Vacuum Expectation Values of Higgs Scalars in a $SU(2)_L \times SU(2)_R \times U(1)$ Gauge Model*

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Abstract

We determine the vacuum expectation values of the Higgs scalars within the framework of a six quark $SU(2)_L \times SU(2)_R \times U(1)$ gauge model after the imposition of discrete symmetrics that are necessary in order to express the Cabibbo angle in terms of quark mass ratios and phases of the vacuum expectation values. We find both real and complex solutions for the vacuum expectation values depending on the relative values of the parameters in the Higgs potential.

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The cabibbo angle has been expressed in terms of quark mass ratios,[1,2,3] phases of the vacuum expectation values (vevs) of Higgs scalars[3], or both[4] within the framework of the gauge group $SU(2)_L \times SU(2)_R \times U(1)$ by requiring the Lagrangian be invariant under specific discrete symmetries. In addition to affecting the magnitude of the cabibbo angle, the phases of the vevs of the Higgs scalars can provide a mechanism for spontaneous CP violation.[5] Since the phases can play a significant role, we determine their values for one of the published models[4] by minimizing the most general Higgs potential which is invariant under the discrete symmetries originally utilized to obtain the quark-Higgs Yukawa interaction. We show that even after the discrete symmetries are imposed, it is possible to obtain real or complex vevs depending upon the relative values of the parameters in the Higgs potential.

We consider the model of Ref. 4 which contains two Higgs scalars ϕ_1 and ϕ_2 that belong to the $(\frac{1}{2}, \frac{1}{2}, 0)$ representation. The most general gauge invariant, renormalizable Higgs potential can be written as,

$$V=-\frac{1}{2}\mu_i^2\,\mathrm{Tr}(\psi^\dagger\Gamma_i\psi)+\frac{1}{4}\lambda_{ij}\,\mathrm{Tr}(\psi^\dagger\Gamma_i\psi)\,\mathrm{Tr}(\psi^\dagger\Gamma_j\psi)+\frac{1}{4}\lambda'_{ij}\,\mathrm{Tr}(\psi^\dagger\Gamma_i\psi\psi^\dagger\Gamma_j\psi), \tag{1}$$

where $\psi=(\phi_1, \phi_2, \tilde{\phi}_2, \tilde{\phi}_1)$, $\tilde{\phi}_n=\tau_2\phi_n^*\tau_2$, and the parameters $\mu_i^2, \lambda_{ij}, \lambda'_{ij}$ are real. The Γ_i are the sixteen hermitian 4×4 matrices,[6] and the summation over i, j has been suppressed. As indicated in Ref. 4, the potential must be independently invariant under the following two discrete symmetries:

$$\psi \rightarrow S_m\psi, \quad m=1,2 \tag{2}$$

where

$$S_1 = -i\rho_3 \tag{3}$$

$$S_2 = \begin{pmatrix} 0 & \eta & 0 & 0 \\ \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & \eta^2 & 0 \end{pmatrix} = -(\sigma_1 + \sqrt{3}\,\sigma_2)/2 \tag{4}$$

and $\eta = \exp(2\pi i/3)$. The further requirement that the potential be CP invariant leads to the relation

$$V = -\tfrac{1}{2}\mu_i^2\,\mathrm{Tr}(\psi^\dagger\Gamma_i^*\psi) + \tfrac{1}{4}\lambda_{ij}\,\mathrm{Tr}(\psi^\dagger\Gamma_i^*\psi)\,\mathrm{Tr}(\psi^\dagger\Gamma_j^*\psi) + \tfrac{1}{4}\lambda'_{ij}\,\mathrm{Tr}(\psi^\dagger\Gamma_i^*\psi\psi^\dagger\Gamma_i^*\psi). \tag{5}$$

To facilitate the construction of the most general potential which is invariant under CP and the discrete transformations S_1 and S_2, we have tabulated in Table I, Γ_i, the complex conjugates Γ_i^* and the eigenvalues s_m of the discrete transformations which are defined by the relation $S_m^\dagger\Gamma_i S_m = s_m\Gamma_i$. We have defined $\sigma_+ = (\sqrt{3}\sigma_1 - \sigma_2)/2$ and $\sigma_- = -(\sigma_1 + \sqrt{3}\sigma_2)/2$ which are simultaneous eigenstates of S_1 and S_2.

It is immediately clear from Table I that there are only two quadratic terms which are invariant under CP, S_1 and S_2, $\mathrm{Tr}\psi^\dagger\mathcal{I}\psi$ and $\mathrm{Tr}\psi^\dagger\rho_3\psi$. To write down the quartic terms, the notation is simplified so that $\Gamma\Gamma$, for example, represents the two invariant quartic terms $\mathrm{Tr}(\psi^\dagger\Gamma\psi)\mathrm{Tr}(\psi^\dagger\Gamma\psi)$ and $\mathrm{Tr}(\psi^\dagger\Gamma\psi\psi^\dagger\Gamma\psi)$. We can immediately construct from Table I the ten invariant quartic terms: $\mathcal{I}\mathcal{I}$, $\mathcal{I}\rho_3$, $\rho_3\rho_3$, $\sigma_3\sigma_3$, $\sigma_3(\rho_3\sigma_3)$, $(\rho_3\sigma_3)(\rho_3\sigma_3)$, $\rho_1\rho_1$, $\rho_2\rho_2$, $(\rho_1\sigma_3)(\rho_1\sigma_3)$ and $(\rho_2\sigma_3)(\rho_2\sigma_3)$. It is possible to construct five more invariant quartic terms as follows. While $\sigma_-\sigma_-$ is invariant under S_1 and S_2, it is not invariant under CP since σ_-^* is a linear combination of σ_- and σ_+. We find $\sigma_-\sigma_- + \sigma_+\sigma_+$ to be invariant. Similarly, the final four invariant terms are $(\rho_1\sigma_-)(\rho_1\sigma_-) + (\rho_1\sigma_+)(\rho_1\sigma_+)$, $(\rho_2\sigma_-)(\rho_2\sigma_-) + (\rho_2\sigma_+)(\rho_2\sigma_+)$, $(\rho_3\sigma_-)(\rho_3\sigma_-) + (\rho_3\sigma_+)(\rho_3\sigma_+)$, and $(\rho_3\sigma_-)\sigma_- + (\rho_3\sigma_+)\sigma_+$.

It would appear from arguments similar to the above that a linear combination of the terms $(\rho_1\sigma_-)(\rho_2\sigma_-)$ and $(\rho_1\sigma_+)(\rho_2\sigma_+)$ would be invariant, but this turns out not to be the case.

After spontaneous symmetry breakdown, the vevs of the Higgs scalars are written as[7]

$$\langle\phi_1\rangle = \begin{pmatrix} x e^{i\rho_1} & 0 \\ 0 & y e^{i\rho_2} \end{pmatrix}, \quad \langle\phi_2\rangle = \begin{pmatrix} u e^{i(\rho_1+\delta_1)} & 0 \\ 0 & v e^{i(\rho_2+\delta_2)} \end{pmatrix}. \tag{6}$$

When the above vevs are substituted into the invariant potential(5), it takes the form $V = V_m + V_p$ where the phase dependence has been isolated in V_p.

$$V_m = -\tfrac{1}{2}\mu^2(x^2+u^2+y^2+v^2) + \tfrac{A}{4}[(x^2+u^2)^2+(y^2+v^2)^2] + \tfrac{B}{2}(x^2+u^2)(y^2+v^2)$$
$$+ \tfrac{D}{4}[(x^2-u^2)^2+(y^2-v^2)^2] + \tfrac{E}{2}(x^2-u^2)(y^2-v^2),$$

$$V_p = xyuv(a\cos\delta + b\cos\eta) + \tfrac{c}{2}[x^2v^2\cos(\eta+\delta) + y^2u^2\cos(\eta-\delta)].$$

The phases δ and η are given, respectively, by $\delta = \delta_2 - \delta_1$ and $\eta = 2\rho_1 + 2\rho_2 + \delta_1 + \delta_2$. The parameters μ^2, A, B, D, E, a, b, and c are linearly independent.

Let us first rewrite V_p in the form,

$$V_p = c\,x^2v^2\{[(1+r)^{\frac{1}{2}}\cos\theta + z(1+s)^{\frac{1}{2}}\cos\phi]^2 + [r^{\frac{1}{2}}\sin\theta + z\,s^{\frac{1}{2}}\sin\phi]^2$$
$$-(r+sz^2) - (1+z^2)/2\}, \tag{7}$$

where $\theta = (\eta+\delta)/2$, $\phi = (\eta-\delta)/2$, and $z = yu/xv$. The constants r and s satisfy the equations

$$(1+r)^{\frac{1}{2}}(1+s)^{\frac{1}{2}} = (a+b)/2c\,, \qquad (r)^{\frac{1}{2}}(s)^{\frac{1}{2}} = (a-b)/2c\,,$$

and are given by

$$r = \{ ab - c^2 \mp [(c^2-a^2)(c^2-b^2)]^{\frac{1}{2}} \} / 2c^2\,,$$

$$s = \{ ab - c^2 \pm [(c^2-a^2)(c^2-b^2)^{\frac{1}{2}} \} / 2c^2. \tag{8}$$

It is particularly simple to find the angles θ and ϕ that minimize V_p with the requirement $(c^2 - a^2)(c^2 - b^2) \geq 0$ which we adopt. It turns out that the vevs can be real or complex. We use the inequality $(c^2 - a^2)(c^2 - b^2) \leq (ab \pm c^2)^2$, and find that there are five cases to be considered independently.

For case I, $ab - c^2 > 0$, $c > 0$ or $ab + c^2 < 0$, $c < 0$, in which case

$$V_p = x^2 v^2 \{ |c| [(|1+r|)^{\frac{1}{2}} \cos\theta + z (|1+s|)^{\frac{1}{2}} \cos\phi]^2 + |c| [(|r|)^{\frac{1}{2}} \sin\theta + z (|s|)^{\frac{1}{2}} \sin\phi]^2 \} \\ - c\, [2(r + s z^2) + (1 + z^2)]/2 \}. \tag{9}$$

For case II, $ab - c^2 < 0$, $ab + c^2 > 0$, $c > 0$, and V_p is given by Eq. (9) except that the sign of the coefficient of the second term is negative. For case III, $ab - c^2 < 0$, $ab + c^2 > 0$, $c < 0$, and V_p is given by (9) except the sign of the coefficient of the first term is negative. For case IV, $ab - c^2 > 0$, $c < 0$ and for case V, $ab + c^2 < 0$, $c > 0$. In cases IV and V, V_p is given by (9) with the sign of the coefficients of the first two terms being negative.

The V_p is minimized in case I when

$$(|1+r|)^{\frac{1}{2}} \cos\theta + z (|1+s|)^{\frac{1}{2}} \cos\phi = 0\,, \tag{10}$$

and

$$(|r|)^{\frac{1}{2}} \sin\theta + z\,(|s|)^{\frac{1}{2}} \sin\phi = 0. \tag{11}$$

Equations (10) and (11) lead to the following solutions for $\delta = \theta - \phi$ and $\eta = \theta + \phi$

$$\cos\delta = [a\epsilon - b - z^2(a\epsilon + b)]/2cz, \tag{12}$$

$$\cos\eta = [b - a\epsilon - z^2(a\epsilon + b)]/2cz\epsilon, \tag{13}$$

where $\epsilon = \pm\left[(c^2 - b^2)/(c^2 - a^2)\right]^{1/2}$. With these values for η and δ, V_p for case I takes the form

$$V^{I}_{p\,min} = x^2 v^2 [(b^2 - c^2)(1 - z^2) - ab\epsilon(1 + z^2)]/2c\epsilon. \tag{14}$$

Now the potential $V = V_m + V^{I}_{pmin.}$ is minimized with respect to x, u, y and v. This leads to the four equations

$$\frac{\partial V}{\partial x} = \left[-\mu^2 + A(x^2 + u^2) + B(y^2 + v^2) + D(x^2 - u^2) + E(y^2 - v^2) + v^2(b^2 - c^2 - ab\epsilon)/c\epsilon\right]x = 0, \tag{15}$$

$$\frac{\partial V}{\partial u} = \left[-\mu^2 + A(x^2 + u^2) + B(y^2 + v^2) - D(x^2 - u^2) - E(y^2 - v^2) - y^2(b^2 - c^2 + ab\epsilon)/c\epsilon\right]u = 0, \tag{16}$$

$$\frac{\partial V}{\partial y} = \left[-\mu^2 + A(y^2+v^2) + B(x^2+u^2) + D(y^2-v^2) + E(x^2-u^2) - u^2(b^2-c^2+ab\epsilon)/c\epsilon\right] y = 0,$$

(17)

$$\frac{\partial V}{\partial v} = \left[-\mu^2 + A(y^2+v^2) + B(x^2+u^2) - D(y^2-v^2) - E(x^2-u^2) + x^2(b^2-c^2-ab\epsilon)/c\epsilon\right] v = 0.$$

(18)

After multiplying (15), (16), (17), and (18) by x, u, y and v, respectively, and summing, one gets,

$$V = -\mu^2(x^2 + u^2 + y^2 + v^2)/4,$$

(19)

where x, u, y and v are solutions to Eqs. (15) - (18).

The solution to Eqs. (15)-(18) with no further requirments is

$$x^2 = v^2 = \mu^2(\beta - \gamma)/(\beta^2 - \alpha\gamma),$$

(20)

$$y^2 = u^2 = -\mu^2(\beta + \gamma)/(\beta^2 - \alpha\gamma),$$

(21)

where

$$\alpha = 2(A+B) - (ab/c),$$

$$\beta = (b^2 - c^2)/c\epsilon,$$

(22)

$$\gamma = 2(D-E) - (ab/c).$$

We note that the relations $x = v$, $y = u$ are incompatible with the quark mass ratios.[4] There are two methods for overcoming this problem. The most natural is to introduce additional Higgs fields which, because of conservation laws, cannot couple to the quark fields. Another way is to require that the parameters in the potential are such that two of the equations become linearly dependent.[8] Case I yields the most interesting solution since, as will be seen, it is the only one for which the phase δ is unequal to zero. In the model of Ref. 4 the Cabibbo angle depends on δ and $\delta \cong \pi/2$ is required to obtain a value which is in agreement with experiment.

For case II, it is obvious that V_p is minimized when $\theta = \phi = \pi/2$ that implies $\eta = \pi$ and $\delta = 0$. If we assume there are no relations among the parameters in the potential such that equations become linearly dependent, we find

$$x^2 = u^2 = y^2 = v^2 = \mu^2 / (2A + 2B + a - b - c).$$

For case III, V_p is obviously minimized when $\theta = \phi = 0$ or $\eta = \delta = 0$ that implies we have only real solutions. Again assuming no equations became linearly dependent, we find

$$x^2 = u^2 = y^2 = v^2 = \mu^2 / (2A + 2B + a + b + c).$$

To solve case IV, we use the inequality $\left[(1+r)(1+s)\right]^{1/2} \geq \left[(1+r)(1+s)\right]^{1/2} \cos\theta\cos\phi + (rs)^{\frac{1}{2}} \sin\theta\sin\phi$ which is valid since $r, s \geq 0$. The V_p is minimized when $\theta = \phi = 0$ and we are led to the solution of case III.

To solve case V, we use the inequality $(rs)^{\frac{1}{2}} \geq \left[(1+r)(1+s)\right]^{1/2} \cos\theta\cos\phi + [rs]^{\frac{1}{2}} \sin\theta\sin\phi$ which is valid since $r,s < -1$. The V_p is minimized for $\theta = \phi = \pi/2$ and we obtain the solution of case II.

We found $x^2 = u^2 = y^2 = v^2$ in cases II - V which is incompatible with the quark mass ratios. The methods suggested for case I are needed to overcome this problem. In any event the Cabibbo angle would not be in agreement with experiment as $\delta = 0$ for the final four cases.

Fritzsch's six quark model[2] does not incorporate spontaneous CP violation because only the case when vevs are real was considered. This model can be readily modified so as to incorporate spontaneous CP violation[9] in which case the Cabibbo angle depends on the phases of the vevs. Techniques employed here can be used to obtain the vevs of the Higgs scalars in the Fritzsch model and we expect conclusions similar to the above.

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6. The Γ_i are the sixteen 4×4 hermitian matrices listed below where I is the 2×2 identity matrix and τ_i are the usual Pauli matrices:

$$\mathbb{1}=\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix},\quad \rho_1=\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix},\quad \rho_2=\begin{pmatrix} 0 & -iI\\ iI & 0\end{pmatrix},$$

$$\rho_3=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \sigma_i=\begin{pmatrix} \tau_i & 0\\ 0 & \tau_i\end{pmatrix},\quad \rho_i\sigma_j,\qquad i,j=1,2,3.$$

7. In the notation of Ref. 4, $x=\beta_1/f$, $y=\beta_2/f$, $u=\epsilon_1/g=b_1/f$, $v=\epsilon_2/g=b_2/f$.
8. The linear combinations $(16)/u+(18)/v-(15)/x-(17)/y$ and $(18)/v+(17)/y-(16)/u-(15)/x$ lead, respectively, to

$$-\delta(x^2-u^2+y^2-v^2)+\beta(x^2+u^2-y^2-v^2)=0,$$

$$\beta(x^2-u^2+y^2-v^2)-\xi(x^2+u^2-y^2-v^2)=0,$$

where

$$\delta=2(D+E)+(ab/c),$$

$$\xi=2(A-B)+(ab/c).$$

If the determinant $\delta\xi - \beta^2 = 0$, the values of x, y, u, and v can be chosen independently.

9. T. Kitazoe and K. Tanaka, Ohio State Preprint, (1978).

Table I

The eigenvalues s_m defined by $S_m^\dagger \Gamma_i S_m$ for discrete symmetries S_m and matrices Γ_i and complex conjugates Γ_i^*.

Γ_i	s_1	s_2	Γ_i^*
$\mathscr{I}$	1	1	$\mathscr{I}$
ρ_3	1	1	ρ_3
σ_-	1	1	$-(\sigma_- + \sqrt{3}\sigma_+)/2$
$\rho_3\sigma_-$	1	1	$-(\rho_3\sigma_- + \sqrt{3}\rho_3\sigma_+)/2$
σ_3	1	-1	σ_3
$\rho_3\sigma_3$	1	-1	$\rho_3\sigma_3$
σ_+	1	-1	$(\sigma_+ - \sqrt{3}\sigma_-)/2$
$\rho_3\sigma_+$	1	-1	$(\rho_3\sigma_+ - \sqrt{3}\rho_3\sigma_-)/2$
ρ_1	-1	1	ρ_1
ρ_2	-1	1	$-\rho_2$
$\rho_1\sigma_-$	-1	1	$-(\rho_1\sigma_- + \sqrt{3}\rho_1\sigma_+)/2$
$\rho_2\sigma_-$	-1	1	$(\rho_2\sigma_- + \sqrt{3}\rho_2\sigma_+)/2$
$\rho_1\sigma_3$	-1	-1	$\rho_1\sigma_3$
$\rho_2\sigma_3$	-1	-1	$-\rho_2\sigma_3$
$\rho_1\sigma_+$	-1	-1	$(\rho_1\sigma_+ - \sqrt{3}\rho_1\sigma_-)/2$
$\rho_2\sigma_+$	-1	-1	$-(\rho_2\sigma_+ - \sqrt{3}\rho_2\sigma_-)/2$

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