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Linear Stability of Self-Similar Flow: 3. Compressional Waves in Imploding Spherical Shells

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A new, potentially disruptive, phenomenon in target pellets irradiated by laser or particle beams is described. Initially weak standing sound waves (vibrations) in an imploding spherical shell can become comparable in amplitude with the shell thickness if the effective adiabatic index satisfies $\gamma < 5/3$. Detailed calculations with a soluble self-consistent model show the relative amplification is stongest for long-wavelength modes in thin shells with small γ, and is insensitive to the density profile.

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Linear Stability of Self-Similar Flow: 3. Compressional Waves in Imploding Spherical Shells

When a pellet or shell of a medium containing standing sound waves undergoes compression, the external force does work in compressing the waves as well as the medium itself. The wave amplitude tends to decrease, but the characteristic thickness d of the compressed medium may decrease faster. When this happens a/d, the amplitude of the oscillations relative to the size of the system, *increases.* If a/d becomes of order unity, the perturbations disrupt the basic state.

We can determine the condition for such "relative instability" using the following simple argument, applicable to systems having spherical symmetry. For motions in which the external pressure $p(t)$ does not increase too abruptly, the action a^2kc is an approximate adiabatic invariant, where k is the wave number and c is the speed of sound. For adiabatic compression, $c = (\gamma p/\rho)^{1/2} \sim p^{\frac{\gamma-1}{2\gamma}}$. The characteristic scale size d is related to p by $d \sim \rho^{-1/3} \sim p^{-1/3\gamma}$ and to k by $kd \sim$ const. Combining these expressions, we find

$$a/d \sim (dc)^{-1/2} \sim p^{\frac{5/3-\gamma}{4\gamma}}, \tag{1}$$

Thus if γ is greater than 5/3, the "geometric" ratio of specific heats, the relative amplitude of the perturbations decreases. If $\gamma < 5/3$, the relative amplitude grows. Similar reasoning, e.g., in slab geometry, leads to the relative growth condition $\gamma < 3$. It should be emphasized that the mechanism described here does not depend on the magnitude of the effective acceleration and is unrelated to the Rayleigh-Taylor instability.

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If $\gamma \rightarrow 1$ (the worst case), $a/d \sim p^{1/6} \sim \rho^{1/6}$. Though weak, this dependence is enough to bring about a state of catastrophic vibration in a target pellet imploding under the action of a laser or particle beam, where compression by as much as 10^6 can take place. The crucial question is the appropriate form for the equation of state. Thermal conduction, incomplete molecular dissociation, and the energy required for ionization tend to reduce the effective adiabatic index γ below its geometric value; condensation at high densities tends to raise it.

The aim of this letter is to illustrate the above ideas by some analytic calculations of the amplification of acoustic modes in imploding spherical shells. For this purpose a self-consistent soluble ideal fluid model previously used by Kidder [1] is employed. The equation of state is taken to be a polytropic adiabatic law with arbitrary γ. The results indicate that amplification is determined by the value of γ, not the form of the density profile.

Consider an initially stationary spherical shell with density $\rho(r)$ in the region defined by $r_- \leqslant r \leqslant r_+$. A pressure loading $p_+(t)$ applied unformly at $r = r_+$ causes it to begin imploding radially with velocity $u(r)$. Assuming the shell material to be an ideal fluid, the motion is described in Lagrangian variables by the equations

$$\dot{\rho} + \rho R^{-2} \frac{\partial}{\partial R} (R^2 u) = 0; \tag{2}$$

$$\rho \dot{u} + \frac{\partial p}{\partial R} = 0; \tag{3}$$

$$(p \rho^{-\gamma})^{\cdot} = 0, \tag{4}$$

where $R(r, t)$ is the position at time t of a fluid element whose initial position was r, and time derivatives are denoted by dots.

The class of solutions of Eqs. (2-4) which describe what is known as homogeneous self-similar motion results from assuming the existence of a function $f(t)$ such that for any fluid element

$$R(r, t) = rf(t), \tag{5}$$

with $f(0) = 1$, $\dot{f}(0) = 0$. From Eq. (5) it follows that $u(r, t) = r\dot{f}$, while the solution of Eq. (2) is

$$\rho(r, t) = \rho_0(r) f^{-3}. \tag{6}$$

Hence from Eq. (4), the pressure must be of the form

$$p(r, t) = s(r)\rho^{\gamma} = s(r)\rho_0^{\gamma} f^{-3\gamma} \tag{7}$$

For self-consistency, the driving pressure must balance the pressure at $r = r_+$ found from Eq. (7), i.e., $p_+(t) = p(r_+, t) = s(r_+)\rho_0^{\gamma}(r_+) f^{-3\gamma}$. Substituting Eqs. (5-7) in the force law (3), we obtain

$$-\ddot{f} f^{\alpha+1} = \tau^{-2} = \frac{1}{\rho_0 r} \frac{dp_0}{dr} = \frac{1}{\rho_0 r} \frac{d}{dr} (s\rho_0^{\gamma}) \tag{8}$$

where $\alpha = 3(\gamma - 1)$ and τ is a separation constant.

The time-dependent part of Eq. (8) can be integrated once to give

$$\tau\dot{f} = -\left[\frac{2}{\alpha} (f^{-\alpha} - 1)\right]^{1/2} \tag{9}$$

for $\gamma > 1$, and

$$\tau\dot{f} = -(2 \ln f^{-1})^{1/2} \tag{9'}$$

for $\gamma = 1$. Equations (9) and (9′) distinguish the motions called Type II by Sedov [2], who first investigated them. Both equations evidently give rise to functions f which vanish at some finite time t_0, when the system becomes singular (implodes completely).

The spatial part of Eq. (8) yields a solution for each non-negative choice of s or of ρ_0. We select two cases for investigation: (i) Isentropic motion, $s = \hat{s} =$ const; (ii) Uniform density, $\rho_0 = \hat{\rho} =$ const. Isothermal ($\gamma = 1$) motion is distinguished as subcases (i)′ and (ii)′, respectively. In case (i) we find

$$\rho_0 = \hat{\rho}\left[\frac{\hat{\rho}(\gamma - 1) r^2}{2\hat{p}\gamma\tau^2} - 1\right]^{1/(\gamma-1)} \tag{10}$$

Here $\hat{\rho}$ and $\hat{p} = \hat{s}\hat{\rho}^{\gamma}$ are constants. In case (i)′

$$\rho_0 = \hat{\rho}\ \exp[\hat{\rho} r^2/(2\hat{p}\tau^2)]. \tag{10'}$$

For case (ii) we find, independently of γ,

$$s = \hat{s}[\hat{\rho} r^2/(2\hat{p}\tau^2) - 1]. \tag{11}$$

In order to restrict the problem somewhat, we choose $r_-^2 = 2\gamma\hat{p}\tau^2/(\gamma - 1)\hat{\rho}$ in case (i), making $\rho_0(r_-) = 0$, and $r_-^2 = 2\hat{p}\tau^2/\hat{\rho}$ in case (ii), making $s(r_-) = 0$, so that in both cases $p_- = p(r_-) = 0$. Then we express r in units of r_-. For case (i)′, $\rho(r)$ nowhere vanishes, so we take $r_- = 0$ and normalize r with respect to $(\hat{p}\tau^2/\hat{\rho})^{1/2}$. Time is conveniently expressed in units of τ and density in units of $\hat{\rho}$.

We begin by investigating linear stability with respect to compressible perturbations for case (i). It is convenient to employ the formulation developed by Book and Bernstein [3,4]. The fluid element whose unperturbed motion is described by $R(r, t)$ is displaced to $\mathbf{R}(r, t) + \xi(r, t)$. Expressing the perturbed density and pressure in terms of the Lagrangian displacement ξ and the unperturbed quantities, we obtain

$$f^{\alpha+2}\ddot{\xi} = \nabla\left[\frac{\partial p_0}{\partial \rho_0}\ (\nabla \cdot \xi + \xi \cdot \nabla\ \ln \rho_0)\right] - \xi \cdot \nabla\left[\frac{\partial p_0}{\partial \rho_0}\ \nabla\ \ln \rho_0\right]. \tag{12}$$

The free surface boundary condition associated with Eq. (12) is just $\nabla \cdot \xi = 0$ at $r = r_{\pm}$.

In Eq. (12) the space and time dependence are completely separated. An equation for $\sigma = \nabla \cdot \xi$ is derived by applying the divergence operator to Eq. (12), assuming $\nabla \times \xi = 0$. The result in normalized units is

$$f^{\alpha+2}\ddot{\sigma} = \frac{\gamma - 1}{2}\ \nabla^2[(r^2 - 1)\sigma] + r\ \frac{\partial \sigma}{\partial r} + \sigma. \tag{13}$$

We look for a solution in the form $\sigma = \mathcal{F}(t)\, W(r)\, Y_{lm}(\theta, \phi)$, where Y_{lm} is the usual spherical harmonic. Two ordinary differential equations result:

$$\frac{\gamma - 1}{2r^2} \left\{ \frac{d}{dr} \left[r^2 \frac{d}{dr} \left((r^2 - 1) W \right) - l(l+1)(r^2 - 1) W \right] \right\} + r \frac{dW}{dr} + (\mu + 1) W = 0 \tag{14}$$

and

$$f^{\alpha+2} \ddot{\mathcal{F}} = - \mu \mathcal{F}, \tag{15}$$

where μ is a separation constant.

Equation (14) is solved together with the boundary conditions $W(r_{\pm}) = 0$. First we show that the eigenvalue μ must be positive. Letting $U = (r^2 - 1) W$ and $x = r^2$, we can rewrite Eq. (14) in the Sturm-Liouville form

$$4 \frac{d}{dx} \left[x^{3/2} (x - 1)^{\frac{1}{\gamma - 1}} \frac{dU}{dx} \right] + \left[\frac{\mu + 1}{\gamma - 1} 2x^{1/2} (x - 1)^{\frac{1}{\gamma - 1} - 1} - \frac{4}{\gamma - 1} x^{3/2} (x - 1)^{\frac{1}{\gamma - 1} - 2} - l(l+1) \right] U = 0. \tag{16}$$

Multiplying Eq. (16) by U and integrating from 1 to r_+^2 in the usual way yields an expression for μ as the ratio of two positive quantities, whence $\mu > 0$.

The radial eigenfunction can be written as a superposition of two independent integrals of Eq. (14), expressible in terms of hypergeometric functions ${}_2F_1$. For example

$$W(r) = C_+ r^l {}_2F_1 \left[\frac{1}{2} (b_+ + d_+),\ \frac{1}{2} (b_+ - d_+);\ 2 + \frac{1}{\gamma - 1};\ 1 - r^2 \right] + C_- r^{-l-1} {}_2F_1 \left[\frac{1}{2} (b_- + d_-),\ \frac{1}{2} (b_- - d_-);\ 2 + \frac{1}{\gamma - 1};\ 1 - r^2 \right], \tag{17}$$

where

$$b_{\pm} = 2 + \frac{1}{\gamma - 1} \pm \left(l + \frac{1}{2} \right), \tag{18}$$

$$d_{\pm} = \left\{ b_{\pm}^{2} - \frac{2\mu + 1}{\gamma - 1} - \frac{11}{4} + \left[l + \frac{1}{2} \right] \left[l + \frac{1}{2} \mp \left(4 + \frac{2}{\gamma - 1} \right) \right] \right\}^{1/2} \tag{19}$$

The eigenvalue μ and the constants $C_{\pm}$ are chosen to satisfy the boundary conditions and a normalization condition.

Evaluation of μ in general must be carried out numerically, for example using the variational principle associated with Eq. (15). Approximate results can be obtained analytically in a number of interesting limits, however. Thus in the thin-shell limit the WKB approximation yields $\mu \simeq \frac{\gamma - 1}{2} \left(\frac{n\pi}{r_+/r_- - 1} \right)^2$, $n = 1, 2, \ldots$. For thick shells ($r_+ \to \infty$) the ground state eigenvalue found using WKB is $\mu \simeq 2(\gamma - 1)l(l + 1)$. The spectrum can readily be shown to have the following properties in general: (1) It is discrete, contains an infinite number of modes and is unbounded above; (2) the lowest level (minimum) value of μ increases with increasing l and γ and (3) decreases with increasing shell thickness, as also does the level separation. Clearly $\mu^{1/2}$, which depends on the value of the indices l and n (but not m) is just the dimensionless characteristic frequency.

Given μ, two independent solutions of the time-dependent Eq. (15) can be written

$$\mathcal{F}(t) = {}_2F_1 \left(\frac{1}{4} + \frac{2 + i\Delta}{4\alpha}, \frac{1}{4} + \frac{2 - i\Delta}{4\alpha}; \frac{1}{2}; 1 - f^{-\alpha} \right); \tag{20}$$

$$\mathcal{G}(t) = \left[\frac{2}{\alpha} (f^{-\alpha} - 1) \right]^{1/2} {}_2F_1 \left(\frac{3}{4} + \frac{2 + i\Delta}{4\alpha}, \frac{3}{4} + \frac{2 - i\Delta}{4\alpha}; \frac{3}{2}; 1 - f^{-\alpha} \right), \tag{21}$$

where $\Delta = [8\mu\alpha - (\alpha + 2)^2]^{1/2}$. At $t = 0$, $\mathcal{F}(0) = 1$, $\dot{\mathcal{F}}(0) = 0$ and $\mathcal{G}(0) = 0$, $\dot{\mathcal{G}}(0) = 1$. As $f \to 0$, the standard asymptotic formulas yield

$$\mathcal{F}(t) \sim \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(-\frac{i\Delta}{2\alpha}\right)}{\Gamma\left(\frac{1}{4} + \frac{2 - i\Delta}{4\alpha}\right) \Gamma\left(\frac{1}{4} - \frac{2 + i\Delta}{4\alpha}\right)} f^{\frac{\alpha + 2 + i\Delta}{4}} + c.c.; \tag{22}$$

$$\mathcal{G}(t) \sim \sqrt{\frac{2}{\alpha}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{i\Delta}{2\alpha}\right)}{\Gamma\left(\frac{3}{4}+\frac{2-i\Delta}{4\alpha}\right)\Gamma\left(\frac{3}{4}-\frac{2+i\Delta}{4\alpha}\right)} f^{\frac{\alpha+2+i\Delta}{4}} + c.c. \tag{23}$$

Evidently, the relative amplification $\mathcal{F}/f$ (or $\mathcal{G}/f$) consists of a rapidly oscillating part multiplied by $f^{\frac{1}{4}(\alpha-2)} = f^{\frac{3}{4}(\gamma-5/3)}$. If $\gamma < 5/3$ $(\gamma > 5/3)$ this factor diverges (vanishes) as $f \rightarrow 0$. For $\gamma = 5/3$, the perturbations and the size of the shell go to zero at the same rate. The constants multiplying the factors containing f depend weakly on μ through Δ, however, both decreasing monotonically as μ increases. Figure 1 displays the absolute values of these constants as functions of μ for $\alpha = 1(\gamma = 4/3)$. Since μ increases with both l and n, lower-order modes have larger amplitudes then higher ones, hence are more disruptive.

We note that by Eqs. (6) and (7), $p \sim f^{-3\gamma}$, and so the limiting dependence of Eqs. (22) and (23) on $p(t)$ is precisely that given by Eq. (1). Of course, to study the temporal development of the perturbations in detail it is necessary to integrate Eq. (15) [or evaluate Eqs. (20) and (21)] numerically. This presents no difficulty until times when the oscillation period becomes very small, at which stage the asymptotic formulas can be used.

Case (i)′ $(\gamma = 1)$ is treated in very similar fashion. Equation (13) is replaced by

$$f^2\ddot{\sigma} = \nabla^2\sigma + r\,\frac{\partial\sigma}{\partial r} + \sigma, \tag{13'}$$

which again separates. The space- and time-dependent equations are both solvable in terms of confluent hypergeometric functions. The spectrum obtained from the former is qualitatively like that found for $\gamma > 1$ in all important respects. The solutions of the time-dependent equation,

$$f^2\ddot{\mathcal{F}} = -\,\mu\mathcal{F}, \tag{15'}$$

may be written as

$$\mathcal{F}(t) = \Phi\left(\frac{1}{2}\mu, \frac{1}{2}; \ln f\right) \tag{20'}$$

and

$$\mathcal{H}(t) = \Psi\left(\frac{1}{2}\mu, \frac{1}{2}; \ln f\right), \tag{21'}$$

where Φ and Ψ are the standard solutions of the confluent hypergeometric equation. As $f \to 0$, the asymptotic forms of the latter yield

$$\mathcal{F}(t) \sim \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu\right)} f\,(\ln f)^{\frac{1}{2}(\mu-1)} \tag{22'}$$

and

$$\mathcal{H}(t) \sim (\ln f)^{-\frac{\mu}{2}}. \tag{23'}$$

Thus only the former solution gives rise to an amplification, with $\mathcal{F}(t)/f$ diverging as a power of $\ln f$.

Turning next to example (ii), the uniform density case, we begin by noting that for an anisentropic unperturbed state, the terms $\frac{1}{\rho_0}\left[\nabla\left(\frac{\partial p_0}{\partial s}\,\underset{\sim}{\xi}\right)\cdot\nabla s - \nabla\cdot\left(\frac{\partial p_0}{\partial s}\,\underset{\sim}{\xi}\right)\nabla s\right]$ must be added to the right hand side of Eq. (12). In scaled variables, the equation for ξ is now

$$f^{\alpha+2}\ddot{\underset{\sim}{\xi}} = \frac{\gamma}{2}\nabla[(r^2-1)\nabla\cdot\underset{\sim}{\xi}] + \nabla\underset{\sim}{\xi}\cdot\mathbf{r} - \mathbf{r}\nabla\cdot\underset{\sim}{\xi}. \tag{24}$$

It is no longer possible to obtain a solution with $\underset{\sim}{\omega} = \nabla\times\underset{\sim}{\xi} = 0$. Instead we find two coupled equations,

$$f^{\alpha+2}\ddot{\sigma} = \frac{\gamma}{2}\nabla^2[(r^2-1)\sigma] - 2\sigma - \mathbf{r}\cdot\nabla\times\underset{\sim}{\omega} \tag{25}$$

and

$$f^{\alpha+2}\ddot{\underset{\sim}{\omega}} = -\underset{\sim}{\omega} + \mathbf{r}\times\nabla\sigma. \tag{26}$$

Elimination of $\underset{\sim}{\omega}$ between Eqs. (25) and (26) yields an equation for σ which is fourth order in time and second order in space:

$$f^{\alpha+2}(f^{\alpha+2}\ddot{\sigma})^{\cdot\cdot} = f^{\alpha+2}\left[\frac{\gamma}{2}\nabla^2[(r^2-1)\ddot{\sigma}] - 3\ddot{\sigma}\right]$$

$$+ \frac{\gamma}{2}\nabla^2[(r^2-1)\sigma] - r^2\nabla^2\sigma + \mathbf{rr}{:}\nabla\nabla\sigma + 2\mathbf{r}\cdot\nabla\sigma - 2\sigma. \qquad (27)$$

A solution is again found by separation of variables, with the time dependence given by Eqs. (15) and (15′), and the radial dependence expressible in the form Eq. (17). The dependence on μ of $d_{\pm}^2$ is now nonlinear. The solution of the eigenvalue problem leads to two values of μ, one positive and one negative, of which only the former can give rise to amplification. No special treatment is required in the limit $\gamma \to 1$, which is now nonsingular. The spectrum is qualitatively similar to that found for case (i). The only changes in the time history of a perturbation are those ascribable to the changed values of μ. It is clear that this time dependence is a general property of the homogeneous self-similar motion on which the present examples are based.

Thus in the two examples treated in detail here, sound waves standing in an imploding spherical shell can become large in amplitude compared with the diminishing shell dimension, at the same rate in each instance. From this it may be concluded that the result is insensitive to the form of the density profile. Since the time dependence agrees asymptotically with the model-independent estimate of Eq. (1), it is unlikely that the shape of the pressure pulse plays an important role. Because the long-wavelength modes have the fastest relative amplification, neither viscosity or other dissipative processes are likely to provide a way of controlling them. Thermal conduction, which reduces the effective γ, only makes matters worse. Layering might help by raising the natural vibration frequencies. The tendency of all materials to behave incompressibly ($\gamma >> 1$) at high densities, when the Fermi energy becomes large, means that the relative amplification cannot really diverge. To determine for a given initial vibration level the actual limit on aspect ratio or compression imposed by the mechanism discussed here requires calculations using a more detailed model.

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References

1. R. E. Kidder, Nucl. Fusion **16,** 3 (1976).
2. L. I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic Press, New York, 1959), pp. 271-281.
3. D. L. Book and Ira B. Bernstein, Phys. Fluids (to be published).
4. Ira B. Bernstein and D. L. Book, Ap. J. (to be published).

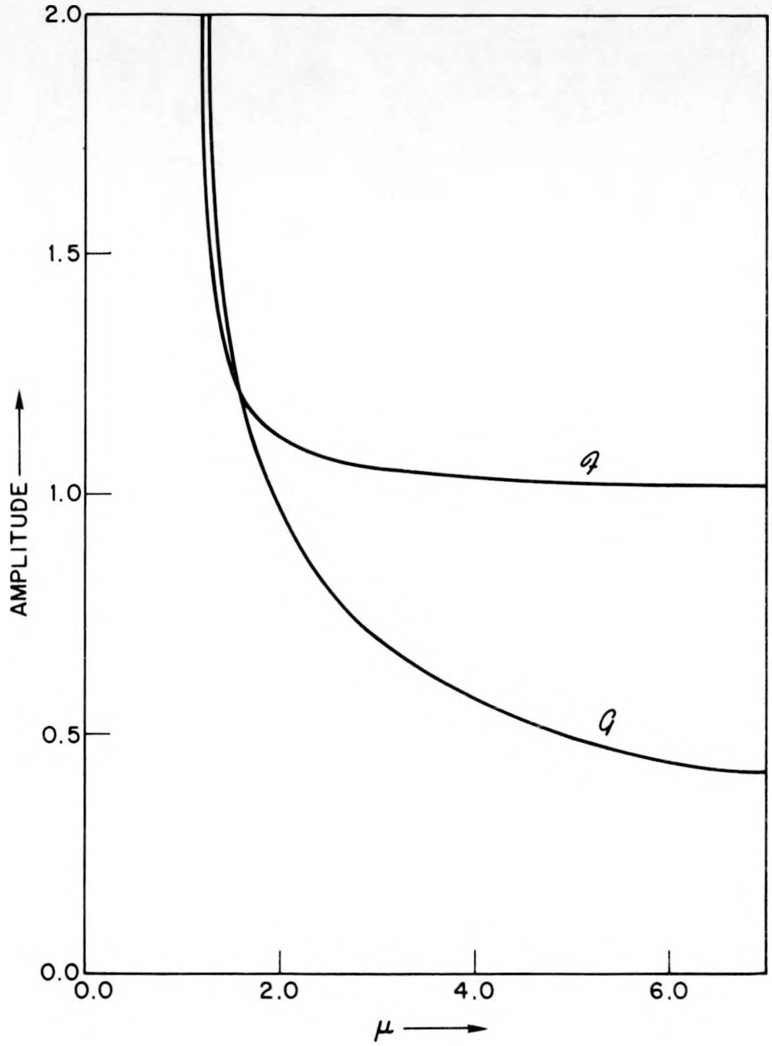

Figure 1 — Asymptotic numerical factors $|F/f^{\frac{\alpha+2}{4}}|$, $|G/f^{\frac{\alpha+2}{4}}|$ defined by Eqs. (22-23) as functions of μ for $\alpha = 1$.