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New Phases of D ≥ 2 Current and Diffeomorphism Algebras in Particle Physics*

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ABSTRACT

We survey some global results and open issues of current algebras and their canonical field theoretical realizations in D≥ 2 dimensional spacetime. We assess the status of the representation theory of their generalized Kac-Moody and diffeomorphism algebras. Particular emphasis is put on higher dimensional analogs of a) fermi-bose corrrespondance b) complex analyticity and c) the phase entanglements of anyonic solitons with exotic spin and statistics.

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0.Introduction

As one who thought deeply about all aspects of symmetries Hermann Weyl[1]had traced their origin in nature to the mathematical character of physical laws. In the last thirty years, the developments in particle physics have been dominated by one single theme, the exploitation of symmetries. They can be either exact or approximate, ultimately fundamental or effective[2]. With its unqualified successes the use of symmetries has become synonymous with that of Lie algebras and groups. In the early seventies the mathematician, Jean Dieudonné [3] wrote : " Les groupes de Lie sont devenus le centre des mathematiques; on ne peut rien faire de sérieux sans eux" . In this era of gauge and string theories, we may, without much exaggeration, assert the preeminent role at the frontiers of physics of infinite dimensional Lie group theory by replacing the words "des mathematiques" above by " de la physique théorique" .

In a broader perspective, with the coming of age of gauge theories, string theories and D=2 conformal field theories, the range of applicable mathematics seems limited only by one's ingenuity and imagination . It spans the gamut of all major branches of 19th century and modern mathematics, from Riemann surfaces to hyperkähler manifolds, from infinite Lie groups to non-commutative geometry, from knots and links to p-adic numbers and analysis. As will be illustrated below, all of these apparently disparate structures are often brought together through the intermediary of one set of physical phenomena. This linkage reflects both the unity of mathematics as well as its unreasonable effectiveness in accounting for the physical world.

We have certainly gone a long way from the Young tableaux and Clebch-Gordan series in finite parameter Lie algebras applied to global (flavor) then local (gauge) symmetries of point particles to the full use of the representation theory of infinite parameter (super-)Virasoro-Kac-Moody algebras in string, 2-dimensional conformal and integrable field theories. Indeed solving for two dimensional quantum field theories is almost equivalent to solving for the representation theories of the loop and/or Virasoro groups. Though the task is rather difficult we dream of a parallel outcome in four dimensions. So while we started out by often invoking symmetries as substitutes for dynamics we have ended up fulfilling the old Einsteinian dictum " symmetry dictates dynamics".

In his instructions to the speakers at this Symposium, Professor Gruber commissioned comprehensive reviews aimed not just at physicists using symmetries in their research but also at experts in other areas of sciences. This criterion has partly guided my choice of topics. My special interests are in algebraic and topological structures in particle physics. Accordingly, I shall take as my main and unifying theme, a few global aspects of Kac-

Moody-Virasoro typed algebras, the representation theory of their current and diffeomorphism groups, seen in the context of a few concrete semi-topological field theories with solitons in D≥2 spacetime dimensions.

In these alotted pages, a truly comprehensive review is admittedly out of the question. I shall therefore not dwell on the numerous well established and extensively reviewed results of two dimensional. Rather using the latter as standards, I will focus on a few basic developments in D≥ 2 dimensions and discuss their open problems. My threefold emphasis will be on a) the question of fermi-boson equivalence or D≥3 bosonization, b) the possibility of anyonic transmutation and c) the role of complex and hypercomplex analyticity. These topics best illustrate some natural directions toward a nonperturbative, algebraic understanding of D≥2 dimensional quantum field theories. Four related topics are singled out for discussion:

1) To introduce the basic concepts and notations, a brief review of 2-cocycles as central extensions of D=2 current algebras, its equivalent fermionic and bosonic representations, via the Wess-Zumino- Novikov-Witten (WZNW) model. The complex analytic structure of the Kac-Moody-Virasoro algebras.

2) Going behond affine Lie algebras, D=4 current algebra with it q-number, non central, Abelian extension, its canonical realization in a Skyrme model with a Wess-Zumino term. Attempts at constructing vertex operators and a representation theory. Generalized fermi-bose correspondence and comments on hypercomplex analyticity of generalized Kac-Moody-Virasoro algebras.

3) An realization of D=3 current and diffeomorphism algebras in the CP_1 σ-model with a Chern-Simons-Hopf term. An anyonic vertex operator construction. A generalized spin and statistics connection by way of the Gauss-Bonnet theorem . Its relation to self-linking, twisting,writhing numbers of Feynman paths.

4) Going behond D=3 anyons, exceptional D= 7, 15 anyonic Hopf 2- and 3-membranes and their connection to division algebras via Adams' theorem. Comments on their current, diffeomorphism algebras.

These topics will be covered respectively in sections 1 to 4 , section 5 encloses some parting remarks. Our treatment of established results will be brief and primarily conceptual . For proofs and greater details we refer the interested reader to our long, though incomplete list of references.

1. D=2 Kac-Moody Groups , Fermionization and Complex Analyticity

1.1 Current Algebras and Cocycles

For those unfamilar with current algebras,a few brief historical remarks may be in order. Comprehensive accounts of current algebras are to be found in the classic books by Adler and Dashen[4] and Ne'eman [5] and in a modern update by Treiman, Jackiw, Zumino and Witten [6]. Before the advent of gauge theories, amids the profusion of hadronic states the introduction of current algebras was motivated by the unifying idea that the basic objects for strong interaction physics should be the observable currents rather than the then still elusive fundamental fields. Thus, while the electromagnetic interactions among all charged particles are governed by the interaction Hamiltonian

$$H_e = e \int d^3x \, j_\mu(x) \, A^\mu(x) \, , \tag{1.1}$$

e being the electric charge , j_μ the electric current and A_μ the elctromagnetic potential, the leptonic and non-leptonic weak decays of hadrons are effectively accounted for by the interaction Hamiltonians

$$H_l = G \int d^3x \, j^h_\mu(x) \, j^l_\mu(x) \, , \quad H_{nl} = G \int d^3x \, j^h_\mu(x) \, j^h_\mu(x) \quad , \tag{1.2}$$

G is the Fermi-coupling constant, $j^h{}_\mu$ (x) and $j^l{}_\mu(x)$ denote the weak currents of the hadrons and the leptons respectively. At a fixed time t , the currents $j_\mu(x)$ are mappings from physical space into some internal space of a symmetry group G .

From these $j^\alpha_\mu(x)$'s , one compute the corresponding charge operators

$$Q^\alpha = \int d^3x \, j^\alpha_0(x) \, . \tag{1.3}$$

Essentially the fundamental hypothesis of current algebra was that , irrespective of the details (or even of the existence) of an underlying quantum field theory, the charges and current close under an algebra of equal- time canonical commutation relations. In order of their reliability, the postulated relations are of the generic forms of

a) a charge-charge algebra

$$\left[Q^{\alpha}(t)\ ,\ Q^{\beta}(t)\right] = if^{\alpha\beta\gamma}Q^{\gamma}(t)\ , \qquad (1.4)$$

b) a mixed charge -current algebra

$$\left[Q^{\alpha}(t)\ ,\ j^{\beta}_{\mu}(\mathbf{x},\ t)\right] = if^{\alpha\beta\gamma}j^{\gamma}_{\mu}(\mathbf{x},t)\ , \qquad (1.5)$$

c) a current-current algebra

$$\left[j^{\alpha}_{0}(\mathbf{x})\ ,\ j^{\beta}_{0}(\mathbf{y})\right] = if^{\alpha\beta\gamma}j^{\gamma}_{0}(\mathbf{x})\ \delta^{3}(\ \mathbf{x} - \mathbf{y}\ ), \qquad (1.6)$$

$$\left[j^{\alpha}_{0}(\mathbf{x})\ ,\ j^{\beta}_{i}(\mathbf{y})\right] = if^{\alpha\beta\gamma}j^{\gamma}_{i}(\mathbf{x})\ \delta^{3}(\ \mathbf{x} - \mathbf{y}\ ) + S^{\alpha\beta}_{0i}(\ \mathbf{x},\mathbf{y})\ , \qquad (1.7)$$

where $f^{\alpha\beta\gamma}$ are the structure constants of the algebra of the symmetry group G , e.g. G ≈ SU(N), SU(N)xSU(N) , N= 2 , 3. The additional matrix valued term $S(\mathbf{x}, \mathbf{x}')$ in (1.7) is the celebrated singular Schwinger term.

One particular feature must be noted. In the old current algebras, the Schwinger terms are highly model dependent and occur only in the space-time current commutator (1.7) while their modern cousins, being of topological origin, have more restricted forms and appear in the time-time current commutator or *local charge algebra* (1.6) . As such these topological extension terms, being the repositories of "good" or "bad" anomalies, have important implications on new physical effects or on the over all quantum consistency of the associated (gauge) field theory.

Clearly (a) and (b) are but special integrated form of (c) . We recall that (1.4) and (1.5) were widely and successfully used. Similarly several sum rules were derived from (1.6) and agreed reasonably well with experiments. Today the above algebras are seen to arise from the underlying dynamics of quantum chromodynamics and the standard model of electroweak interactions.

Looking back ,what physicists missed during the 60's was the possible topological significance of the Schwinger term(s). At one time Gell-Mann even banned these terms by decree from his universal current algebra. Yet they are necessarily present for consistency with Lorentz invariance and energy positivity. Its form is constrained by the associativity of the algebra, i.e. the Jacobi identity . It turns out that these singular terms are the residual local signatures of nontrivial 2-cocycles or projective representations of quantum systems with an infinite numbers of degrees of freedom and with topologically nontrivial configuration spaces.

This fact was realized early on by I.M. Gelfand and his followers [7]who pioneered the representation theory of current groups in arbitrary dimensions. Unfortunately the relevance of their works musts await the coming of age of affine and loop algebras, ushered in by the advent of superstring and conformal field theories.

In the title of this review, by "new phases" we mean the nontrivial phases of the projective representations of infinite dimensional algebras, the 2-cocycles or Berry's phases[8]. We next recall their mathematical and physical meanings.

It is commonly said that the phrases of the complex Schrödinger wave function Ψ of a quantum system do not matter since physically observable effects are determined by the real norm $|\Psi|^2$. This statement is true provided Ψ is the wave function for the whole system, which is seldom the case in practice or even in principle[9]. As the Aharonov-Bohm effect [10]and the rich theoretical analyses and experimental confirmations of the Berry phase recently show, the relative phases of wave functions describing a part of the entire system are most relevant and physically detectable. Moreover they often have drastic and stunning effects on the properties of the subsystem(s) in question. These*anomaly phenomena* to be illustrated here by the fermi-bose equivalence in D=2 quantum field theory, the emergence of the D=4 baryonic topological soliton, the Skyrmion, from QCD, by the anyonic membrane excitations in odd dimensional , semi-topological field theories etc...

What are cocycles ?[9]Consider a quantum system Σ with a symmetry group G of transformations T(g) . For each fixed g , T(g) can be represented up to a phase factor $e^{i\omega_1(q,\, g)}$ by an (anti-) unitary operator U(g) in a Hilbert space *H*. Let q be the dynamical variable(s) on which g acts : $q \to q^g$, A wave function $\Psi(q)$ transforms as

$$U(g)\, \Psi(q) = e^{i\omega_1(q,\, g)}\, \Psi(q^g) \quad . \tag{1.8}$$

Consistency with the group composition law

$$U(g_1)\, U(g_2) = U(g_1 g_2) = U(g_{12}) \tag{1.9}$$

implies

$$\omega_1(q^{g_1};\, g_2) - \omega_1(q;\, g_{12}) + \omega_1(q;\, g_1) = 0 \;\; (\bmod Z) \, . \tag{1.10}$$

The real phase $\omega_1(q;g)$, a 1-cocycle, depends generally on both g and q . Specifically if Σ is a non-Abelian gauge (chiral field) theory , then in the Hamiltonian formalism, the g's, q's and $\Psi(q)$ correspond respectively to local gauge (global chiral) transformations, the spatial components of the gauge potential **A** (chiral current **J**) and the Schrödinger wavefunctional $\Psi(\mathbf{A})$.

Similarly, to the group relation (1.9) corresponds the composition law

$$U(g_1)\, U(g_2) = e^{-2\pi i\, \omega_2(q; g_1, g_2)} U(g_{12}) \tag{1.11}$$

Associativity of (1.11) leads to the consistency condition

$$\omega_2(q^{g_1}; g_2, g_1) - \omega_2(q; g_{12}, g_3) + \omega_2(q; g_1, g_2) - \omega_2(q; g_1, g_2) = 0 \pmod{Z}\,. \tag{1.12}$$

Such a phase is a 2-cocycle and the unitary representations bearing it are the *ray or projective* representations of frequent occurence in quantum theory.

One could continue this process and abstractly define higher cocycles. Thus the 3-cocycle is given through

$$(U(g_1)\, U(g_2))\, U(g_3) = e^{-2\pi i\, \omega_3(q; g_1, g_2, g_3)}\, U(g_1)\, (U(g_2) U(g_3))\,. \tag{1.13}$$

However it violates associativity; moreover nonassociative entities cannot be represented by linear operators in a Hilbert space. As of now no physical effect is attributable to 3 - or higher cocycles. So we shall limit ourselves here to 2-cocycles as we consider next the global aspects of the affine Kac-Moody algebras in 2-spacetime dimensions.

1.2 D= 2 Kac-Moody Groups

There exist several approaches to construct D=2 Kac-Moody groups. We adopt the simple and instructive construction of Mickelsson [11] as it readily generalizes to higher dimensions; specifically for the case of D=4. The latter's papers should be consulted for greater details. Consider (suitable smooth) mappings where the target space is a finite dimensional Lie group G and the base space, the unit circle $S^1 \approx \partial D = \{ z \in \mathbf{C} \mid |z| = 1 \}$. This S^1 is seen as a boundary of an unit disc $D = \{ z \in \mathbf{C} \mid |z| \leq 1 \}$. Let LG be the space of loops $f : S^1 \to G$ and $\Omega G \approx \{ f \in LG \mid f(1) = 1 \}$ the space of based loops. While LG and ΩG both has a natural group structure under point-wise multiplication ,namely given two maps γ_1 and $\gamma_2 : S^1 \to G$, their product composition is $\gamma_1 \cdot \gamma_2 : S^1 \to G$ such that $\gamma_1 \cdot \gamma_2(z) = \gamma_1(z) \cdot \gamma_2(z)$, only ΩG is a C^∞- manifold . Now let $DG = \{ f : D \to G \mid f(1)=1 \}$, the space of based smooth maps from D into G and let $\pi : DG \to \Omega G$ be the natural projection, $\pi(f) \equiv f|_{S^1}$. Then the triple (DG , π , ΩG) is a principal fibre bundle with as its structure group $\mathcal{G} = \{ f : D \to G \mid f(1)=1 , x \in S^1 \}$ acting on DG frcm the right..

By contracting $S^1 \approx \partial D$ to a single point , the North pole of S^2, $\mathcal{G}$ can alternatively be the space $\{ f : D \to G \mid f(\text{North pole of } S^2) =1 , x \in S^1 \}$. Then for $f \in DG$, $g \in \mathcal{G}$ define the 1-cocycle ω_1

$$\omega_1(f,g) = \frac{\psi^2}{16\,\pi^2}\int_D \langle f^{-1}\partial_\alpha f , \partial g_\beta\, g^{-1}\rangle dx_\alpha \wedge dx_\beta + C(g) \qquad (1.14)$$

with

$$C(g) = \frac{-\psi^2}{48\,\pi^2}\int_{D_3} \left\langle \tilde{g}^{-1}\partial_\alpha \tilde{g} , \frac{1}{2}\left[\tilde{g}^{-1}\partial \tilde{g}_\beta, \tilde{g}^{-1}\partial_\gamma \tilde{g}\right]\right\rangle dx_\alpha \wedge dx_\beta \wedge dx_\gamma \qquad (1.15)$$

where < ... > denotes the Kiling form on **g** , the Lie algebra of G . In (1.15) the map $\tilde{g}$: $D_3 \to G$, D_3 being a 3-dimensional unit ball , is now an arbitary extension of $g : S^2 \to G$. ψ^2 is the length squared of the longest root of **g** . If $\tilde{g}_1$ and $\tilde{g}_2$ are two extensions of the same g , then $C(\tilde{g}_1) - C(\tilde{g}_2) \in Z$, so that the phase $\exp(2\pi i\, \omega_1)$ is well defined. The 1-cocycle ω_1 allows one to define for $g \in \mathcal{G}$ in the foregoing extension DG x U(1) the following equivalence relation " ~ "

$$(f , \lambda) \sim (f g , \lambda \exp\{ 2\pi\, i\, \omega_1(f, g)) \qquad (1.16)$$

whose transitivity property is but the 1-cocycle consistency condition satisfied by ω_1 (1.10) . The Kac-Moody group $\mathcal{G}$ is then a principal U(1) bundle P over the loop apace $\Omega L \approx$ Map(S^1 , G) , $P \approx \{DG \times U(1) / \sim\}$. The right action of U(1) in DG x U(1) commutes with the g-action; U(1) acts on P . So the KM group $\mathcal{G}$ can be defined by the pairs (f, λ) , $f \in$ Map(S^1 , G) , with a multiplication law

$$(f, \lambda) (f',\lambda') = (f f' , \lambda \lambda' \exp\{ 2\pi\, i\, \omega_2(f , f') \} , \qquad (1.17)$$

where

$$\omega_2(f, f') = \frac{\psi^2}{16\pi^2}\int_D \langle f^{-1}\partial_\alpha f , \partial_\beta f'\, f'^{-1}\rangle dx_\alpha \wedge dx_\beta \qquad (1.18)$$

ω_2 (f, f') satisfies (1.12) so $\exp (2\pi i\, \omega_2)$ is then a U(1) valued 2-cocycle in DG . The bundle P is a group; a central extension of ΩG by $U(1)$.

So the Kac-Moody algebra **g** of G is a 1-dimensional central extension of loop algebra Map (S^1 , **g**) . Given f_1 and f_2 ε Map (S^1 ,G) the Lie algebra cocycle corresponding to (1.18) is simply given by Map (S^1 , **g**) with the commutator is defined point-wise as

$$[f_1 , f_2] (\theta) \equiv [f_1(\theta) , f_2(\theta)] . \tag{1.19}$$

The central extension is given by the Lie algebra 2-cocycle $c(f_1, f_2)$ corresponding to the group cocycle ω_2

$$c(f_1, f_2) = 4\pi \frac{d^2}{d\sigma\, d\tau} \omega_2(e^{\sigma f_1}, e^{\tau f_2})) |_{\sigma = \tau = 0}$$

$$= \frac{\psi^2}{4\pi} \int_D \langle \partial_\alpha \tilde{f} , \partial_\beta \tilde{f} \rangle d\, x_\alpha \wedge dx_\beta = = \frac{\psi^2}{4\pi} \int_0^{2\pi} \left\langle f_1(\theta) \frac{d}{d\theta} f_2(\theta) \right\rangle d\theta . \tag{1.20}$$

So if G = SU(N) , < X, Y > = Tr (XY) and $\psi^2 = 2$. Then (1.20) , which defines a symplectic, nondegenerate and closed Kirillov 2-form on ΩG leads to a modified commutator

$$[f_1(\theta) , f_2(\theta)] + i\, x\, c(f_1, f_2) . \tag{1.21}$$

Alternatively it takes the more familiar form of

$$\left[T^a_n , T^b_m\right] = f^{abc} T^c_{n+m} + \frac{i x \psi^2}{4\pi} m\, \delta^{ab} \delta_{m,-n} \tag{1.22}$$

Here $T^a_n = T^a e^{in\theta}$ are the Fourier components of f near the identity map in an orthonormal basis { T^a } (a = 1, ...dim **g**) of **g** with structure constants f_{abc} and where $x\,\psi^2 \,\varepsilon\, Z$. (1.22) shows the 1-dimensionality of the central extension. It is called a level k =1 Kac-Moody algebra (KMA) . A level k KMA is simply gotten by multiplying ω_2 in (1.18) by k ε Z .

The pervasive phenomenon of fermi-bose equivalence in 2-dimensions is best illustrated by two equivalent representations , one fermionic, the other bosonic of the same untwisted affine KMA (1.21) . Witten[12] considered a conformal invariant system of N free Majorana fermions with a non-Abelian chiral symmetry $G \approx O(N)$. In light cone or conformal coordianates $z \equiv z_+ = x + i\,y$ and $\bar{z} \equiv z_- = x - i\,y$, the Euclidean action reads

$$S(\psi, \overline{\psi}) = \frac{1}{2}\int d^2x \sum_{i=1}^{N} \left[\psi^i \partial_{\bar{z}} \psi^i + \overline{\psi^i}\, \partial_z \overline{\psi^i}\,\right]. \tag{1.23}$$

From second quantization the anticommutation relations read $\left\{ \psi^i , \psi^j \right\} = \hbar\, \delta(x-y)\, \delta^{ij}$.
Then equivalent to the Dirac equations for $\psi_i = \psi^i = \begin{pmatrix} \psi^i_+ \\ \psi^i_- \end{pmatrix}$, are the conservation laws

$$\partial_{\bar{z}} J^a_+ = \partial_z J^a_- = 0 \tag{1.24}$$

for the chiral currents

$$J^a_\pm = \frac{1}{2} :\psi^T_\pm M^a \psi_\pm: \quad , \tag{1.25}$$

M^a (a= 1,2,...N) are real skew symmetric NxN O(N) representation matrices. Consequently J^a_+ is only a function of z and J^a_- a function of $\bar{z}$, they also mutually commute, so they can be taken as independent. This shows the theory to be invariant under a much larger infinite invariance group $G(z) \times G(\bar{z})$ whose generators are J^a_+ and J^a_-. The resulting two commuting ∞-dimensional Kac-Moody algebras are

$$\left[J^a_\pm (z_\pm) \; , J^b_\pm(w_\pm) \right] = i\hbar\, f^{ab}{}_c\, J^c_\pm(w_\pm)\, \delta\,(z_\pm - w_\pm) + \frac{i\,\kappa_\lambda}{4\pi} \hbar^2 \delta^{ab}\, \delta'(z_\pm - w_\pm) \tag{1.26}$$

where κ_λ is, up to a representation free normalization, the Dynkin index of the representation: $-\kappa_\lambda\, \delta_{ab} = \mathrm{tr}(M^a M^b)$. In fact $c_\lambda d_\lambda = \kappa_\lambda \dim G$; $-c_\lambda = (M^a)^2$ is the value of the quadratic Casimir in the representation λ and $d_\lambda = N$ is the dimension of λ.

The existence and physical origin of the Schwinger term were in fact known to P. Jordan[13, 14] long before the works of Goto, Imamura [15] and Schwinger [16]. Indeed the validity of (1.26) presupposes a Dirac vacuum (i.e, 2nd quantization) ; specifically the condition for the global existence of such a fermion ground state is encoded in a local " deformation" of the algebra of currents by the addition of a Schwinger term. Furthermore, Jordan et al pointed out that the current commutator derives from their D=2 quantum massless spinor field

$$[J_1 \, , J_0] = \frac{i}{\pi}\partial_1\delta(x-y) \tag{1.27}$$

is reproduced exactly by the commutator $[\partial_1\phi \, , \partial_0\phi] = i \, \hbar \, \partial_1\delta(x-y)$ resulting from the Heisenberg relation $[\phi \, , \partial_0\phi] = i \, \hbar \, \delta(x-y)$ for a Bose field ϕ, provided one sets

$$J_\mu = : \overline{\psi} \, \gamma_5\gamma_\mu\psi := \left(\frac{\hbar}{\pi}\right)^{1/2}\partial_\mu\phi \, . \tag{1.28}$$

This mapping is the first example of fermi-bose equivalence or abelian *bosonization*. The $\hbar$ dependent factor in (1.28) testifies to its purely quantum character. Later on, another canonical example was established by Coleman[17]: the equivalence between the fermion of the massive Thirring model and the quantum soliton of Sine-Gordon model. The corresponding lagrangian densities are

$$L_{MT} = \frac{i}{2} \, \overline{\psi}\partial\!\!\!/\psi - m \, \overline{\psi} \, \psi - \frac{1}{2}g \, (\overline{\psi}\gamma_\mu\psi)^2 \tag{1.29}$$

$$L_{SG} = : \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \alpha\beta^{-1}(\, 1 - \cos(\beta\phi)): . \tag{1.30}$$

For subsequent comparison we only write down the "vertex operator" $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ for creating a point-like fermion as a topologically nontrivial bose field coherent state excitation. Pioneered by Skyrme ,its construction was completed by Mandelstam[18]

$$\psi_1 = N : \exp\left\{-2i \, \beta^{-1}\int_{-\infty}^{x} d\xi \, \dot{\phi} - \frac{1}{2} \, i\beta\phi\right\}: \tag{1.31a}$$

$$\psi_2 = - iN : \exp\left\{-2i \, \beta^{-1}\int_{-\infty}^{x} d\xi \, \dot{\phi} + \frac{1}{2} \, i\beta\phi\right\}: \tag{1.31b}$$

Characteristic features can be inferred from these explicit expressions. Here we observe that while the fermionic currents are local, the fermion fields themselves are nonlocal in terms of the field ϕ with nontrivial topology. Actually this short (local) to long (global) distance connection reflects a quite general a trademark of anomalies or quantum symmetry breaking. For an elaboration of this intriguing phenomenon, we recommend the excellent reviews of Morozov [19] and Shifman [20].

Subsequently Witten[12, 21] put forth a non-abelian extension of the above fermi-bose equivalence . His model is governed by the following semi-topological action [22]

$$S_{\lambda,k}(g) = \frac{1}{4\lambda^2}\int d^2\xi(\partial_\mu g^{-1}\partial_\mu g) + k\,\Gamma(g)\ . \tag{1.32}$$

$$\Gamma(g) = \frac{1}{24\pi^2}\int_D d^3Y\, \varepsilon^{\alpha\beta\gamma}\, Tr\left(\tilde{g}^{-1}\partial_\alpha\tilde{g}\tilde{g}^{-1}\partial_\beta\tilde{g}\tilde{g}^{-1}\partial_\gamma\tilde{g}\right), \tag{1.33}$$

with $g \varepsilon G$, namely a G_Lx G_R (say $G \approx O(N)$) D=2 invariant chiral model made up of the sum of the standard geometrical nonlinear σ-model and a topological action $\Gamma(g)$. This added Wess-Zumino term is defined over a 3-dimensional ball D (with coordinates Y^α) whose boundary is 2-spacetime. The boundary values of $g(\xi)$ determine (1.23) modulo 2π . Γ is an example of a multiplevalued action; the singlevaluedness of the Feynman action $\exp(iS_{\lambda,\,k})$ implies that the quantization of $k = n\hbar$.

A renormalization group analysis shows that the Wess-Zumino-Novikov-Witten (WZNW) model (1.32) has an infrared fixed point when $\lambda = \frac{4\pi}{k}$, it then reads

$$S_k(g) = \frac{k}{16\,\pi}\left\{\int d^2\xi(\partial_\mu g^{-1}\partial_\mu g) + \Gamma(g)\right\} \tag{1.34}$$

which is now invariant, exactly like the system (1.23), under the infinite dimensional Kac-Moody group $G(z)_L \times G(\bar{z})_R$, namely under the transformation $g(\xi) \rightarrow \Omega(z)\, g(\xi)\bar{\Omega}^{-1}(\bar{z})$

Indeed the equations of motions for (1.34) are the same as (1.24) if the $J^a_\pm$ are defined as

$$T^aJ^a_+ = -\,i\sqrt{2}\frac{n\hbar}{4\pi}g^{-1}(\partial_+ g) \tag{1.35}$$

$$T^aJ^a_- = -\,i\sqrt{2}\frac{n\hbar}{4\pi}(\partial_- g)g^{-1} \tag{1.36}$$

The T^a are generators of G . Then the obtained canonical Poisson brackets promoted to Dirac brackets yield in the case of n =1 the same KMA (1.17) of the massless O(N) fermion theory. The nonabelian bosonization rules are given by equating the $T^aJ^a_\pm$ from (1.35) with (1.25) . It can on fact be proved that the two theories (1.23) and (1.34) are dynamical identical.

Yet the translation dictionary for this generalized fermi-bose equivalence is still to our knowledge incomplete. Despite attempts, we still do not have the non-abelian counterparts of the vertex operators (1.31) giving the fermionic field in term of exponential of the non-abelian currents. What we have is the powerful Frenkel-Kac construction to be recalled subsequently.

1.3 Virasoro -Kac-Moody algebra : representations

Examples of conformally invariant field theories(CFT) are statistical mechanical systems at their critical points[23, 24]. The representation theory of the conformal group places contraints on the critical exponents and on correlation functions[25]. Since the two dimensional conformal group , Vir , the Virasoro-Botts group of diffeomorphisms [26, 27] of the circle is infinite dimensional, it has a very rich and powerful in structure [24, 28]. One could actually realized[25] the conformal bootstrap program of Polyakov[29] in two dimensionns. It amounts to solving for the representation theory of the Dirac-Schwinger algebra of the energy momentum tensor components, the generators of Vir . This will give a complete classification of all possible D=2 conformal field theories.

The basic objects of a CFT are the primary fields $\phi(z, \bar{z})$. They transform as tensors

$$\phi(z, \bar{z}) \rightarrow \phi(z, \bar{z})' = (\partial_z z')^h \left(\partial_{\bar{z}} \bar{z}'\right)^{\bar{h}} \phi(z', \bar{z}') \qquad (1.37)$$

under conformal transformations $z \rightarrow z' = f(z)$, $\bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z})$. h and $\bar{h}$ are the conformal weights. Since under rescaling $z \rightarrow \lambda z$, λ real , and under a rotation $z \rightarrow \exp\{-i\,\theta\}z$, $\phi \rightarrow \lambda^{h+\bar{h}} \phi$ and $\phi \rightarrow \exp\{ -i\,(h - h)\,\theta\}\,\phi$, $d = h + \bar{h}$ and $s = h - \bar{h}$ are called the scaling dimension and the conformal spin of ϕ respectively.

The tracelessness and conservation of the energy momentum tensor $T_{\mu\nu}$ of a CFT imply that

$$\partial_{\bar{z}} T = 0 \quad , \quad \partial_z \bar{T} = 0 \qquad (1.38)$$

namely the two nonzero components of $T_{\mu\nu}$, $T(z) \equiv T_{zz}(z)$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$ are holomorphic and anti-holomorphic functions respectively.

Now any primary field ϕ has the following operator product expansion (OPE) with $T(z)$

$$T(z)\,\phi(\xi) = \frac{h\,\phi(\xi)}{(z-\xi)^2} + \frac{\partial_\xi \phi(\xi)}{(z-\xi)} + \text{finite terms} \qquad (1.39)$$

as for the OPE of T with itself

$$T(z)\,T(\xi) = \frac{\frac{c}{2}}{(z-\xi)^4} + \frac{2\,T(\xi)}{(z-\xi)^2} + \frac{\partial_\xi T(\xi)}{(z-\xi)} + \text{finite terms} \tag{1.40}$$

The anomalous first term is due to the famous nonvanishing D=2 trace anomaly . For example c = n , if the field theory is a free massless theory of n scalar field. Indeed (1.40) and its barred counterpart are together another expression of the Virasoro algebra of Vir realized quantum mechanically by the central extension of the algebra of the diffeomorphisms of the circle S^1. It is given by the product of two commuting Virasoro algebras $Vir_L \times Vir_R$, the first of which is

$$[L_n\,,\,L_m] = (n-m)\,L_{n+m} + \frac{c}{12}\,n(n^2-1)\,\delta_{m,-n}\,; \tag{1.41}$$

the second obtains by the mere replacement $L_n \rightarrow \overline{L_n}$ in (1.41) with the same c since $T + \overline{T}$ is real. The L_n and $\overline{L_n}$, $n \in Z$, are respectively the hermitian ($L_n^\dagger = L_{-n}$) moments of T(z) and $\overline{T}(\overline{z})$: $T(z) = \sum_n z^{-n-2} L_n$.

While (1.41) describes the infinitesimal transformation $\delta z = z^{n+1}$, T(z) obeys the following composition law for finite transformations $z \rightarrow z' = f(z)$:

$$T(z)\,dz^2 = T(f)\,df^2 + \frac{c}{12}\{f,\,z\}\,dz^2 \tag{1.42}$$

where

$$\{\,f,\,z\}\;dz^2 \equiv d^3f\,df^{-1} - \frac{3}{2}(\,d^2f\,df^{-1}\,)^2 \tag{1.43}$$

is the Schwarzian quadratic differential . Its properties, unique for a weight 2 conformal object, are

$$\{f,\,z\} = 0 \quad \text{if} \quad f = \frac{\alpha\,z + \beta}{\gamma z + \delta} \in SL(2,\,R)\;, \tag{1.44}$$

$$\{\,f,\,z\}\;(dz)^2 = \{\,f,\,\xi\}(d\xi)^2 + \{\,\xi,\,z\}\;(dz)^2\;, \tag{1.45}$$

SL(2, R) is the maximal subalgebra of Vir and generated by L_0 and $L_{\pm 1}$.

There are numerous reviews of the representation theory of the Virasoro algebra are [24, 30, 31]. For later reference, we only mention the following facts. In a conformal field theory such as (1.34), the Hilbert space must be partioned into irreducible representations of the

Virasoro algebras. The dictate of physics, i.e. energy positivity, requires the representations to be of highest weight i.e. such that

$$L_0 |h> = h |h> \quad , L_n|h> = 0 \ , n > 0 \ . \qquad (1.46)$$

Such a Verma module V(c,h) is spanned by the linear independent vectors

$$L_{-1}^{n_1} L_{-2}^{n_2} \ldots L_{-nr}^{n_r} |h> \qquad (1.47)$$

and is graded by the level $\sum_j j\, n_j$. For unitary representations it is necessary that either

$$c \geq 1 \text{ and } h \geq 0 \qquad (1.48)$$

or

$$c = 1 - \frac{6}{(m+2)\,(m+3)} \quad \text{and} \quad h = \frac{[(m+3)p - (m+2)q]^2 - 1}{4(m+2)\,(m+1)} \qquad (1.49)$$

where $m = 0,2,2,\ldots$; $p = 1,2,\ldots, m+1$; $q = 1,2,\ldots p$.

A conformal field theory is thus characterized by the value of its central charge and the set of highest weights { h , $\bar{h}$ } of its irreducible representations. In addition the Wilson operator product algebra for these fields should also be specified. Having in mind a WZNW theory at its critical point , the possible values of h and $\bar{h}$ can be determined and formulas for the characters in a Kac Moody highest weight representation have been computed by Kac and Petersen[32].

To every Kac-Moody algebra is associated a Virasoro algebra as a derivation algebra. Thus since $\lambda = \frac{4\pi}{k}$ corresponds to a conformal invariant fixed point , the Wess-Zumino-Novikov-Witten model is also invariant under the Virasoro-Bott conformal group. The L_n 's are given through the generic Sugawara-Sommerfield form[33, 34] of the system energy momentum tensor

$$T(z) = \frac{1}{2\,K + C_2} \sum_m :J^a J^a: \qquad (1.50)$$

whence

$$L_n = \frac{1}{2\,K + C_2} \sum_m :J^a_{n-m} J^a_m: \qquad (1.51)$$

J^a_m are the moments of the current $J^a_+ = \sum_m J^a_m z^{-(m+1)}$ and the KMA (1.21) i.e. (1.26) reads

$$\left[J^a_n , J^b_m\right] = f^{ab}{}_c J^c_{n+m} + \frac{K}{2} n \delta_{m,-n} . \qquad (1.52)$$

K , a real constant in each representation in general, is called the *level* of the KMA . The central charge c of (1.41) is given by

$$c = \frac{2K \dim G}{2K + C_2} \qquad (1.53)$$

where C_2 is the quadratic Casimir operator for the adjoint representation of G. So when (1.52) and (1.41) are combined together with $\left[L_n , J^a_n\right] = - m J^a_{n+m}$, and similar relations for the barred counterparts, the full invariance algebra of the WZNW model is the semi-direct products $(Vir_L \times KMA_L) \times (Vir_R \times KMA_R)$.

If G is simply -laced and of rank n, then Witten's result (K =1) (1.34) implies that for level K=1 , the corresponding c = n , an integer , and hence that the level 1 KMA currents of G should be reproducable from n free bosonic fields. This bose field realization is the Frenkel-Kac construction. Its results are as follows:

In a Cartan-Weyl basis and self-explained standard notations, a KM algebra (1.52) of a simply-laced G with rank n reads

$$\left[H^i_n , H^j_m\right] = m\, \delta^{ij}\delta_{m+n} \qquad , \left[H^i_m , E^\alpha_n\right] = \alpha^i E^\alpha_{m+n}$$

$$\left[E^\alpha_m , E^\beta_n\right] = \varepsilon(\alpha,\beta)E^{\alpha+\beta}_{m+n} \qquad \alpha\cdot\beta = -1$$

$$= \alpha\cdot\mathbf{H}_{m+n} + K\, m\, \delta_{m+n} \qquad \alpha\cdot\beta = -2$$

$$= 0 \qquad \alpha\cdot\beta \geq 0 \qquad (1.54)$$

i, j = 1,2,... rank G , by hermiticity $H^{i\dagger}_n = H^i_{-n}$, $E^{\alpha\dagger}_n = E^{-\alpha}_{-n}$. it admits an explicit realization from n free bosonic fields

$$X^i(z) \equiv q^i - ip^i \ln z + i \sum_{n \neq 0} \frac{1}{n}\alpha^i_n z^{-n} , \qquad (1.55)$$

namely

$$H^i_n = \oint_C \frac{dz}{2\pi i} z^n H^i(z) \quad \text{where} \quad H^i(z) = i\,\partial_z X^i(z), \tag{1.56}$$

$$E^\alpha_n = \oint_C \frac{dz}{2\pi i} z^n E^\alpha(z) \quad \text{where} \quad E^\alpha(z) = c_\alpha :\exp\{i\,\alpha\cdot X(z)\}, \tag{1.57}$$

c_α are Klein factors or cocycles obeying $c_\alpha \cdot c_\beta = (-i)^{\alpha\cdot\beta} c_\beta \cdot c_\alpha$ and $c_\alpha \cdot c_\beta = \varepsilon(\alpha,\beta) c_{\alpha+\beta}$.

Since a D=2 field theory of a free Majorana-Weyl (i.e real chiral) fermion corresponds to a CFT with c= 1/2 , we expect to be able to realize a K =1 KMA for 2n real fermions in a vector representation say of S0(2n) by a CFT of n scalar fields with momentum vector being the vectors of SO(2n). Thus using the complex basis $\Psi^{\pm a} \equiv \frac{1}{\sqrt{2}}\left(\psi^{2a-1} \pm i\,\psi^{2a}\right)$
one has for the Cartan subalgebra S0(2n) currents

$$J^{a,-a}(z) = :\Psi^a \Psi^{-a}: = i\,\partial_z X^a(z) \qquad (a < b) \tag{1.58}$$

while the other non-commuting SO(2n) currents read

$$J^{\pm a,\pm b}(z) = c_{\pm a, \pm b} : \exp\{\, i\,(\pm X^a \pm X^b): \qquad (a < b) \tag{1.59}$$

In fact one has a generalized fermi-bose equivalence , a generalized Mandelstam -Halpern, vertex construction in

$$\Psi^{\pm a}(z) = c_{\pm a} : \exp\{\pm i\, X^a\} \tag{1.60}$$

This Frenkel-Kac bosonization[35] is key to the incorporation of Yang-Mills symmetries in the heterotic string and allows for enormous simplifications in handling vertex operators of fermionic CFT's

The key question is how and how much can the above sampling of the rich representation and analyticity structures be generalized to four and higher dimensions. We survey the various excursions toward higher dimensional worlds next.

2. Beyond the affine and diffeomorphisms algebras of the circle

2.1 D=4 Gauge and Current Groups

We have recalled the tremendous successes of affine Lie algebras realized as loop algebras in D=2 quantum field theories. A natural next question concerns how much of these structures carries over to a four dimensional setting by replacing the circle S^1 by a higher dimensional arbitrary Riemannian manifold M[36]. Indeed the group Map(M;G) of smooth maps $M \rightarrow G$ is an infinite dimensional Lie group and appears almost as simple as the loop group Map(S^1, G). There had been results on the representation theory of these algebras and groups; they were reviewed in 1983 by R.S. Ismagilov[37]. However it is a remarkable fact of loop algebras that all positive energy irreducible representations are both unitary and necessarily projective. It would therefore be most interesting to seek their higher dimensional analogs in D≥2 counterpart of affine Lie algebras, namely algebras with nontrivial extensions. From the standpoint of physics where $M \approx S^3$, the compactified physical space, such groups are of primary importance in quantum field theory as the "gauge groups" and their special cases, the"current groups' [6]. They are the algebraic structures underlying current gauge theories and effective chiral theories of strong interactions at the Gev[38]as well as the Tev energy scales[39]. Unfortunately not much is known after several ongoing efforts. Here we assess the results and mention the novel directions some have been undertaking to make further progress.

As shown by the works of Bars[40] and of Bruce and Bose[41] it is an easy matter as far as obtaining the algebras with extensions, say of the sphere group Map (S^d, G) . As illustrations we summarize the results for the simplest case of d = 2 and 3 respectively. Generically the current algebra reads:

$$\left[J^a(x) \ , J^b(x')\right] = f^{abc}J^c(x)\ \delta(x - x') + S^{ab} \tag{2.1}$$

It is a very straightforward matter to find the most general Schwinger terms S^{ab} consistent with the Jacobi identity . They are

For $M = S^2$ parametrized by the Euler angle θ and ϕ , $z = \cos\theta$

$$S^{ab} = \delta^{ab}\,[\, f_1(\theta, \phi)\ \delta(z - z')\ \delta'(\phi - \phi') + f_2(\theta, \phi)\ \delta'(z - z')\delta(\phi - \phi')\,] \tag{2.2}$$

where

$$f_1 = \frac{\partial h(\theta,\phi)}{\partial z} \ , f_2 = -\frac{\partial h(\theta,\phi)}{\partial \phi} \ , \tag{2.3}$$

$h(\theta, \phi)$ is an arbitary function on S^2 .

For $M = S^3$

$$S^{ab} = \delta^{ab} [f_1\delta(z - z')\, \delta(\gamma-\gamma')\delta'(\alpha-\alpha') + f_2\delta'(z - z')\delta(\gamma-\gamma')\delta(\alpha-\alpha')$$

$$+ f_3\delta(z - z')\, \delta'(\gamma-\gamma')\, \delta(\alpha-\alpha')] \tag{2.4}$$

with

$$f_1 = \frac{\partial h_3}{\partial z} - \frac{\partial h_2}{\partial \gamma}, \quad f_2 = \frac{\partial h_1}{\partial \gamma} - \frac{\partial h_3}{\partial \alpha}, \quad f_3 = \frac{\partial h_2}{\partial \alpha} - \frac{\partial h_1}{\partial z}. \tag{2.5}$$

f_i or h_i ($i = 1,2\ 3$) are three arbitrary functions of $z = \cos\beta$ and the Euler angles α, β, γ . These arbitrary functions h and h_i are in fact identifiable with components of closed 1-forms on S^2 and S^3 respectively .

Paralleling the algebra of the Fourier moments J^a_m of Map (S^1 , g) , one expands the G-algebra valued currents $J^a(\theta, \phi)$ on S^2 and $J^a(\alpha,\beta,\gamma)$ on S^3 in spherical harmonics $Y_{l,m}$ and Wigner's $D_{l,m,m'}$ functions

$$J^a(\theta,\phi) = \sum_{l,m} J^a_{l,m} Y_{l,m}(\theta,\phi) \tag{2.6}$$

$$J^a(\alpha,\beta,\gamma) = \sum_{l,m,m'} J^a_{l,m,m'} D_{l,m,m'}(\alpha,\beta,\gamma) \tag{2.7}$$

The notable features distinguishing the above algebras from the affine Lie algebras are the following :

a) the resulting algebras of moments will clearly have an <u>infinite</u> number of central elements corresponding to the number of components of the function h for S^2, and h_i (i= 1,2, 3) . So for d > 1 , the central extension is no longer one dimensional ; there are an infinite number of central extensions. This new phenomenon for Dim M > 1 agrees with a general cohomological theorem of Feigin[42]. The latter states that if g^M is the Lie algebra of Map(M;G) , then the second cohomology group $H^2(g^M)$ is infinite dimensional for M with Dim M > 1 . One can interpret the space $H^2(g^M)$ as an infinite set of classes of independent 1-dimensional central extensions.

b) Focusing of the moments $J^a_{l,m}$ and $J^a_{l,m,m'}$, it was shown that while grading operators can be constructed by the indices m and m' , none exists for the index l . This feature implies that in contrast to the D=2 affine case it is <u>not</u> possible to asociate with these sphere algebras a root vector system in a finite dimensional root vector space .

To construct the corresponding KM groups, their representation theory and make contact with physics ,we return to Mickelsson's bundle formulation . To be specific we restrict to D=3+1 dimensions ; extension to higher dimensions being straightforward[43]. Let us consider the case of the "gauge group" Map($S^3 \rightarrow G$) , specifically with $G \approx SU(N)$, $N \geq 3$. Let $\mathcal{A}$ be the space of gauge connections on S^4 and $\mathcal{G}$ be the gauge group of point based maps $f : S^4 \rightarrow G$ with f(p) =1 for some fixed $p \varepsilon S^4$. Let $D = \{ x \varepsilon R^4 \mid |x| \leq 1 \}$ be the unit disk so that $S^3 \approx \partial D$ .and let $DG = \{ f : D \rightarrow G \mid f(p) =1 \}$ for some fixed $p \varepsilon \partial D$. Now the space $\Omega_3 G = \{ f : S^3 \rightarrow G \mid f(p) =1 \}$ is infinitely connected since $\pi_3(\Omega_3 G) \approx \pi_3(G) \approx Z$, its connected components $\Omega_3^n G$ are labelled by the instanton number n . As in the two dimensional case, each connected component of the bundle $\mathcal{A} \rightarrow \mathcal{A} / \mathcal{G}$ on S^4 is homotopically equivalent to $DG \rightarrow \Omega_3^0 G$ where the restriction to the zero instanton sector is of no consequence as all sectors are homeomorphic.

Now there is a D=4 analog of the principle bundle P discussed earlier. This bundle P_3 on $\Omega_3 G$ consists of equivalence classes (f,λ) in DGx U(1) w.r.t.

$$(f , \lambda) \sim (f g , \lambda \exp\{ 2\pi i\, \omega_1(f,g) \}) \tag{2.8}$$

$$\omega_1 (f,g) = \overline{\omega_1} (f^{-1}df , g) \tag{2.9}$$

for $g \varepsilon \mathcal{G}$,

$$\overline{\omega_1} (A , g) = \frac{i}{24\pi^3} \int Tr\left\{-dg\, g^{-1}\left(\frac{1}{2} (\frac{1}{2}AdA + dAA + \frac{1}{2}A^3\right) + \frac{1}{4}(dg\, g^{-1} A)^2 + \frac{1}{2} (dg\, g^{-1})^3 A\right\} + C_5(g) \tag{2.10}$$

and

$$C_5(g) \equiv \frac{i}{240\, \pi^3} \int_{D_5} Tr(dg\, g^{-1})^5 \tag{2.11}$$

Here $A = f^{-1}df$. $\overline{\omega_1}$ is a 1-cocycle of the group DG.

Now unlike the bundle P in the 2-dimensional case, P_3 has <u>no</u> natural group structure. To see this, define a 2-cocycle ω_2 for the group DG as

$$\omega_2(A\,;\,g_1,g_2) \equiv -\frac{i}{48\pi^3}\int_{S^3} \mathrm{Tr}\{[(dg_2\, g_2^{-1})(g_1^{-1} A\, g_1)(g_1^{-1} dg_1)$$

$$- (dg_2\, g_2^{-1})(g_1^{-1} dg_1)(g_1^{-1} A\, g_1)]\} + R_3(g_1, g_2) \qquad (2.12)$$

where the A independent term R_3 is of no importance to our subsequent discussion. We defer the details of the cohomological derivation of (2.10) to a large body of literature. We note that ω_2 differs from its two dimensional counterpart (1.17) by being a function of the gauge potential A . Consequently a proper extension of DG is <u>not</u> simply a U(1) but rather the infinite Abelian group (by point wise multiplication) Map $(\mathcal{A}_3, U(1))$ where $\mathcal{A}_3$ is the space of g-valued vector potential in S^3 . We have then a non central , operator valued extension by an abelian ideal, Map $(\mathcal{A}_3, U(1)$. This is in accord with the cited Feigin's theorem .The group composition rule reads

$$(f, \lambda)\cdot(f', \lambda') = (f f', \lambda\, \lambda'_f \exp\{2\pi i\, \omega_2(A\,;\,f, f'\}) \qquad (2.13)$$

where $\lambda'_f(A) \equiv \lambda\,(f^{-1}A\,f + f^{-1}d\,f)$. The associativity of this product is guaranteed by the 2-cocycle nature of $\omega_2\,(A,;\,g_1, g_2)$. The gauge transformation for $A \in \mathcal{A}_3$ is defined by the restriction of f to $S^3 \approx \partial D$, the boundary of the unit 4-disk. As in the D=2 case, one can now define a <u>group</u> Q_3 by way of the abelian extension mod out the equivalence relation " ~ " (2.13) . $Q_3 = (DG \times \mathrm{Map}(\mathcal{A}_3, U(1))/\sim)$, the obtained set of equivalence classes , is thus the principal bundle on $\Omega_3 G$ with as structure group Map($\mathcal{A}_3$, U(1)) . Q_3 is seen as an associated bundle to P_3 through the natural action of U(1) in the space Map($\mathcal{A}_3$, U(1)), its group structure being inherited from that of DG x Map($\mathcal{A}_3$, U(1)).

The Lie algebra 2-cocycle c_3 in Map(D, g) corresponding to ω_2 (2.12) can be computed to be

$$c_3(A\,;\,f_1, f_2) = \frac{1}{12\,\pi^2}\int \mathrm{Tr}\,\{A\,(df_1 df_2 - df_2 df_1)\} \qquad (2.14)$$

where $A = A_i^a T^a dx^i$, df_1 and df_2 are three matrix-valued 1-forms. As we are solely interested here in current groups arising from chiral σ-models, we see that (1.14) defines the extension of the current algebra Map(S^3 , G) by the ableian ideal Map($\mathcal{A}_3$, U(1)) . If we define our smeared current J (f), in an evident generic notation

$$J(f) \equiv \int dx\, f^a(x)\, J^a(x) \qquad , \quad f = f^a T^a \,, \tag{2.15}$$

then the D=4 Kac-Moody current algebra reads

$$[J(f_1)\,,\, J(f_2)\,] = i\, J\,(\,[f_1,\, f_2]) + c_3(\,A;\, f_1,\, f_3) \tag{2.16}$$

where the integration in (2.15) is over S^3 and A reduces to the flat connection $\omega = U^{-1}dU$, $U \varepsilon G$ as we will illustrate next.

2.2. Canonical realization , soliton operator and representation theory

We saw that a salient and powerful feature of affine Lie algebras is the existence of equivalent fermionic and bosonic representations. The existence of the Skyrmion testifies to the existence of a similar phenomenon in four dimensions. A question of great theoretical and phenomenological interest is the full extent and exact mathematical nature of this analogy , namely its proper place in the representation theory of D=4 Kac-Moody algebras.

Any physically motivated current algebra A has too many representations only a subset of which is of physical relevance. So to better single out these physical representations, which must be adopted to the dynamics at hand, one is led in practice to assume some concrete dynamics underlying A such as an effective field theory. After all the vertex operator representation of the Virasoro algebra was first discovered by physicists in the dual resonance model. A concrete point starting for looking at the representation problem is the D=4 analog of the D=2 WZNW model , which may emerge from a large N , low energy limit of QCD. Its manifestly SU(N)xSU(N) chiral invariant action reads

$$S = S_{\sigma m} + S_{WZ} \tag{2.17}$$

where

$$S_{\sigma m} = \frac{-1}{16\, f_\pi^2} \int_M d^4x \mathrm{Tr}\left(\partial_\mu U^{-1} \partial_\mu U\right) \tag{2.18}$$

$$S_{WZ} = \frac{-i N_c}{240 \pi^2} \int_\Omega Tr (U^{-1}dU)^5 . \qquad (2.19)$$

The SU(N) matrix field U parametrizes the field space $G \approx \frac{SU(N)x\ SU(N)}{SU(N)} \approx SU(N)$. With strong interactions in mind, N is the number of flavor. $U^{-1}dU \equiv \omega = \omega^a T^a$ is the Maurer-Cartan, left-invariant current 1-form. The $\{T^a\}$ denotes an anti-Hermitian basis of SU(N). $S_{\sigma m}$ is the standard nonlinear σ-model action . S_{WZ} is the D=4 Wess-Zumino term where for the same reason as its D=2 counterpart , the integration is over Ω , a 5-dimensional disk with as its boundary the spacetime M., N_c is similarly quantized .

With the boundary condition that $U(x) \to I$ at spatial infinity, 3-space is effectively compactified onto a 3-sphere S^3. The configuration space of our model is then the infinite Lie group Map (S^3 ; G≈SU(N)) . From here we can obtain the corresponding current algebra of (2.17) either by the cohomological or the canonical field theory method . By way of the field equations of (2.17) expressed as a current conservation law

$$\partial_\mu J^a_\mu = 0 \qquad (2.20)$$

for the current

$$J^a_\mu = \frac{1}{8f_\pi^2} Tr (T^a\omega_\mu) + 5\, \lambda\, \varepsilon_{\mu\nu\rho\sigma} Tr\{ T^a\omega_\nu\omega_\rho\omega_\sigma \}. \qquad (2.21)$$

Its first term is of the usual Sugawara-Sommerfield form, its second term derives from the Wess-Zumino anomaly, $\lambda \equiv \frac{-i N_c}{240 \pi^2}$. In particular the resulting *local charge density algebra* reads

$$[J^a_o(\mathbf{x}) , J^b_o(\mathbf{x})] = i f^{abc} J^c_o(\mathbf{x})\, \delta^3(\mathbf{x} - \mathbf{y}) + S^{ab}(A ; \mathbf{x}, \mathbf{y}) \qquad (2.22)$$

where

$$S^{ab} = 10\, i\, \lambda\, \varepsilon_{ijk} Tr\left(\{T^a, T^b\}\omega_i\omega_j\right) \partial_k\, \delta^3(\mathbf{x} - \mathbf{y}) \qquad (2.23)$$

(2.22) is an exact realization of the commutator (2.16) given previously. The operator-valued Schwinger term or Abelian extension S^{ab} , originates from the Wess-Zumino action. It is the flat connection ($\omega = U^{-1}dU$) limit of the anomalous gauge generator algebra of

Faddeev and Shatashvili[44] for a quantum theory of left-handed fermions coupled to an external gauge field $A_{\mu}(x)$. In the case of global chiral symmetry, the presence of S^{ab} signals the possibility of projective representations, new sectors in the model's Hilbert of physical states. It is thus called a "good" anomaly. In the gauge theory case on the other hand, we have an inconsistent quantum gauge theory since a nonvanishing S^{ab} for $N \geq 3$ is a topological obstruction to the implementation of Gauss's law or local gauge invariance. We have here a "bad" anomaly.

What is most remarkable about the algebra (2.22) and its canonical bosonic realization (2.17)) is that the later admit a fermionic soliton. It is well known that a D=4 dimensional σ-model i.e. the action $S_{\sigma m}$ augmented by suitable stabilizing higher field derivative terms admits topological $S^3 \rightarrow G$ solitons, the Skyrmions[45]. Since the works of Balachandran et al [46, 47, 48], there has been an explosion of phenomenological applications to hadronic physics[38, 49]. But of interest to us is Witten's proof that the added topological Wess-Zumino action induces the realization of a projective fermionic representation of the current algebra. It confirmed Skyrme's conjecture existence of a D=4 bosonization .

We recall in a nutshell Witten's semi-classical argument . Take the static classical 1-Skyrmion map $U_S(\mathbf{x}) : S^3 \rightarrow SU(2)$, seen as a suitable SU(2) embedding in $G \approx SU(3)$ with topological charge B

$$B = \frac{1}{24\pi^2} \int_{S^3} d^3x \, \mathrm{Tr} \, (U^{-1}dU)^3 = 1 \tag{2.24}$$

the generator of $\pi_3 (G) \approx Z$. Using the time dependent ansatz $U(\mathbf{x},t) = g(t) \, U_S \, g(t)^{-1}$, $g(t) \in SU(3)$ being the collecfive coordinate matrix, we now adiabatically rotate the skyrmion by an angle of 2π around some axis . The resulting contribution coming solely from the Wess-Zumino term is $(i \pi N_c)$, giving a geometrical phase factor of $\exp (i \pi N_c) = (- 1)^{N_c}$, the spin phase , to the quantum mechanical wavefunctional . So the soliton is a fermion for N_c = odd integer. N_c is identified as the number of colors by matching the flavor anomalies of the effective chiral model (2.17) with those of its underlying gauge theory, quantum chromodynamics.

To go beyond the semi-classical description of the Skyrmion, one would like to obtain the D-=4 counterpart of the local Skyrme-Mandelstam[18, 50, 51] fermionic operator (1.31) for creating a point like soliton out of the vacuum. The first effort was due to Skyrme himself[52]. It is at least of conceptual interest to sketch the essential elements of his construction . For the Sine-Gordon model with field denoted by α , he showed that the operator

$$S = FK = \exp(\pm \frac{i}{2}\alpha(x_0)) \exp\left[\frac{i}{2}\int_{x_0}^{\infty} \frac{\partial\alpha}{\partial t}\, dx\right] \tag{2.25}$$

obeys a massless Dirac equation. After normal ordering or renormalization , (2.25) was to become the Mandelstam operator . It later generalized to the vertex operator representation of affine Lie algebras. In analogy to (2.25) , Skyrme argued that the corresponding Weyl-liked operator S should also be made of two factors

$$S = S_2 S_1 = \exp\left\{i\int I_\alpha \frac{e_{\alpha\, i}(x-x_0)_i}{r}\,\omega(r)\, d^3x\right\} \exp\{it_\alpha\theta_\alpha\} \tag{2.26}$$

with $r = |\vec{x} - \vec{x_0}|$, $I_\alpha(x)$ is the time -component of the isospin current , $e_{\alpha\, i}$ is a proper orthogonal matrix interlocking spatial and isospin directions and ω a suitable angular function of r . S_2 identifies the auxiliary momentum p_0 with a suitable field expression and $e_{\alpha\, i}$ with a matrix characterizing the field orientation. In S_1 , t_α is a rotation operator conjugate to $e_{\alpha\, i}$ relating to the internal symmetry index α,the θ_α(α =1,2,3) are functions of the soliton map . By applying the collective coordinate method to a static point like soliton and naively manipulating with S seen as canonical transformation, Skyrme partially diagonalized the nonlinear field system turning it into an effective, rotator Hamiltonian $H_{eff} = At_\alpha t_\alpha + B +$ interactions, with its isobaric spectrum so typical of old strong coupling theories. Then by a rather unconvincing argument he projected out the spin 1/2 state with H_{eff} leading to a Dirac hamiltonian of a free point particle plus interactions.

Much later Rajeev[53] took as operator which would create a soliton state from the vacuum the unitary operator

$$U(g_1) = \exp\{i \int d^3x I_0^a \theta^a(x)\} \tag{2.27}$$

where $g(x) = e^{i\lambda^a\theta^a(X)}$. U is to implement a projective or 2-cocycle representation of the 3-sphere current group $\Gamma \approx \mathrm{Map}(S^3, G)$ in that $\hat{U}(g_1)\,\hat{U}(g_2) = e^{-2\pi i\,\omega_2(q;g_1,g_2)}\hat{U}(g_{12})$ ( ) . As is well known [54], there exist nontrivial projective representations of Γ provided its 2nd cohomology $H^2(\Gamma)$ is nontrivial . This is the case for $G \approx SU(N)$, $N \geq 3$ as $\pi^5(SU(N)) = Z$ for $N \geq 3$ due to the isomorphism $H^2(\Gamma) \approx H^5(G) \approx \pi^5(G)$. Nontrivial $H^5(G)$ is [55]precisely the condition for the existence of the Wess-Zumino anomaly . In a rather sketchy analysis, it was argued that for largely separated Skyrmions , two U (2.27)at different spatial points anticommute. In any case it is clear that to obtain a true local soliton operator obeying some spinor wave equation etc...greater kinematical and dynamical inputs through a canonical quantization of a definite model need to be brought to bear on Rajeev 's program . In our opinion, it may be more fruitful to try extracting such a soliton operator from a Skyrmion wavefunctional seen as a section of a Dirac determinant bundle.

The attempts by Skyrme and Rajeev while embodying the necessary central ideas are at best heuristic and incomplete . A technically rigorous construction has yet to be performed .One must face such ignored yet crucial and difficult issues such as regularizations, the meaning of exponentials of D>2 non-abelian field operators and other new topological ingredients. Surely one could profit from the recent experience (see Sect.C.1) in establishing certain exact operatorial boson fermi correspondence in three spacetime dimensions.

From an algebraic viewpoint, the existence of a kind of D=4 quantum fermi-bose correspondence has provided a strong inducement to attack a larger problem, that of the representation theory of the D=4 current algebra (2.16). Indeed there is also an pressing phenomenological need to do so. With the profusion of Gev hadrons, the possibility of a strongly coupled Higgs sector at SSC energies and the still intractable infrared structure of QCD, we may revive and seek to further advance the old program of current algebra , this time with an added topological twist, the Wess-Zumino chiral anomaly. The hope would be that , knowing the physical representations of such extended current algebra based on QCD, such an approach would provide a systematic nonperturbative (albeit effective) handle to portray strong interactions.

Compared to the rich developments of the representation theory of affine algebras what , if anything, is known about the representations of Map(M; G) ? The answer is " rather surprisingly little" . Till recently, there was only one irreducible representation due Gelfand , Graev and Vershik [7],but it has no apparent physical relevance. Another physically well founded attempt is by Mickelson and Rajeev[56]. The goal is to construct a suitable (3+1) dimensional generalization of the fermionic Fock space of D=2 current algebra with a Schwinger term (say (2.22).). To do so they generalized to the case of a larger linear group modelled by rank p ($o < p < \infty$) Schatten classes the methods of Pressley and Segal [57]for constructing cocycle representations of the infinite dimensional restricted general linear groups. This is no place for involved technical details, we can only sum up their results.

Working in a D =<u>odd</u> spatial manifold M such as S^d, they consider the system of a Dirac spinor coupled to an external Yang-Mills field with gauge group G and its corresponding algebra **g**. Its current algebra is just Γ = Map (S^d ; **g**) is just (2.26) without the extension term C_3 . Let f : $S^d \rightarrow$ **g** be Lie algebra valued functions , λ^i the representation matrices of **g**. Since the Dirac Hamiltonian is unbounded below, the "1st quantized" unitary representation of Γ given by

$$[J(f) , \psi_\alpha(x)] = \lambda_\alpha^{i\beta} f^i(x) \psi_\beta(x) \qquad (2.28)$$

is physically unsuitable as it has no vacuum state or highest weight vector . Second quantization cures this instability i.e. by constructing a Dirac vacuum as the highest weight state. The operator product $J^i(x) = \psi^\dagger \lambda^i \psi$ needs a short distance say a point splitting regularization, which involves substracting the VEV of J{f} . As a result, one no longer has a representation of Γ but a central extension of it .

In contrast to the situation in one spatial dimension where normal ordering is enough to a well-defined quantum theory , for d >1 further renormalizations are required. Thus for d >1 $J\{f\}^2$ are still not well defined after substraction . For d=3 , J{f} requires an additional multiplicative renormalizaton, implying that such an operator is meaningless within the purely fermionic Fock space as it creates out states of infinite norm of the vacuum . A larger Hilbert space is then introduced . It include the fermionic states which no longer form a complete set and new bosonic states created from the vacuum by J{f} and having the quantum numbers of a two fermion states. In this manner Mickelsson and Rajeev [56] found a nonunitary representation of an Abelian extension of Map(S^d ; **g**) i.e. (2.12) . Their procedure illustrates the flip "local " side of the anomaly or of the ray representations of the KM group Map (S^d ; **g**) . Specifically they found a linear representation with highest weight vector, essentially including these bosonic states . Very recently [56] they did manage to construct unitary representations in certain special cases of a 3-parameter family of deformations of the abelain extension $\widehat{\mathbf{gl}_2}$ of the general linear algebra $\mathbf{gl}_2$. It would be of great interest to see physical applications of such results.

We note that the necessity to include bosonic states along with fermionic ones, say to implement unitarity, seems consistent with the more recent conclusion on D=3 bosonization . In fact while D=3, 4 purely bosonic field theories do admit fermionic solitons, only in D=2 are such theories exactly equivalent to a local fermion model. Luscher [58] has shown that there exists an analogous exact quantum correspondence between certain D=3 interacting field theories, but this equivalence is between a purely bosonic model and one involving not only a basic local fermion field but also other bosonic fields. Similarly an illustration of exact of D=4 fermi-bose corespondence was put forth by Mickelsson[59]. He showed that on a spatial manifold with topology $S^2 \times S^1$ a D=4 Yang-Mills system coupled to a U(1) monopole and the mixed field system of a 4-component fermion coupled to a U(1) monopole have identical KMA. These results should motivated further work on physical representations of D>2 current algebras.

2.3 D > 2 Diffeomorphisms

As noted before, solving for quantum field theory is often equivalent to knowing all the unitary representations of its invariance groups. In two dimensions CFT testifies to the truth of the above assertion. In higher dimensions the place of the Virasoro-Bott conformal group is taken by the group of diffeomorphisms of a given manifold. In particular the diffeomorphism invariant topological quantum field theories should naturally take the place of CFT 's.

In the 60's diffeomorphism groups were considered in the motion of incompressible fluids by V. Arnold[60]. Subsequently applications of representations of the group SDiff(R^n) of volume preserving diffeomorphisms of R^n (n=2, 3) have been made in classical and quantum fluids, specifically to vortex filaments and other topological defects [61, 62]. In particle physics, recent attempts to quantize relativistic closed p-(super)branes[63], which generalize (super)strings, have led to the analysis of the algebras of SDiff(S^2) , of SDiff(M_g), M_g being a Riemann surface of genus g , of SDiff(S^3) etc.... and their possible central extensions. We briefly survey the status of these algebras as the D>2 analogs of the Virasoro algebra [64].

A p-brane M_p is a bosonic extended object of p-spatial dimensions propagating in d-spacetime dimensions according to the Polyakov action

$$S_p = \int d^{p+1}\sigma\sqrt{g}\left[g^{ij}\partial_i X\cdot\partial_j X - (p-1)\right] \qquad (2.29)$$

$\sigma^i = (\sigma^a , \tau)$, σ^a (a= 1,...,p) are the coordinates on the p-brane, τ parametrizes the latter's time evolution. Working in the light cone gauge, well tested in string theory, means imposing first the following condition on the p-brane metric g_{ij}

$$g_{0a}= 0 , \qquad g_{00} = - \det h_{ab} \equiv h \qquad (2.30)$$

where h_{ab} is the spatial metric on the p-brane. One can then choose the light cone gauge

$$X^+ = p^+\tau \ , \tag{2.31}$$

$X^\pm \equiv \frac{1}{\sqrt{2}} (X^0 \pm X^{d-1})$. What remains of the (p+1)-dimensional general coordinate invariance group is its subgroup which preserves (2.30) and (2.32) and consists of reparametrizations of the spatial variables $\sigma^a \rightarrow \sigma^{a'} (\sigma^b)$ with the restriction that they preserve the volume

$$\det \left(\frac{\partial \sigma^{a'}}{\partial \sigma^b} \right) = 1 \ . \tag{2.32}$$

The existence of this reparametrization invariance relates to that of the constraint

$$\partial_{[a} X^m \partial_{b]} P^m \equiv \phi_{ab} \approx 0 \ , \tag{2.33}$$

$m = 1,2,...d\text{-}2 \ , \ P^m(\sigma) = \frac{\partial X^m}{\partial \tau} \ .$

In mathematics the above subgroup is called the group of volume-preserving diffeomorphisms which we denote by ($VPDiffM_p$). Its general classical algebra has been computed, valid for any topologies and geometries. Infinitesimally (2.33) is equivalent to the variation $\delta\sigma^a = \eta^a(\sigma^b)$ where $\nabla_a(\alpha^{-1}\eta^a) = 0$. The latter is solved by

$$\eta^a = \xi^a + \sum_{r=1}^{b_1} c_{(r)} \omega^a_{(r)} \tag{2.34}$$

in terms of coexact ($\xi^a d\sigma^a$) and harmonic ($\omega^a_{(r)} d\sigma^a$) 1-forms on M_p . b_1 denotes the 1st Betti number of M_p , $c_{(r)}$ are constant coefficients,

$$\xi^a \equiv \frac{1}{(p-2)!} \varepsilon^{an_2n_3...n_p} \partial_{n_2} \Lambda_{n_3...n_p} \ . \tag{2.35}$$

The classical algebra reads

$$[L_{\Lambda_1} , L_{\Lambda_2}] = L_{\Lambda_{12}} \ , \tag{2.36a}$$

$$[P_{(r)} , L_{\Lambda_2}] = L_{\Lambda_{(r)}}, \tag{2.36b}$$

$$[P_{(r)} , P_{(s)}] = L_{\Lambda_{(r)(s)}} \tag{2.36c}$$

where the generators are $L_\Lambda = \xi^a \partial_a$, $P_{(r)} = \omega^a_{(r)} \partial_a$ and the parameters on the RHS of (2.36 a-

c) are given by

$$(\Lambda_{12})_{a_3\ldots a_p} = -\varepsilon_{aba_3\ldots a_p}\xi_1^a\xi_2^b \tag{2.37}$$

$$(\Lambda_{(r)})_{a_3\ldots a_p} = -\varepsilon_{aba_3\ldots a_p}\omega_{(r)}^a\xi^b \tag{2.38}$$

$$(\Lambda_{(r)(s)})_{a_3\ldots a_p} = -\varepsilon_{aba_3\ldots a_p}\omega_{(r)}^a\omega_{(s)}^b \tag{3.39}$$

In the quantum theory of membranes (p=2), the constraints $\phi^{ab} \approx 0$ become $\hat{L}_\Lambda|phys\rangle = \hat{P}_{(r)}|phys\rangle\ 0$ on physical states through the operators

$$\hat{L}_\Lambda = -i\int d^2\sigma\xi^a\partial_a X^m p^m \qquad , \hat{P}_{(r)} = -i\int d^2\sigma\ \omega_{(r)}^a\ \partial_a X^m p^m \tag{2.40}$$

They satisfy at the level Poisson brackets the classical algebra (2.36) . For an arbitrary closed bosonic membrane $\Sigma = M_{2,\ g}$, namely a 2-sphere with g handles , the area-preserving diffeomorphism algebra was found with the most general central extension consistent with the Jacobi identy. It reads

$$\left[\hat{L}_{\Lambda_1}, \hat{L}_{\Lambda_2}\right] = \hat{L}_{\Lambda_{12}} + \int_\Sigma d^2\sigma V^a\Lambda_1\overleftrightarrow{\partial}_a\Lambda_2\ , \tag{2.41a}$$

$$\left[\hat{P}_{(r)}, \hat{L}_\Lambda\right] = \hat{L}_{\Lambda_{(r)}} - 2\int_\Sigma d^2\sigma\ \ V^a\varepsilon_{ab}\omega_{(r)}^b\Lambda\ , \tag{2.41b}$$

$$\left[\hat{P}_{(r)}, \hat{P}_{(s)}\right] = \hat{L}_{\Lambda_{(r)(s)}} + \int_\Sigma d^2\sigma\ \ \varepsilon_{ab}\omega_{(r)}^a\omega_{(s)}^b W\ . \tag{2.41c}$$

where W is any scalar and $\nabla_a(\alpha^{-1}V^a) = 0$, namely $\alpha^{-1}V^a$ is any divergenceless vector field on Σ . Consequently , from (2.42a-c) we see that the most general central extension is specified by one arbitrary scalar function and by 2g arbitarry constant coefficients of the harmonic forms on Σ [65] . This is so since the dimension of the space of harmonic 1-forms on Σ , $\dim H^1(M_{2,g}, R) = \dim R^{2g} = 2g$ [66].

From the above analysis it follows that , for the 2-torus (indeed the n-torus) and the 2-sphere, there are no nontrivial central extensions . The algebra for the torus was one found long ago by Arnold in hydrodynamics[67]. It is also remarkable that the Lie algebra of SU(∞)

as well as the large N limit W_∞ of the operator algebra W_N generated by primary fields with integer spin 1,2,...N are isomorphic to the L- subalgebra (2.41a) of the area-preserving diffeomorphisms algebra of of the 2-plane ,whether or not it is compactified i.e. to S^2 [68].

As far as explicit representations, Figueirido and Ramos[69] have constructed Fock space representations of the algebra of vector fields of the n-torus, i.e of Diff(T^n). They used a generalization of the infinite wedge representation of Kac, Petersen, of Feigin and Fuchs for Diff (S^1)[32, 70]. Further renormalizations beside normal ordering were necessary to make everything well defined. Their representations are generated not by linear operators but bilinear forms so they do not arise from a Verma module. The unitarity question remains to be answered before physical applications. Of course, if one could manage to quantize membranes, this would amount to finding interesting unitary representations of their corresponding diffeomorphism groups with or without central extensions.

In two dimensions one can form the semi-direct sum of the Kac-Moody and Virasoro algebras where the first appears as an ideal. The join structure underlies conformal field theories with internal symmetries such as the WZNW model. A natural question is whether Map(S^n, g) and Diff(S^n) can be so combined into a larger algebra. This question was investigated for n=2,3 . It was shown [41] on the basis of consistency with the Jacobi identity that a) there exist no such a larger algebra containing Diff(S^n) and the centrally extended S^n-algebra e.g. (2.2) for n=2 and (2.4) for n=3 , b) however such a structure does exist if the S^n- algebra is not centrally extended . An example of (b) could be the current algebra Map(S^3; su (N)) of D=4 WZNW (2.17) non centrally extended by an abelian ideal.

From the standpoints both of physics and mathematics, the representation theory of higher dimensional analogs of Kac-Moody-Virasoro algebras is an object of great mathematical fascination and much potential physical importance. Since the D> 2 conformal groups are finite dimensional, to analyze D> 2 system at their critical points one should target infinite dimensional subgroups of the general D-dimensional diffeomorphism groups, ones which have the conformal group as a subgroup . In four dimensions several attempts to define and study four dimensional structures endowed with the richness of D=2 CFT. One group [71, 72] studies representations of infinite self-dual and anti-self dual subalgebras of the Diff(R^4) making use of quaternionic Schwartzian derivative and Fueter quaternionic analyticity. Inspired by the connection between D=3 Chern-Simons theory and D=2 CFT ,another group [73] seeks by descending from a D=5 Chern-Simons theory to find D=4 analogs of 2d CFT . Another group[74, 75] has undertaken a more radical approach à la Penrose. They propose replacing the Riemann surfaces of D=2 CTF by twistor spaces and complex analyticity by holomorphic sheaf cohomology. In this manner a classical infinite algebra has been found, one which has as its subalgebras both the D=4 SU(2,2) conformal and D=2 conformal algebras. The hope is to apply the corresponding quantum algebra with extension to classify fields and field theories reformulated in terms of twistors. If any of these programs succeeds, very enticing mathematical vistas will surely lie ahead.

3 . Odd phenomena in odd dimensions

3.1 Anyons Revisited : current algebra and vertex operator

Two overlapping topics in field theory have attracted a great deal of attention. They are the topological quantum field theories (TQFT) in $D \geq 3$ spacetime dimensions[76], ushered in by the mathematical works of Donaldson and Jones and field theories[77] with excitations bearing*any* spin and statistics, *anyons*, which may well account for the fractional quantum Hall effect and high temperature superconductivity.

In his trail brazing analysis of D=3 Chern-Simons theories, Witten [78] showed the beautiful correspondence between the expectation values of the Wilson loops traced by 'colored' *point* sources in spacetime and Jone's polynomials for knots. To obtain the fundamental Skein relation of knot theory , he had to regularize or *frame* the Wilson loops inaddition to performing the standard regularization. A different form of such a regularization had been discovered by Polyakov[79, 80] in his proof of the fermi-bose transmutation of D=3 "baby Skyrmions" in their geometric *point-like limit.* It is at that juncture that an interesting overlap with the theory of anyonsoccurs . Our work[81, 82] on which this section is based takes off at this intersection between physics, the biology of DNA molecules, differential geometry, topology, representation theory of current algebras and division algebras.

To be specific we shall choose without loss of generality the model par excellence for anyons ,one governed by the action [77]

$$A = \int d^3x \left\{ |D_\mu Z|^2 + \frac{\theta}{8\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} + A_\mu J^\mu \right\} . \tag{3.1}$$

the $\mathbf{CP}_1$ σ- model with a Chern-Simons term. Its basic field is a two component complex spinor $Z^T = (Z_1, Z_2)$ with $|Z|^2 = 1$, consequently it lives on S^3. The more familiar unit normed Wegner vector field $\mathbf{n}$ is given by the complex Hopf projection map taking $Z \in S^3$ to $\mathbf{n} = Z^+ \sigma Z \in S^2 \approx CP_1$. D_μ is the covariant derivative w.r.t the composite U(1) gauge field $A_\mu = i Z^\dagger \partial_\mu Z$, $F_{\mu\nu}$ being the associated field strength . θ is a free parameter ($0 \leq \theta \leq \pi$). This model should best be viewed as the low energy limit ($e^2 \rightarrow \infty$) of a system with a

Maxwellian kinetic term $\frac{1}{2e^2} F_{\mu\nu}F_{\mu\nu}$ added on to (3.1).

It is well known that (3.1) (with $\theta = 0$) to admit exact $S^2 \to S^2$ general solitons. While the standard third term in (3.1) is the Aharonov-Bohm term, the second or Chern-Simons (C-S) term also reads for $\theta = \pi$ as $S_H = \frac{-1}{2}\int d^3x\, A_\mu J^\mu$, namely as an interaction between the field A_μ and the conserved topological current

$$J_\mu = \frac{1}{8\pi}\varepsilon_{\mu\nu\lambda}\varepsilon_{abc}n^a\partial^\nu n^b\partial^\lambda n^c . \qquad (3.2)$$

The soliton (electric) charge $Q = \int d^2x\, J_0$ is an integer labelling the elements of $\pi_2(S^2) = Z$. The field boundary condition is such that spacetime is $R^3 \cup (\infty) \approx S^3$; the C-S action is in fact the Whitehead form of the Hopf invariant for the maps $\mathbf{n} : S^3 \to CP_1 \approx S^2$, classified by the generators of $\pi_3(S^2) \approx Z$. The configuration space of fields is the infinte Lie group of 2-sphere base preserving smooth mappings $\Gamma \approx \{\, \mathbf{n}: S^2 \to S^2 \}$. The homotopic relations $\pi_0(\Gamma) \approx \pi_2(S^2) \approx Z$ and $\pi_1(\Gamma) \approx \pi_3(S^2) \approx Z$ allow the topological possibilities of solitons and exotic spin -statistics connection respectively[84]. The latter option is implemented dynamically by a topological Chern-Simons action .

What effect does the Chern-Simons term have on the chiral soliton ? Wilczek and Zee[83] showed that either the interchange of two Q=1-solitons or a 2π rotation of one of them around the other gives a statistical (alias spin) phase factor $e^{i\theta} = (-1)^{2s}$ to the wave function. Hence the soliton has spin $s = \frac{\theta}{2\pi}$ and intermediate statistics, it is an *anyon.* A key ingredient for their proof is that the soliton map giving raise to this phase be of Hopf invariant 1 . It will be a guiding criterion in our subsequent $D \geq 3$ generalization of the θ-spin and statistics connection .

Being a σ-model , (3.1) with $\theta = 0$ has a canonical realization [84]of the following pseudo- Sugawara-Sommerfield equal time algebra of currents

$$\left[I_0^a(x) , I_0^b(y) \right] = -i\, \varepsilon^{abc} I_0^c(x)\delta^2(\vec{x}-\vec{y}) , \qquad (3.3a)$$

$$\left[I_0^a(x) , I_i^b(y) \right] = -i\, \varepsilon^{abc} I_i^c(x)\delta^2(\vec{x}-\vec{y}) + i f^{-1}(\delta^{ab} - n^a(y)n^b(y))\, \partial_{\vec{x}_i}\delta^2(\vec{x}-\vec{y}) , \qquad (3.3b)$$

$$\left[I_0^a(x)\,,\,I_0^b(y)\right] = 0\,. \tag{3.3c}$$

where $I_\mu \equiv f^{-1}\varepsilon^{abc}\partial_\mu n^b\, n^c$. It is a pseudo-current algebra in it does not close since the operator-valued Schwinger term in (3.3b) is explicitly a function of the field **n**. Actually a larger close algebra can be obtained from (3.3a)-(3.3c) by introducing a new rank two symmetric tensor operator $S^{ab}(y)$ which is realized as $S^{ab} =(\delta^{ab} - n^a(y)n^b(y))$ by the model (3.1) . However one feature which distinguishes even and odd -dimensional current algebras seem to be the following . In contrast to the 4-dimensional σ-model with Wess-Zumino term leading to a noncentral extension of its algebra of currents (vis 2.22) by an abelian ideal, the model (3.1) with its Chern-Simons terms in fact leads to exactly the same algebra (3.3) where, say in the local charge density algebra (3.3a) we replace I_0^a by the new canonical momentum

$$L^a= I_0^a + \frac{\theta}{2\pi}\epsilon_{ij}A_i(x)\partial_i n^a(x). \tag{3.4}$$

In other words the current algebra is without extension and is independent of the Chern-Simons term; it is θ-independent. While in odd dimension locally there is no signature of the Chern-Simon anomaly, globally at the level of representations, Semenoff ,Sodano [85]and Karabali [86]have shown that at the level of the wavefunctional ,when one exponentiates the algebra and boundary conditions then enter, it does make a big difference in whether one uses à la Skyrme as static vertex soliton creation operator

$$U_I(\vec{x}) = \exp\left\{i\int d^2y\; I_0^a(y)\;\omega^a(\vec{x}-\vec{y})\right\} \tag{3.5}$$

or

$$\widehat{U}_L(\vec{x}) = \exp\left\{i\int d^2y\; L^a(y)\;\omega^a(\vec{x}-\vec{y})\right\} \tag{3.6}$$

where the soliton profile map

$$\omega(\vec{x}) = g(r)\begin{pmatrix}\sin\phi\\ -\cos\phi\\ 0\end{pmatrix} \tag{3.7}$$

twists the vacuum configuration $n^a = (0, 0,1)$ into a soliton with charge $Q \neq 0$. Indeed while (3.6) does <u>not</u> transform like a scalar under rotation (3.7) provide a proper representations of the rotation group SO(2). In fact it can be shown that one has the graded commutation relations

$$\hat{U}(\vec{x})\,\hat{U}(\vec{y}) = e^{i\frac{\theta}{\pi}\Delta}\;\hat{U}(\vec{y})\hat{U}(\vec{x}) \qquad (3.8)$$

where the multi-valued phase $\Delta(\mathbf{x}, \mathbf{y}) \equiv \Theta(\mathbf{x},\mathbf{y}) - \Theta(\mathbf{y},\mathbf{x}) = \pi \bmod 2\pi n$, $\Theta(\vec{x},\vec{y}) \equiv \tan^{-1}\frac{(x_2-y_2)}{(x_1-y_1)}$. (3.8) is the signature of exotic statistics. As with all such very ill-defined vertex operators, (3.6) is a topologically nontrivial coherent state operator with its classical soliton profile, it does not create a state of definite spin. As Skyrme was already keenly aware, it is in general not an easy task to use collective coordinates and proper regularization (!) procedures to project out from (3.6) operators with definite spin. All this and more remain to be done if vertex operators are to be useful entities in formulating effective Hamiltonian theories where anyonic excitations are the basic quasi-particles.

3.2 Geometry of a phase: linking the soliton's twist and writhe to exotic spin and statistics

In the last section, we discussed the problem of zero in on some important nonperturbative states or degrees of freedom of a highly nonlinear theory e.g. projecting out anyons of definite spin. It is clear that by going to the pointlike limit one can get to the lowest energy and spin states of a extended object. It is very much in this spirit that Polyakov[79] pioneered a tractable Wilson loop approach to the low energy behavior of soliton Green functions of model (3.1). To study the effects induced by the long range Chern-Simons interactions, he approximated the partition function **Z** by

$$\mathbf{Z} = \sum_{(P)}^{\text{all closed paths}} e^{-m\,L\,(P)} \left\langle \exp\!\left(i \int_P dx^{\mu} A_{\mu} \right) \right\rangle \; . \qquad (3.9)$$

P is a Feynman path of a pointlike soliton , hence a curve, in spacetime R^3, L(P) is the total path length .

The first exponential in (3.9) is just the action of the path P of a free relativistic point soliton of mass m. Let

$$\Phi(P) = \left\langle \exp(i \oint_P A^{\mu}\, dx_{\mu}) \right\rangle , \qquad (3.10)$$

the bracket < ...> denotes functional averaging w.r.t. the Hopf action. It embodies the Aharonov-Bohm effect , characteristic of topologically massive gauge theories: namely the

Chern-Simons-Hopf action induces magnetic flux on electric charges and vice versa, thus producing dyonic objects. Being Gaussian, this phase $\Phi(P)$ is exactly calculable, thereby the analytic appeal of the point soliton approximation. By direct integration of the equation of motion , $\Phi(P)$ is given by exponentiating the effective action :

$$\Phi(P) = \frac{1}{N} \exp\left\{ iS_0 + i \int d^3x \left(\frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \right) \right\} . \tag{3.11}$$

S_0 is the free point particle action , N a suitable normalization. The conserved current of a Q=1 point source $J_\mu(x) = \int d\tau\, \delta^3(x-y(\tau)) \frac{dy_\mu(\tau)}{d\tau}$ is given geometrically. From (3.1) the equation for J_μ reads

$$J_\mu(x) = - \frac{\theta}{2\pi^2} \epsilon_{\mu\nu\rho} \partial^\nu A^\rho \tag{3.12}$$

which on substitution in (3.11) with $\theta = \pi$ gives $\Phi(P) = \exp\left(\frac{i}{2} \int d^3x\, A_\mu J^\mu \right)$. Then for the above *given* point current and in the gauge $\partial^\alpha A_\alpha = 0$, A_μ can be solved to give :

$$\Phi(P) = \exp \{ i\pi\, I_G(P) \} , \tag{3.13}$$

where

$$I_G(C_\alpha \rightarrow C_\beta) = \frac{1}{4\pi} \oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \frac{\varepsilon_{\mu\nu\lambda}(x^\lambda - y^\lambda)}{|x-y|^3} \tag{3.14}$$

in the limit where the two smooth closed 3-space curves C_α and C_β <u>coincide</u>, namely $C_\alpha = C_\beta = P$, the soliton worldline.

Were C_1 and C_2 in R^3 (or S^3) *disjoint* curves , (3.14) would be their Gauss linking coefficient. If we denote by $\Omega(M_2)$ the solid angle subtended by C_1 at the point M_2 of C_2, Stoke's theorem gives $I_G = \frac{1}{4\pi} \int_{C_2} d\,\Omega(M_2)$, which measures the variation of this solid angle divided 4π as M_2 runs along C_2 ; it is an integer, the algebraic number of loops of one curve around the other.

However though the integrand in (3.14) is that of Gauss' invariant , the integration is over <u>one and the same</u> curve. $I_G(P)$ is therefore *undetermined.* This artifact of the point-limit approximation must be amended by a proper definition or regularization of $I_G(P)$.

Polyakov's regularization consists in trading the δ-function in $\int_P dx^\mu \iint d^2y_\mu \delta(x-y)$, an equivalent expression for $4\pi I_G$, for the Gaussian $\delta_\varepsilon(x-y)=(2\pi\varepsilon)^{-\frac{3}{2}}\exp\left(\frac{-|x-y|^2}{\varepsilon}\right)$. He found that $I_G(P)_{Reg} = -T(P)$, the total torsion or twist of the curve P in spacetime with $T(P) = \frac{1}{2\pi}\oint_P d\mathbf{x}\cdot(\mathbf{n}\times\frac{\partial\mathbf{n}}{\partial s}) \equiv \frac{1}{2\pi}\int_P \tau(s)\, ds$. s and **n** denote the arc length and the principal normal vector to P at the point $\mathbf{x}(s)$.

What is the meaning of this regularization ? By substituting the Gaussian, the dominant contribution to the surface integral comes from an infinitesimal strip Σ_P ; so this procedure effectively turns a spacetime curve into a *ribbon*. Precisely in 1961 such a process was used by Calugareanu [87]in his search for new invariants of the knot. The entity $I_G(C_\alpha \rightarrow C_\beta)$ turns out to be perfectly well defined and gives a new topological invariant SL , the self–linking number for a simple closed *ribbon.* SL is in fact the linking number of C_β with a twin curve C_α moved an infinitesimally small distance ε along the principle normal vector field to C_β. As disjoint curves they can be linked and unlinked exactly the strands of a circular supercoiled DNA molecule[88]. In modern knot theory this construction is termed the *framing* of a curve C_β . Of special importance to us is the existence of the "conservation law":

$$SL = T + W \tag{3.15}$$

explicitly

$$SL(P) = \frac{1}{4\pi}\int_{P\times P} d\Omega_2 + \frac{1}{2\pi}\int_P \tau\, ds \tag{3.16}$$

whereby SL, is the algebraic sum of two *differential geometric* characteristics of a closed ribbon, its total torsion or twisting number T and its *writhing number* or *writhe* W. . W is also the Gauss integral for the map $\phi : S^1\times S^1 \rightarrow S^2$, is the element solid angle, the pullback volume 2–form $d\Omega_2$ of S^2 under φ. While their sum SL is a topological invariant so must be an integer, T and W are metrical properties of the ribbon and its "axis" respectively, they can take a continuum of values. A coiled phone cord best illustrate the relation W + T = SL for a ribbon . When unstressed with its axis curling like a helix in space, its writhe is large while its twist is small. When stretched with its axis almost straight, its twist is large while its writhe is small.

By way of the dilatation invariance of W and the map $\mathbf{e}(s,u)$ $(\mathbf{e}^2 = 1)$, a local Frenet-Serret frame vector attached to the curve, we can write $W = \frac{1}{4\pi}\int_0^L ds \int_0^1 du\, \varepsilon_{abc}\, e^a \partial_s e^b \partial_u e^c$, a,b,c = (1,2,3) and $\partial_s = \partial/\partial s$, $\partial_u = \partial/\partial u$. A conformally invariant action for the frame field **e**, W is manifestly a WZNW term as well as a Berry phase upon exponentiation[8]. Since W = – T (mod Z), (3.15) explains Polyakov's double integral representation (modulo an integer) for the torsion T(P).

By way of (3.15) the alternate form $\Phi(P) = \exp(-i\pi\, T(P)) \exp(+i\pi\, n)$ is the "spin" phase factor, essential to Polyakov's proof that the 1-solitons of model (3.1) with $\theta=\pi$ are fermions by obeying a Dirac equation in their point-like limit [80]. For arbitrary θ, we go over to the more general case of pointlike anyons. So we see the relation W = -T + SL as the very mathematical expression of the connection between statistics and spin in the geometric point soliton limit.

We now recall that in the geometry of 2-surfaces, a form of the Gauss-Bonnet theorem says $K = 2\pi\chi$. Like (3.15), it relates a topological entity such as the Euler characteristics χ of a closed surface M to a metrical entity such as the total Gaussian curvature K for M. In applying (3.15) to supercoiled DNA, Fuller[89] in fact showed (3.15) to follow from the Gauss-Bonnet formula, one of the simplest examples of an index theorem. Thus it is pleasing to see how a fundamental physics principle, the relation between spin and statistics is mirrored by such a fundamental theorem of geometry, indeed the simplest of index theorem.

4. Anyonic Membranes

4.1 Hopf's Essential Fibrations and Division Algebras

By 1935, Hopf [90, 91]discovered an unique link between topology and the four division algebras **K= R, C, H** and **Ω**, namely the real, complex numbers, the quaternions and octonions by connecting the latter and the fibrations of S^{2n-1} by a great S^{n-1}-sphere, n=1, 2, 4 and 8 respectively.

Hopf's construction [91],[90]of his maps is most instructive. It can be directly inferred from Hurwitz's theorem which states: the only dimensions n of R^n with a multiplication R^n x

$R^n \rightarrow R^n$, denoted by $F(X,Y) = X\overline{Y}$ with $X\overline{Y}=0 \leftrightarrow X=0$ or $Y=0$ are n=1, 2, 4, 8. Namely these multiplications can be realized by the four division algebras over the reals **R**, **K** ≈ **R**, **C**, **H** and **Ω**, the real, complex numbers, quaternions and octonions respectively, $X, Y \in$ **K** i.e. $R^n \approx$ **K**.

Next by a linear identification of the product space **K**x **K** with R^{2n}, the product F(X,Y), X, Y ε **K**, defines a bilinear map, the Hopf map

$$H: R^{2n} \rightarrow S^{n+1} \quad (4.1)$$

with

$$H(X,Y) = (|X|^2 - |Y|^2, 2F(X,Y)) = (|X|^2 - |Y|^2, 2X\overline{Y}). \quad (4.2)$$

It follows that for $|X|^2 + |Y|^2 = 1$, $|H(X,Y)|^2 = (|X|^2 - |Y|^2)^2 + 4|XY|^2 = 1$. Considers two spheres, S^{2n-1} as the space of pairs (X, Y) of **K** with $|X|^2 + |Y|^2 = 1$ and S^n as the space of all pairs (s, **k**) of a real number $s = |X|^2 - |Y|^2$ and $\mathbf{k} = 2X\overline{Y} \in$ **K**. Thus H restricts to the map $H: S^{2n-1} \rightarrow S^n$ with S^{2n} and S^{n-1} as base space and fiber respectvely and S^{2n-1} as the fibre space, .

We parametrize S^n by a unit (n+1)-vector parametrizing $\vec{N}$, $\vec{N}^2=1$. Let $K^T = (K_1, K_2)$, $K_1, K_2 \in$ **K**, $K^\tau K=1$, be a unit normed **K**-valued 2-spinor parametrizing S^{2n-1}. The Hopf map (4.1) then reads

$$\vec{N} = Sc\left(K^\dagger \vec{\gamma} K\right) \quad (4.3)$$

with $K^\dagger = (\overline{K}_1, \overline{K}_2)$ and $\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ \overline{e_\mu} & 0 \end{pmatrix}$, μ= 0, 1,...m-1 and $\gamma_m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, m =1, 2, 4 and 8.

Alternatively, with S^{2n-1} in **K**x**K** and $S^n =$ **K** ∪ {∞}, the Hopf projection map, π : $S^{2n-1} \rightarrow S^n$ also reads

$$\pi(X,Y) = \begin{cases} X/Y & \text{provided } Y \neq 0 \\ \infty & \text{if } Y=0 \end{cases} \quad (4.4)$$

where $|X|^2 + |Y|^2 = 1$, X, Y ε **K**. $\pi^{-1}(X,Y)$, the pre-image (or inverse) of this Hopf map, is the geometric intersection of S^{2n-1} with an n-subspace of **K**x **K**, namely a great (n-1) sphere S^{n-1} or an (n-1) *cycle.* So the image of any point on S^n is a S^{n-1}-sphere on S^{2n-1}. This is apparent since $\vec{N}$ (or X/Y) is invariant under the phase transformation $K \to KU$ ($X \to XU$, $Y \to YU$). $U=\overline{U}$, $|U|^2=1$, is a unit normed, pure imaginary **K**-number, i.e. $U \varepsilon S^0 \approx Z_2$, $S^1 \approx U(1)$, $S^3 \approx SU(2)$ and S^7, an (n-1)-cycle for n = 1, 2, 4 and 8 respectively.

(4.2) The Hopf invariant in its many disguises

The Hopf invariant $\gamma(\Phi)$ classifies the maps $\Phi : S^{2n-1} \to S^n$. As an added topological action in the model (3.1) (n=2) it is essential to a dynamical realization of exotic spin and statistics. Our work[81] is in essence about the many faces of $\gamma(\Phi)$, its mathematically different and physically telling expressions[92]. First there is the connection to the abelian Chern-Simons invariant .

Let $V^{(p)}(M)$ be the space of p-forms on a manifold M, $p \leq \dim M$. On S^n, we select a normalized n-form area element ω_n, $\int_{S^n} \omega_n = 1$. On S^{2n-1}, by pulling back the Hopf map F, we define a second induced n-form $\widetilde{F}_n = \Phi^*\omega_n \ \varepsilon\ V^{(n)}(S^{2n-1})$ which is closed ($d\widetilde{F}_n = 0$) since $d(F^*\omega_n) = F^*(d\omega_n) = 0$ and $d\omega_n=0$. By de Rham's 2nd theorem $H^n(S^{2n-1}) \approx 0$, all closed n-forms on S^{2n-1} are exact, there is a non-unique (n-1)-form $\widetilde{A}_{n-1}\ \varepsilon\ V^{(n-1)}(S^{2n-1})$ such that $d\widetilde{A}_{n-1} = \widetilde{F}_n$. So the integral

$$\gamma(\Phi) = \oint_{S^{2n-1}} \widetilde{A}_{n-1} \wedge \widetilde{F}_n \qquad (4.5)$$

is defined. The following features hold:
a) $\gamma(\Phi)$ is independent of the choice of either $\widetilde{A}_{n-1}$ ($d\widetilde{A}_{n-1}=\widetilde{F}_n$) or of ω_n, b) $\gamma(\Phi) = 0$ for all maps $\Phi : S^{2n-1} \to S^n$ with n odd, c) $\gamma(\Phi)$ is invariant for any two smooth and homotopic maps $S^{2n-1} \to S^n$.

(4.5) is the Whitehead form of the Hopf invariant $\gamma(\Phi)$. For physicists this form is the Chern-Simons term for the Kalb-Ramond field $A_{n-1} \equiv 2\pi\, \widetilde{A}_{n-1}$ and property (a) translates simply into the gauge invariance of this antisymmetric Abelian gauge field F.

There are variants of the Hopf invariant. Let us first parametrize the map $F: S^{2n-1} \to S^n$

by a (n+1) component unit vector $\vec{N} \varepsilon S^n$, ($\vec{N}^2=1$). If $\vec{N}_0$ is an arbitrary fixed point on S^n, then as in the case of the complex Hopf fibration , $\vec{N}(x) = \vec{N}_0$ is thus the equation of a closed hypercurve $C_0 \approx S^{n-1}$on S^{2n-1}. Equivalently, the preimage of $C_0 = F^{-1}(\vec{N}_0)$ of $\vec{N}_0$ is an (n-1)-cycle in S^{2n-1}. If S_0 is some n-dimensional closed connected submanifold on S^{2n-1} with, as its boundary δS_0, C_0, then $\vec{N}(x)$ maps S_0, a Seifert surface, onto the whole n-sphere. The Hopf invariant $\gamma(\vec{N})$ can be defined as the number of times $\vec{N}$ maps S_0 onto S^n. It is the mapping degree of $\vec{N}(x)$ restricted to S_0, from S_0 to S^n, $\vec{N}(x) : \Sigma_0 \rightarrow S^n$ and is independent of the point $\vec{N}_0$ of S^n. With $\pi_n(S^n) \approx Z$, the Hopf invariant is then an integer . By a theorem of Eilenberg and Niven, representative maps $S^n \rightarrow S^n$ for n=2,4 and 8 with winding number m are given simply by X^m with X ε **C**, **H** and **Ω** respectively. So we also have a generalized flux and loop integral representation of $\gamma(\vec{N})$

$$\gamma(\vec{N}) = \oint_{\Sigma^0 \sim S^n} \widetilde{F}_n = \oint_{C^0 \sim \partial\Sigma^0} \widetilde{A}_{n-1} \tag{4.6}$$

where $\widetilde{F}_n = d\widetilde{A}_{n-1}$ is the area element nform of S^n mapped by $\vec{N}$ into S^{2n-1}. As it should be, these $\widetilde{F}_n$ and $\widetilde{A}_{n-1}$ are the same ones occuring in the Whitehead form of $\gamma(\vec{N})$. As (), the Hopf invariant gives, upon exponentiation, a generalized Aharonov-Bohm-Berry phase factor associated with its antisymmetric U(1) gauge field .

The connection, due to Hopf himself, between his invariant and Gauss' linking number cannot be simpler : γ (Φ) was originally defined as a linking number ! The map **N** represents an element in $\pi_{2n-1}(S^n)$. Pick two distinct points $\mathbf{N}_1$ and $\mathbf{N}_2$ on S^n, then their pre-images $F(\mathbf{N}_a) = C_a$ (a=1,2) are (n-1)-manifolds in S^{2n-1}. After assigning a natural orientation to these hypercurves we get two (n-1)-spheres in S^{2n-1} or (n-1)-cycles C_1 and C_2. They can be linked or unlinked ; $\gamma(\alpha)$ is just the linking numbers $Lk(\alpha_1,\alpha_2)$ of C_1 and C_2 and depends only on α . $\gamma(\alpha)$ is thus a homomorphism :

$$H : \pi_{2n-1}(S^n) \rightarrow Z \tag{4.7}$$

with the generalized Gauss linking coefficient to be given .

We finally list some useful properties of the Hopf invariant :

a) For n odd, H is zero in consequence of the anticommutativity of linking numbers

b) For n even , Hopf proved that maps of an *even* H always exist.

c) If the map $\Gamma : S^{2-1} \to S^{2n-1}$ has degree p the $\gamma(\Phi \circ \Gamma) = p\,\gamma(\Phi)$.

d) If the map $\Psi : S^n \to S^n$ has degree q then $\gamma(\Psi \circ \Phi) = q^2 \gamma(\Phi)$, where the degree of the map $S^n \to S^n$ is an element of $\pi_n(S^n)$.

4.3 Combining White and Adams' Theorems

In his 1969 thesis the mathematician White [93] derived the D>3 version of Calugareanu's formula as a byproduct of a reformulation of the Gauss-Bonnet-Chern theorem for Riemannian manifolds. In view of the established connection in section 3.2 it was natural to for us to extend Polyakov's approach to the D>3 counterparts of Wilcek-Zee σ-model (3.1) . First we need to generalize Gauss linking number to higher dimensional manifolds.

Extending to D>3 manifolds the procedure for linking 3-space curves, we consider two continuous maps f(M) and g(N) from two smooth, oriented, non intersecting manifolds M and N, dim(M) = m and dim(N) = n, into R^{m+n+1}. Let S^{m+n} be a unit (m+n)–sphere centered at the origin of R^{m+n+1} . Let $d\Omega_m$ be the pull–back S^{m+n} volume form under the map $e : M \times N \to S^{n+m}$ where to each pair of points $(\mathbf{m},\mathbf{n}) \in M \times N$ is associated the unit vector e in R^{m+n+1} : $e(\mathbf{m},\mathbf{n}) = \frac{g(\mathbf{n})-f(\mathbf{m})}{|g(\mathbf{n})-f(\mathbf{m})|}$. The degree of this map

$$L(f(M),g(N)) \equiv L(M,N) = \frac{1}{\Omega_{n+m}} \int_{M \times N} d\Omega_{n+m} \qquad (4.8)$$

is the Gauss linking number of M and N. Ω_n $(=2\pi^{(n+1)/2}/\Gamma((n+1)/2))$ is the volume of S^n. Due to the non-commutativity property $L(M, N) = (-1)^{(m-1)(n-1)} L(N, M)$, L (M,N) vanishes for <u>even</u> dimensional submanifolds M and N .

White's main theorem states: Let $f : M^n \to R^{D=2n+1}$ be an smooth embedding of a closed oriented differentable manifold into Euclidean (2n+1) space. Let v be an unit vector along the mean curvature vector of M^n . If n is <u>odd</u> (i.e. D=3, 7, 11,15 etc...) then

$$SL(f,f_\varepsilon) = \frac{1}{\Omega_{2n}}\int_{M\times M} d\Omega_{2n} + \frac{1}{\Omega_n}\int_M \tau^* \, dV \qquad (4.9)$$

is the self-linking number of a hyper-ribbon. The latter consists of M^n and the same manifold deformed a small distance ε along v. The two terms on the RHS of (4.9) are respectively the generalized writhing and twisting numbers, W and T, of the hyper-ribbon. The cases of even n (D= 1,5,9,...) are of no interest to us since both W and T are zero and hence also SL =0.

The universality of the formula SL = W + T (3.15) mirrors that of Gauss-Bonnet-Chern theorem. As a possible physical application, we expect that for solitons in suitable D> 3 models White's general formula T = -W (mod Z) similarly links their spin and statistical phases . It would define and relate the twisting and writhing of odd dimensional closed S^3-, S^5-, S^7-... hyper-ribbons, the world volumes of topological S^2 -, S^4-, S^6- membranes solitons in D=7, 11, 15... dimensional spacetime respectively. The first problem is to cut down this infinity of choices ? What are the natural D >3 σ-model counterparts of (3.1) which may admit solitons with exotic spin and statistics ?

In seeking for exact analogs of θ-spin and statistics among D>3 extended objects, at least three key features of the CP(1) model (3.1) should be maintained:

1) the existence of topological solitons,
2) the presence in the action of an Abelian Chern-Simons-Hopf invariant;
3) the associated Hopf mappings $S^{2n-1} \rightarrow S^n$ include ones with Hopf invariant 1.

The first two requirements are embodied in the time component of the key equation (3.12) . Upon integration over all of space of both members of this equation, one obtains the topological charge-magnetic flux coupling which is at the very basis of the fractional statistics phenomenon in (2+1) dimensions. As to the third requirement, essential to the proof of the fractional spin and statistics for one soliton, the following striking feature holds true for these Hopf mappings. While for any n even there always exists a map f : $S^{2n-1} \rightarrow S^n$ with only even integer Hopf invariant $\gamma(f)$, the existence question of Hopf maps of invariant 1 received the final answer in the celebrated theorem of Adams[94]:

If there exists a Hopf map Φ: $S^D \rightarrow S^{(D+1)/2}$ of invariant $\gamma(\Phi) = 1$, in fact of $\gamma(\Phi)$ = any integer, then D must equal 1, 3, 7 and 15 (m = (D+1)/2 = 1, 2, 4 and 8)

So there can only be four and only four classes of Hopf maps with $\gamma(\Phi) = 1$. They along with their associated hidden (or holonomic) gauge field structures are best displayed through the following diagram of spheres bundles over spheres:

$$
\begin{array}{l}
\qquad\qquad U(1) \approx SO(2) \\
\qquad\qquad\qquad \| \\
Z_2 = O(1) \approx S^0 \to \mathbf{S^1} \to \mathbf{S^1/Z_2} \approx \mathbf{RP(1)} \approx SO(2)/Z_2 \\
\qquad\qquad\qquad \| \\
SO(2) \approx U(1) \approx S^1 \to \mathbf{S^3} \to \mathbf{S^2} =\approx \mathbf{CP(1)} \approx SU(2)/U(1) \\
\qquad\qquad\qquad\qquad \| \\
SU(2) \approx Sp(1) \approx S^3 \to \mathbf{S^7} \approx S0(8)/SO(7) \to \mathbf{S^4} \approx \mathbf{HP(1)} \approx Sp(2)/Sp(1)\psi Sp(1) \\
\qquad\qquad\qquad\qquad\qquad \| \\
Spin(8)/Spin(7) \approx S^7 \to \mathbf{S^{15}} \approx Spin(9)/Spin(7) \to \mathbf{S^8} \approx \mathbf{\Omega P(1)} \approx Spin(9)/Spin(8).
\end{array}
$$

The four rows reflect the one to one correspondence between the four division algebras over **R** and the real (**R**), complex (**C**), quaternionic (**H**) and octonionic (**W**) Hopf bundles (displayed in bold letters). The first three principal bundles are actually the simplest members of the three infinite sequences of the **K**= **R**, **C**, **H** universal Stiefel bundles over Grassmannian manifolds . The fourth bundle stands alone, a fact connected to the non-associativity of octonions.

The spheres S^p, p = 0, 1, 3 and 7 are the fibers,the first three are Lie groups while S^7 is a very special coset space, the space of the unit octonions. The latter has been an exotic object of fascination and discoveries in mathematics and in the Kaluza-Klein compactification of D=11 supergravity and supermembrane theories . An n-sphere S^n is parallelizable if there is a continous family of n orthonormal vectors at each its points. The fact that S^1, S^3 and S^7 are the only parallelizable spheres is yet another corollary to Adams' theorem. S^r, r = 1, 3, 7, 15 constitute the corresponding fibre spaces. Finally the sequence of base spaces S^n, n = 1, 2, 4, 8 are equally interesting as **K**-projective lines, as is clear from their coset forms. With their holonomy groups Z_2, SO(2), SO(4) and SO(8) being the norm groups of **R**, **C**, **H** and **Ω** they can be said to have a real, complex, quaternionic and octonionic Kähler structures .

The Hopf maps f : $S^{2n-1} \to S^n$, n=1, 2, 4, 8 with *Hopf invariant one* have found important physical applications in condensed matter physics and in quantum field theory. Even the connection between Hopf maps and nonstandard spin and statistics had been lurking in the background. Thus, in the D=2 ϕ^4 field theory , it was shown that the n=1 real Hopf map realizes the 1-kink soliton , which carries intermediate spin and admits exotic statistics. Besides being the Dirac 1-monopole bundle, the n=2 complex Hopf map underlies the θ spin and statistics of D=3 CP(1) model. The n=4 quaternionic Hopf map is the embedding map for the SO(4) invariant, D=4 SU(2) BPST 1-instanton or the SO(5) invariant, D=5 SU(2) Yang monopole with eg=1/2. The n=8 octonionic Hopf map appears as a SO(9) invariant, D=8 SO(8) 1-instanton. The latter two maps admit further realizations in terms of U(1) tensor gauge fields associated with extended Dirac monopoles with eg = 1/2 in p-form Maxwellian electrodynamics.

We have noted that the field theory realizations of the **R**- and **C**- Hopf fiberings both admit exotic spin and statistics. It is then only natural to ask whether this pattern persist in suitable theories built on the two remaining Hopf fibrations, $S^{2n-1} \rightarrow S^n$ for n= 4, 8? Clearly the answer should be sought within the quaternionic D=7 HP(1) ($\approx S^4$) and the octonionic D=15 ΩP(1)$\approx$ (S^8) σ-models augmented with their respective Hopf invariant term. We consider them next

4.4 Division algebra σ- models with a Hopf Term

In mathematics, the standard nonlinear σ-models are well known as *harmonic maps.* One associates with the map $\Psi : M \rightarrow N$ between two Riemannian manifolds M and N an action

$$S = \frac{1}{2}\int_M |d\Psi(x)|^2\, d^m x .$$

$d\Psi(x)$ is the differential of Ψ at the point x Θ M and $d^m x$, the volume element of M. In a coordinate patch, $|d\Psi|^2 = g^{ij}\frac{\partial\Psi^\alpha}{\partial x^i}\frac{\partial\Psi^\beta}{\partial x^j}h_{\alpha\beta}$ is the pullback on M of the metric $ds^2 = h_{\alpha\beta}d\Psi^\alpha d\Psi^\beta$ on N. Ψ is called harmonic if it leads to a vanishing Euler-Lagrange operator (or tension field) div($d\Psi$) $\equiv$ 0. The quadratic Hopf map $\Psi(X,Y) : S^{2n-1} \rightarrow S^n$ (n= 2, 4, 8) is in fact a harmonic polynomial map, with constant Lagrangian density $|d\Psi(x)|^2 = 2n$. As such it is the simplest harmonic representative of maps with Hopf invariant 1 .

While the D=3 CP(1) σ-model [83]) admits exact finite energy static solitons, the corresponding D=7 HP(1)($\approx S^4$) and the D=15 ΩP(1)($\approx S^8$) σ-models do not . This is clear from the Hobart-Derrick scaling argument . In practice, as in the Skyrme model , dynamical stability can be insured either by coupling the model to a gauge field or by adding to the standard KP(1) σ-model action with suitable chiral invariant terms of higher order in the field derivatives. Taking the second alternative, the generic σ-model action with the added Hopf term then reads

$$S_{(n)} = \int_M \partial_\mu \vec{N} \cdot \partial^\mu \vec{N} \, d^{2n-1}x + \frac{\theta}{a} \int_M A_{n-1} \wedge dA_{n-1} + \text{ suitable Skyrme terms;} \qquad (4.10)$$

$$n = 4, 8, \quad M = S^7, S^{15}$$

where the unit vector **N** with **K** = **H** and **Ω** is given (4.3). The composite U(1) ATGF A_{n-1}, nonlocal in N, is local in the 2-spinor K (4.3). Its expression in terms of K will be given later.

The θ-term can be rewritten as

$$S_I = \frac{1}{(n-1)!} \int d^{2n-1}x \; J^{\mu_1 \ldots \mu_{n-1}} A_{\mu_1 \ldots \mu_{n-1}} \qquad (4.11)$$

i.e. an interaction of the potential A_{n-1} with the topological current $J_{n-1} = -\frac{(n-1)!\theta}{4\pi^2} {}^*F_n$ (n=4, 8). The latter's conservation and expression in terms of **N** will be shortly deduced solely from the field topology. Since the sources of J_{n-1} are charged solitons, we shall first determine what types of solitons are allowed in our KP(1) models.

In condensed matter physics our σ-models are the familiar Wegner's n-vector models . As field theories of a 5- and 9- unit-vector order parameter $\vec{N}$, they are the quaternionic and octonionic counterparts of the isotropic Heisenberg ferromagnet, albeit in rather exotic higher dimensions and with an added Chern-Simons terms. In consequence the nature and dimensionality of their allowed topological defects should only depend on the dimensionalities of the order parameters and of the compactified spacetime. They should obey the defect formula of Toulouse and Kleman [95].

Consider a topological defect of spatial dimension d' in D-space or D-Euclidean spacetime. To measure its homotopic charge, we need to completely "surround" this defect by a submanifold of dimension r such that d' + r + 1 = D. The meaning of the contribution 1 on the LHS of this last relation is evident for a vortex line; it corresponds to the distance in 3-space (D=3) between the line defect (d'=1) and its surrounding loop (r=1) . The topological charge labels the equivalence classes of the group $\pi_r(S^n)$ of mappings $S^r \to S^n$, of the spatial submanifold S^r into the space of the (n+1) unit vector order parameter $\vec{N}$. With r

< n and $\pi_r(S^m) = 0$ for $r < m$, $\pi_m(S^m) = Z$, it follows that topologically stable ($\pi_r(S^n) \neq 0$) defects must have spatial dimension $d' = D - 1 - r = D - (n+1)$. So there exist no stable defects for $(n+1) > D$ and $(n+1)<0$, but for $0 < (n+1) < D$, the so called *triangle of defects* in the $((n+1), d)$ plane, there are defects of various kinds, points, vortices, membranes etc... Furthermore if $D > 4$ such as in Kaluza-Klein-typed theories and if $r > m$ $\pi_r(S^m)$ is generally nontrivial, even a richer variety of defects are possible.

Applying the Toulouse-Kleman formula to our cases of (D, r=n) = (3, 2), (7, 4) and (15, 8) we find that the allowed topological defects in the CP(1), HP(1) and ΩP(1) σ-models (3.16) to be 0-, 2- and 6-membrane solitons, their topological charges being the generators of

$\pi_n(S^n) \approx Z$, n=2, 4 , 8.

Since our solitons are charged 2- and 6- membranes, we expect the associated σ-models to possess a rank 3 and rank 7 topological conserved current $J^{\mu\rho\sigma}$ and $J^{\mu\rho\sigma\alpha\beta\gamma\lambda}$. Their conservation follows solely from the constraint $N^2 = 1$, hence $\mathbf{N}\cdot\partial_\mu \mathbf{N} = 0$, and the fact that n the dimension of the unit vector N is less or equal the dimension D of spacetime. Since here (D, n) = (7, 5), (15, 9), the latter condition is satisfied. Indeed

$$\partial_{\mu_1} J^{\mu_n \mu_{n+1} \ldots \mu_D} = 0 \qquad (4.12)$$

with $J^{\mu_n \mu_{n+1} \ldots \mu_D} = \varepsilon^{\mu_1 \ldots \mu_D} \varepsilon_{\alpha_1 \ldots \alpha_n} \left(\partial_{\mu_1} N^{\alpha_1} \ldots \partial_{\mu_{n-1}} N^{\alpha_{n-1}} \right) N^{\alpha_n}$

As with the CP(1) model, these 3 and 7-form conserved currents, suitably normalized, are just the D=7 and 15 Hodge duals of the respective 4- and 8-forms antisymmetric gauge fields $F_n = dA_{n-1}$ appearing in the Hopf invariant action in (3.17): $J_n = -\frac{n!\theta}{4\pi^2} {}^*F_{n+1}$, with the star operation denoting the Hodge dual ${}^*F_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} F^{\mu_{n-p+1} \ldots \mu_n}$.

The conserved current (4.12) can be converted into a conservation law. Two equalities are used : a) Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$, ω is a p-form and M is an oriented compact manifold with boundary ∂M, D = dim M = (p+1) and dim(∂M) = p; b) the relation between the divergence and the exterior derivative: $\partial_\alpha \omega^{\alpha}_{\mu_1 \ldots \mu_p} = (-1)^{Dp} [{}^*d^* \omega]_{\mu_1 \ldots \mu_p}$. With the latter identity 4.12) becomes $d^*J = 0$. Its integration over a (D-p+1)-dimensional manifold M with boundary ∂M gives

$$\oint_{(\partial M)} {}^*J = 0 \ . \tag{4.13}$$

If ∂M consists of two spacelike hypersurfaces Σ (with $\dim(\Sigma) = D-p$), connected by a remote timelike tube ∂T and since the topological current J in our σ-models is localized in space, the integral over ∂T vanishes and (4.13) gives the Lorentz invariant and conserved charge $Q = \int_\Sigma {}^*J$, its value being independent of Σ. Applied to our KP(1) σ-models where the equations of motion , e.g.(3.12), forces a θ-dependent linear relation between topological charge and flux , for (D, p) =(3, 1) ( ) reduces to the Skyrmion winding number, the generator $\pi_2(CP(1)) \approx Z$

$$Q = \frac{-1}{4\pi}\int_{S^2} d\Sigma^{ij} F_{ij} = \frac{-1}{2\pi}\int_{S^1} dx^i A_i = C_1 \quad (\text{for } q = p). \tag{4.14}$$

(Note that we are using Roman indices for the spacelike components.) It is also the first Chern index C_1 of the U(1)-bundle, the complex $S^3 \rightarrow S^2$ Hopf fibration. For (D,p) = (7,3) and (15,7), the topological charges of the membrane S^4- and S^8- solitons and the generators of $\pi_4(HP(1)) \approx Z$ and $\pi_8(\Omega P(1)) \approx Z$ are similarly given for $\theta = \pi$ by

$$Q = \int_{S^3} J^{0 i_1 i_2} d\Sigma_{i_3 \cdots i_6} = \frac{-1}{4\pi}\int_{S^4} F_{i_3 \cdots i_6} d\Sigma^{i_3 \cdots i_6} = \frac{-1}{2\pi}\int_{S^3} A_{i_3 i_4 i_5} d\Sigma^{i_3 i_4 i_5} \tag{4.15}$$

and

$$Q = \int_{S^7} J^{0 i_1 \cdots i_6} d\Sigma_{i_7 \cdots i_{15}} = \frac{-1}{4\pi}\int_{S^8} F_{i_7 \cdots i_{15}} d\Sigma^{i_7 \cdots i_{15}} = \frac{-1}{2\pi}\int_{S^7} A_{i_7 \cdots i_{14}} d\Sigma^{i_7 \cdots i_{14}} \tag{4.16}$$

respectively.

That Q is equal to n, an integer can be seen through the mentioned gauge field connection between our problem and the D=2 complex, D=4 quaternionic and D=8 octonionic instanton. We take for the KP(1) field coordinate, the mapping $K(x) = x^n$, where x is the space position K-number in Σ, K = C, H and Ω. While these maps are not 0-, 2- or 6-membrane solutions to the systems (4.10), they are the simplest harmonic representatives maps : $S^m \to S^m$ (m =2, 4 and 8) with topological number Q= n . As will be clear the charges (4.15) and (4.16) can be identified with the 2nd and 4th Chern index which reflects the relation between the U(1) ATGF and the hidden non-Abelian gauge structure of the σ-models, namely the Sp(1) quaternionic and associated Spin(8) octonionic Hopf fibrations respectively. Though our subsequent analysis of the thin soliton limit deals primarily with the θ-term in (4.10), the Hopf term, to make clear the hidden gauge connection we now consider the $HP(1) \approx S^4$ model in greater detail. An analogous discussion of the ΩP(1) model can be carried out.

As the coset space Sp(2)/Sp(1)x Sp(1), the quaternionic projective line HP(1) can be parametrized in two ways (see Sect.2f.3 for details) . Either we have by two real quaternions q_1 and q_2 with $|q_1|^2 + |q_2|^2 = 1$, i.e by a 2-component H-spinor $Q^T = (q_1, q_2)$, $Q^\tau Q = 1$, coordinatizing the sphere S^7 or by one quaternionic inhomogeneous coordinate $h = q_2 q_1^{-1}$. An alternative parametrization is by the unit 5-vector **N** defined by the Hopf projection map (4.3) from S^7 to S^4, $\vec{N} = Sc\left(Q^\dagger \vec{\gamma} Q\right) = \left(N = \frac{2h}{1+\bar{h}h}, N_5 = \frac{1-\bar{h}h}{1+\bar{h}h} \right)$. To make manifest the local $Sp(1) \approx SU(2)$ gauge invariance

$$q_\alpha' = U(x)\, q_\alpha \qquad \alpha = 1, 2 \;;\; U(x) \in Sp(1) \tag{4.17}$$

of the HP(1) model, we introduce the covariant derivative $D_\mu Q = (\partial_\mu + a_\mu)Q$. The holonomic Sp(1) gauge field is $a_\mu = \mathbf{a}_\mu \cdot \mathbf{e} = Q^\tau \partial_\mu Q = \frac{1}{2} \bar{q}_\alpha \partial_\mu q_\alpha = \frac{1}{2} \frac{\bar{h} \overleftrightarrow{\partial_\mu} h}{1+\bar{h}h}$ is purely vectorial and takes the ADHM form [96] for the 1-SU(2) instanton solution. So the first term in (4.10) reads

$$S_{(4)\,0} = Sc\left\{(D_\mu Q)^\dagger (D^\mu Q)\right\} \tag{4.18}$$

and similarly for the Skyrme terms.

As for the Hopf term, we can check that the 3-form $A_{(3)}$ to be the D=4 Sp(1) Chern-Simons form of Lüscher et al.:

$$\begin{aligned} A_{(3)} &= \frac{1}{3!} A_{[\mu\nu\lambda]} dx^\mu dx^\nu dx^\lambda \;, \\ &= Tr(A \wedge dA + \tfrac{2}{3} A^3) \end{aligned} \tag{4.19}$$

$$F_{(4)} = dA_{(3)} \tag{4.20}$$

where $dx^\mu dx^\nu = dx^\mu \Lambda dx^\nu$ etc... In terms of the 2-spinor Q, they read

$$A_{(3)} = Sc\left\{Q^\dagger dQ\, dQ\, dQ + \frac{1}{3}(Q^\dagger dQ)^3\right\}, \tag{4.21}$$

$$F_{(4)} = Sc\left\{dQ^\dagger dQ + (Q^\dagger dQ)^2\right\}. \tag{4.22}$$

These forms clearly show the *local* nature of the Hopf term when written in terms of quaternionic -valued field Q of the bundle space S^7. It is thus locally a total divergence as is already clear from (4.6).

A parallel derivation of $A_{(7)}$ and $F_{(8)} = dA_{(7)}$ can be done for the D=15 $\Omega P(1)$ σ-model. In fact the connection to the D=8 octonionic instanton problem identifies the 7-form $A_{(7)}$ as the D=8 Chern-Simons term of a Spin(8) gauge field :

$$A_{(7)} = Tr\left\{ A(dA)^3 + \frac{4}{3}A^3(dA)^2 + \frac{6}{5}A^5 dA + \frac{4}{7}A^7\right\}. \tag{4.23}$$

The above specifics of the σ-models are sufficient for our analysis. Being essentially nonlinear, our models are analytically highly intractable in their field theoretic details. Besides, there is much arbitrariness in the choice of Skyrme terms which, being higher order in the field derivatives, control the shorter distance solitonic structure. As in the D=3 case, the latter structure is not relevant to the problem of phase entanglements of the solitons. Only the existence but not the details of the Skyrme terms matter. It is enough to analyse the effective theories obtained in the geometrical Nambu-Goto limit of widely separated membranes.

Referring to and for the relevant details, we can show using the method of Umezawa et al that our membrane solitons have a thin London limit . Thus Polyakov's approximation for the models (4.10) translates into the Chern-Simons-Kalb-Ramond electrodynamics of Nambu-Goto membranes. Ti regularate the ultraviolet divergences of the theory we can also add a Maxwellian kinetic terms for the antisymmetric gauge field.

To obtain the statistical phase [97], we consider the propagation of two pairs of membranes-antimembrane and compute the phase resulting from adiabatically exchanging the two membranes. We get

$$\left\langle \exp\left(\frac{i}{3!}\oint_{P_1} A^{[\mu\nu\lambda]}dx^\mu\wedge dx^\nu\wedge dx^\lambda\right)\exp\left(\frac{i}{3!}\oint_{P_2} A^{[\mu\nu\lambda]}dx^\mu\wedge dx^\nu\wedge dx^\lambda\right)\right\rangle \qquad (4.24)$$

P_1 and P_2 are S^3 hyper-curves. The functional average $\langle\cdots\rangle$ is taken over the Hopf action (9). As in D=3 case , the resulting phase here is the sum of three phases. The first contribution gives the phase factor $\exp\{2i\,(\pi^2/\theta)\,L\}$ with L being the generalized Gauss' linking coefficient (4.8) for two S^3-loops. We get π^2/θ for the statistical phase. The other two phase factors $\Phi(P_i)$ are given by the expectation value of one hyperloop:

$$\Phi(P) = \left\langle \exp\left(\frac{i}{3!}\oint_{P} A^{[\mu\nu\lambda]}dx^\mu\wedge dx^\nu\wedge dx^\lambda\right)\right\rangle . \qquad (4.25)$$

In the London-Nielsen-Olesen limit the effective action reads [98]

$$S = S_0 + \frac{\theta}{(3!)^2 a}\int_{S^7} d^7x\, \varepsilon_{\mu\nu\lambda\alpha\beta\gamma\delta}\, A^{\mu\nu\lambda}\partial^\alpha A^{\beta\gamma\delta} + \frac{1}{3!}\int_{S^7} d^7x\, J_{\mu\nu\lambda}A^{\mu\nu\lambda} , \qquad (4.26)$$

where S_0 is free Nambu-Goto action for a 2-membrane, $0 \le \theta \le \pi$ and a is a constant to be freely chosen . Direct integration of the equation of motion

$$J_{\mu\nu\lambda} + 2\,\frac{\theta}{3!a}\varepsilon_{\mu\nu\lambda\alpha\beta\gamma\delta}\partial^\alpha A^{\beta\gamma\delta} = 0 \qquad (4.27)$$

with $J^{\mu\nu\lambda}(y) = \int d^3x\, \delta^7(x-y)\, \frac{\partial(x^\mu,x^\nu,x^\lambda)}{\partial(\tau,\sigma_1,\sigma_2)}$. , in the Lorentz gauge $\partial^\alpha A_{\alpha\beta\gamma} = 0$, gives

$$i\frac{1}{2\cdot 3!}\int d^7x\, J_{\beta\gamma\delta}A^{\beta\gamma\delta}(x) = = i\,\frac{a}{4\theta\,\Omega_6}\frac{1}{}\int_{S^3} d\Sigma_{\beta\gamma\delta}\int_{S^3} d\Sigma_{\mu\nu\lambda}\frac{\varepsilon^{\mu\nu\lambda\alpha\beta\gamma\delta}(x-y)_\alpha}{|x-y|^7} \qquad (4.28)$$

Here the double 3-sphere integration is over one and the same hypercurve S^3 ; the phase (4.25) is *undetermined* unless we regulate[99] the short distance divergence say by including the Maxwellian kinetic term for the gauge field $A_{\mu\nu\rho}$. The regularized phase is then

$$\Phi\,(\,P \approx S^3) \;=\; \exp\left(i\,\frac{a}{4\theta}\,W(P)\right) \qquad (4.29)$$

where $W(P) = \frac{1}{\Omega_6}\int_{S^3\times S^3} d\Omega_6$ is the writhe of the Nambu-Goto S^2-membrane tracing the Feynman path P, a S^3 hyper-ribon in 7-spacetimes. A parallel computation gives the same expression as (4.29) for Φ in the octonionic case of the 6-brane S^6 with $P \approx S^7$ in S^{15} - spacetime .

Setting $a = 4\pi^2$, $\Phi(P)=\exp(\pi^2 iW/\theta)$. Invoking White's formula (4.9) we obtain for $\theta = \pi$ the exact S^3- (S^7-) counterpart of Polyakov's phase factor $\Phi(P) = \exp(-\pi i\, T(P)) \exp(\pi i\, n)$, T(P) being the generalized torsion for an S^3-(S^7) ribbon P. This phase factor presumably embodies the thin membrane's spin in a functional integral formalism. If this reasonable expectation is realized by an explicit construction à la Polyakov[80] of the spin factor for membranes, we will then have a 7- (15-) dimensional analog of the D=3 Fermi-Bose transmutation. With the value of θ not being fixed by the gauge invariance of the antisymmetric tensor gauge field, we have in general the possibility of fractional statistics and spin via the relation W = -T (mod Z) for solitonic membranes.

Finally, without knowing the short distance soliton structure or performing a detailed canonical quantization of the above $\mathbf{KP}_1$ σ-models, the case for the θ-spin and statistics among our Hopf- membranes can be made on topological grounds. We focus on the topology of the configuration space of fields Γ of the above **KP**(1) σ-model . In the Schrödinger picture, the space Γ of finite energy static solutions is the *mapping space* of all based preserving smooth soliton maps $\vec{N}$ (x) : $\mathbf{x} \in S^n \;\rightarrow\; \vec{N}(x) \in S^n$, n = 2, 4, 8 . Γ is an infinite Lie group with the nontrivial connectivity property:

$$\pi_0(\,\Gamma = \{\vec{N}: S^n \rightarrow S^n\}) \approx \pi_n(S^n) \approx Z. \qquad (4.30)$$

So Γ is split into an infinite set of pathwise-connected components Γ_α, $\alpha \in Z$, corresponding to the various soliton sectors labelled by the charge Q. For our membranes, as with Skyrmions and Yang-Mills instantons, each sector Γ_α has <u>further</u> topological obstructions. G.W. Whitehead showed that all the Γ_α's in Γ have the *same* homotopy type i.e. $\pi_i(\Gamma_\alpha) \approx \pi_i(\Gamma_\beta)$. The relations

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx Z \text{ for } (i, n) = (1, 2), (3, 4), (7, 8). \qquad (4.31)$$

are of particular relevance to the question of exotic spin and statistics, for the 1–soliton sector

.

They result from the Whitehead and Hurewicz's isomorphisms, the latter stating $\pi_i(\Gamma_\alpha) \approx \pi_{i+n}(S^n)$, and reflect the *multi-valuedness* of Γ_i. (4.31) imply the possibility of adding to the KP(1) σ-model action a Hopf invariant $\gamma(\vec{N})$, the generator of the torsion free part of $\pi_{i+n}(S^n)$ ($\pi_3(S^2) \approx Z$, $\pi_7(S^4) \approx Z \oplus Z_{12}$ and $\pi_{15}(S^7) \approx Z \oplus Z_{120}$). Generalizing the CP(1) model $\{(i, n) = (1,2)\}$, the nontriviality of these $\pi_i(\Gamma_1)$ implies the possibilities of Aharonov-Bohm effects of a multiply connected configuration space Γ and signals for the membrane solitons the existence of a higher dimensional analog of a θ spin and statistics connection .

In the CP(1) case,upon a 2π rotation P of the Skyrmion or an interchange of two Skyrmions, the Hopf term induces a spin phase factor $\Phi(P) = \exp\{i\theta\} = \exp(i\,2\pi s)$, s being the soliton spin .The equality $\theta = 2\pi s$ for this process of rotation is a physical realization of the homomorphism:

$$\pi_1(SO(2)) \approx \pi_3(S^2) \approx \pi_1(\Gamma_1) \approx Z. \qquad (4.32)$$

It establishes the equality of the kinematically allowed exotic spin to the dynamically induced θ-spin by way of the Hopf term. Notably (4.32) is but a special case of the Hopf-Whitehead J-homomorphism $\pi_k(SO(n)) \approx \pi_{k+n}(S^n)$. Generally we have the following chain of homomorphisms :

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx \pi_i(SO(n)) \approx Z \qquad (4.33)$$

with (i, n) = (1, 2), (3, 4), (7, 8) . $\pi_3(SO(4)) \approx \pi_7(S^4) \approx Z$, $\pi_7(SO(8)) \approx \pi_{15}(S^8) \approx Z$. Clearly the most natural physical interpretation of these topological relations is a dynamically induced exotic spin and statistics connection for the 2-and 6-membranes.

The foregoing analysis represents a first assault on the problem of the spin and statistics connection for higher dimensional topological extended objects. It is a small step both in the bosonic functional integral formulation for spinning extended objects and in the study of the θ-vacuum phenomenon in Kaluza-Klein compactification. Paralleling the study of p-branes it would be of interest to supersymmetrize the above theories, to study the canonical quantization of anyonic membranes, attempt the construction of a thin membrane soliton operator, tackle the representation theory of associated algebras of diffeomorphisms with their specific central extensions. In a broader framework the systems discussed here are parts of higher dimensional topological field theories[100].

5. Parting Remarks

We have entered into a new phase of extensive developments and applications of algebraic methods to physics. In this review we try to illustrate in the context of field theoy some deep interconnections between topology, geometry, division algebras and the representation theory of certain infinite algebras. The unifying entity is a geometrical phase carried by solitonic excitations realized as projective representations of certain current algebras. Kac-Moody groups and their higher dimensional counterparts have provided the common thread for seemingly disparate areas of physics and mathematics.Why is there such unreasonable effectiveness of mathematics in accounting for physical phenomena ? Writing in the Notices of the American Mathematical Society, Weinberg[101] advanced a tantalizing explanation : " It is because some mathematicians have sold their soul to the Devil in return for advance information about what sort of mathematics will be of scientific importance". If he is right, some of us should perhaps consider taking this Faustian path in order to make significant headway on the problems outlined here.

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