic case experiment is illustrated in Fig. 1 by contour plots of the near-surface streamfunction fields

$\psi(x, y, z_1, t) = \psi_1(x, y, t)$ where z_1 corresponds to the center of the top grid cell at 50 m depth. Fields from the solutions of the primitive equations PE and the L1 model are shown every 20 days from day 10 to day 90. The corresponding vorticity fields ζ_1/f from PE and L1 are shown in Fig. 2. The close agreement of the ψ_1 and ζ_1/f fields from the L1 model with the corresponding fields from the PE is apparent.

The normalized errors for the streamfunction field $\psi(x, y, z, t)$ from the L1 model relative to PE for the weak jet and basic case experiments are shown in Fig. 3. Similar error characteristics are found for the other variables. For comparison, the errors from other intermediate model solutions as reported in Allen and Newberger (1993) are included. The other models are the quasigeostrophic (QG) approximation (Pedlosky, 1987), the geostrophic momentum (GM) approximation (Hoskins, 1975), the linear BEM model (Allen and Newberger, 1993), the IG2 iterated geostrophic model (Allen, 1993), and the balance equations based on momentum equations BEM model (Allen, 1991; Holm, 1996).

The errors are plotted with two different scales. An expanded scale plot is included at the bottom to show clearly the relative errors of the more accurate models. In both experiments, the errors from the L1 model are substantially lower than those from QG, GM or LBEM. That fact is further illustrated by a comparison of the streamfunction fields $\psi_1(x, y, t)$ at day 70 of the basic case experiment from PE, L1, GM, and QG (Fig. 4). The BEM model gives the most accurate approximate solutions. The balance equations BE (Gent and McWilliams, 1983) (errors not plotted) also give accurate solutions comparable to those of BEM. The errors from the new L1 model, however, are generally relatively small and remain so during the experiment. In the basic case experiment, the errors

from L1 and IG2 are comparable for $t < 50$ days with IG2 slightly lower. After day 50, the errors for IG2 become considerably larger than those from L1. Similar qualitative behavior is found in the weak jet experiment. The inviscid IG2 model does not have exact analogues of potential vorticity conservation on fluid particles or of conservation of volume integrals of energy (Allen, 1993). The L1 model, of course, does have analogues of these conservation equations, given in (3.29) and (3.31). It seems possible that the tendency of the IG2 model errors to increase at large time, while the L1 errors remain relatively constant, may be related to the differences in the models with regard to possession of analogue conservation equations for potential vorticity and energy.

The L1 model has the smallest errors of any of the approximate models in Allen and Newberger (1993) that advect a vorticity of the form $\zeta_1 = \nabla^2\phi$ (4.12). Those models include QG, LBEM, and the linear balance equations (LBE). All of the more accurate models advect a higher $O(\epsilon)$ approximation to the vorticity similar to that of IG2, where the advected vorticity ζ_A is

$$\zeta_A = \nabla^2\phi - \epsilon 2J(\phi_x, \phi_y)\,. \tag{5.2}$$

From a comparison of the model errors in Allen and Newberger (1993), it appeared that advection of vorticity with $O(\epsilon)$ accuracy was a necessary property for an accurate model. The relatively small errors found with the L1 model do not seem to fit with that idea. In the L1 model, the vorticity of the advecting velocities

$$\zeta = \nabla^2\psi = \nabla^2\phi - \epsilon 2J(\phi_x, \phi_y) + O(\epsilon^2) \tag{5.3}$$

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is accurate to $O(\epsilon)$ and that feature may contribute to the relatively small errors found with L1. Additionally, the requirement in L1 that $\langle \nabla^2 \psi \rangle = \langle \nabla^2 \phi \rangle$ may provide additional accuracy.

6. Summary Comments

The L1 model produces generally accurate approximate solutions for the idealized, moderate Rossby number, mesoscale oceanographic flow problems examined in Allen and Newberger (1993). These solutions are not quite as accurate as those from the BEM or BE models, but are substantially more accurate than those from GM or QG. Overall, the results are encouraging and provide motivation for further work extending the L1 derivation to variable bottom topography and to variable Coriolis parameter $f = f(x, y)$. In addition, motivation is also provided for the derivation of a presumably more accurate extended-geostrophic model that systematically includes $O(\epsilon)$ terms in the approximation for **u** in (3.14) following Allen and Holm (1996).

In summary, we note that the L1 model appears to realize some of the potential anticipated for approximate equations derived from Hamilton's principle. The accuracy of the L1 model solutions seems better than expected based on the asymptotics involved in its derivation. In addition, the errors remain small as time increases, which may be the desired consequence of retaining analogue energy and potential vorticity conservation laws.

APPENDIX

Numerical Methods

The numerical finite-difference approximations for the model equations in section 4 are discussed in this appendix. For consistency, the difference approximations are presented here in terms of the dimensionless variables of sections 3 and 4, although the numerical solutions are obtained in corresponding dimensional variables. The finite difference grid for the variables (ψ, χ, w, ϕ) and the corresponding spatial difference operators are identical to those described in appendix A of Allen and Newberger (1993). Thus, those definitions are not repeated here.

The governing equations in difference form, corresponding to (4.10), (4.11), (4.12) and (4.13) are

$$\delta_z w + \nabla^2 \chi = 0 \,, \tag{A1}$$

$$\nabla^2 \phi_t + \delta_z(S^{-1}\delta_{\hat{z}}\phi_t) = RQ, \tag{A2}$$

$$\nabla^2 \chi = -\nabla^2 \phi_t + R\zeta, \tag{A3}$$

$$\nabla^2 G = RG \,, \tag{A4a}$$

$$\nabla^2 \hat{\psi} + \delta_z(S^{-1}\delta_{\hat{z}}\hat{\psi}) = \delta_z(S^{-1}G), \tag{A4b}$$

where (4.5) holds and where

$$RQ = R\zeta + RDZ\theta \,, \tag{A5a}$$

$$R\zeta = -J(\psi, \zeta_1) - \epsilon\left[\nabla \cdot \overline{\left[w\nabla\delta_{\hat{z}}\phi\right]}^z + \zeta_1\nabla\chi\right] + \overline{J}^z(I, \delta_{\hat{z}}\phi)\right] - \nu\nabla^6\phi, \tag{A5b}$$

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$$RDZ\theta = -\delta_z\left\{S^{-1}\left[J(\overline{\psi}^{\hat{z}}, \delta_{\hat{z}}\phi) + \epsilon\left[\nabla\cdot(\delta_{\hat{z}}\phi\overline{\nabla\chi}^{\hat{z}}) + \delta_{\hat{z}}(\overline{w\delta_{\hat{z}}\phi}^z)\right]\right]\right\}, \qquad (A5c)$$

$$RG = -2\delta_{\hat{z}}J(\delta_x\overline{\phi}^x, \delta_y\overline{\phi}^y) - \epsilon\left\{\delta_{\hat{z}}J(\zeta_1, \chi) + \delta_{\hat{z}}\overline{J}^z(w, \delta_{\hat{z}}\phi)\right.$$

$$+ \delta_{\hat{z}}\nabla\cdot(\zeta_1\nabla\hat{\psi}) - \delta_{\hat{z}}\nabla\cdot\overline{(I\nabla\delta_{\hat{z}}\phi)}^z$$

$$\left. - \nabla^2[\overline{\nabla\hat{\psi}}^{\hat{z}}\cdot\nabla\delta_{\hat{z}}\phi + J(\delta_{\hat{z}}\phi, \overline{\chi}^{\hat{z}}) - \delta_{\hat{z}}\overline{(I\delta_{\hat{z}}\phi)}^z + \delta_{\hat{z}}\overline{I}^z\delta_{\hat{z}}\phi]\right\}. \qquad (A5d)$$

The function

$$I(z) = I(z_{k+(1/2)}) = \int_{-1}^{z} dz'\, \nabla^2\hat{\psi}, \qquad (A6)$$

where the z integration is calculated as described for w in (A11) of Allen and Newberger (1993).

We advance in time by using the implicit time difference scheme described for the balance equations (BE) in Allen and Newberger (1993). The equations (A2) and (A3) are time differenced as

$$\nabla^2\delta_t^{n+\frac{1}{2}}\phi + \delta_z(S^{-1}\delta_{\hat{z}}\delta_t^{n+\frac{1}{2}}\phi) = \overline{RQ}^{n+\frac{1}{2}}, \qquad (A7a)$$

$$\nabla^2\overline{\chi}^{n+\frac{1}{2}} = -\nabla^2\delta_t^{n+\frac{1}{2}}\phi + \overline{R\zeta}^{n+\frac{1}{2}}. \qquad (A7b)$$

Equations (A4a,b) are assumed to hold at each time level $t = n\Delta t$.

It follows from (A7a,b) that

$$\nabla^2\phi^{n+1} + \delta_z(S^{-1}\delta_{\hat{z}}\phi^{n+1}) = RIQ, \qquad (A8a)$$

$$\nabla^2\chi^{n+1} = RI\zeta, \qquad (A8b)$$

where

$$RIQ = \nabla^2\phi^n + \delta_z(S^{-1}\delta_{\hat{z}}\phi^n) + \Delta t\,\overline{RQ}^{n+\frac{1}{2}} \qquad (A9a)$$

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$$RI\zeta = -\nabla^2\chi^n + 2\left[-\nabla^2\delta_t^{n+\frac{1}{2}}\phi + \overline{R\zeta}^{n+\frac{1}{2}}\right]. \qquad (A9b)$$

In solving (A8a) for ϕ^{n+1} and (A4b) for $\hat{\psi}^{n+1}$, an expansion for these variables in terms of vertical linear normal modes is utilized as described in equations (A22) and (A23) in Allen and Newberger (1993).

With all variables known at $t = n\Delta t$ and at previous time levels, we solve (A8a,b) and (A4a,b) by iteration. Estimate all variables at $t = (n+1)\Delta t$ by extrapolation, e.g., $\phi^{n+1} = 2\phi^n - \phi^{n-1}$. Use these estimates in the rhs of (A8a) and (A8b). Solve (A8a) for ϕ^{n+1}. Substitute the new value of ϕ^{n+1} in the time derivative term $(-\nabla^2\delta_t^{n+\frac{1}{2}}\phi)$ on the rhs of (A8b) and solve (A8b) for χ^{n+1}. Calculate w^{n+1} from (A1). Substitute ϕ^{n+1}, w^{n+1} in the rhs of (A4a) and solve (A4a) for G^{n+1}. Substitute G^{n+1} in the rhs of (A4b) and solve (A4b) for $\hat{\psi}^{n+1}$. Return to the step where (A8a) is solved for ϕ^{n+1} and substitute the latest values for the variables at $t = (n+1)\Delta t$ in the rhs. Repeat the cycle until convergence is obtained for $\phi^{n+1}, \chi^{n+1}, w^{n+1}$, and $\hat{\psi}^{n+1}$.

At $t = 0$, ϕ^0 is specified. Initial-values for $\hat{\psi}^0, \chi^0$, and w^0 are found by an iterative procedure. Estimate $\hat{\psi}^0 = \chi^0 = w^0 = 0$. Calculate $\hat{\psi}^0$ from (A4a,b), ϕ_t^0 from (A2), χ^0 from (A3), and w^0 from (A1). Repeat the calculations until convergence for $\hat{\psi}^0, \chi^0, w^0$, and ϕ_t^0 is obtained. For the first time step, estimate $\phi^1 = \phi^0 + \Delta t\phi_t^0, \hat{\psi}^1 = \hat{\psi}^0, \chi^1 = \chi^0, w^1 = w^0$ and proceed with the general implicit time difference scheme.

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FIGURE CAPTIONS

Fig. 1. Contour plots of the ψ_1 fields from PE and L1 as a function of (x, y) every 20 days from $t = 10$ to $t = 90$ days for the basic case experiment. The distance between tick marks on the axes is 50 km. Positive (negative) values are contoured by solid (dashed) lines with the zero contour a heavy solid line. The contour interval is 3000 m^2 s^{-1}.

Fig. 2. Contour plots of the ζ_1/f_0 fields from PE and L1 every 20 days from $t = 10$ to $t = 90$ days for the basic case experiment. The contour interval is 0.1 and the zero contour line is omitted. Scaling of axes as in Fig. 1.

Fig. 3. The normalized rms errors E for ψ as a function of time from the QG, GM, LBEM, IG2, L1, and BEM models compared to PE for the weak jet experiment (left) and for the basic case experiment (right). Note the different scales for the errors in the top and bottom panels. The type of line representing the errors from each model is consistent for both experiments.

Fig. 4. Contour plots of the ψ_1 fields at day 70 of the basic case experiment from PE, L1, GM and QG. Contour interval and scaling of axes as in Fig. 1.