In-situ Response Time Testing of Platinum Resistance Thermometers

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In-situ Response Time Testing of Platinum Resistance Thermometers

Noise Analysis Method

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SUMMARY

Estimation of temperature sensor characteristics by the noise analysis method is presented in this part of the final report on In-Situ Response Time Testing of Platinum Resistance Thermometers. Temperature fluctuations in hotleg and coldleg fluid during normal reactor operation are recorded and random data analysis techniques are applied to obtain information about the sensor performance. The noise signal is modeled using autoregressive time series modeling strategy of the form

$$y_k = \sum_{i=1}^{n} a_i \, y_{k-i} + v_k$$

where $\{y_k\}$ is the stationary sensor output and $\{v_k\}$ is the temperature fluctuation noise. The model evaluation consists of the following four steps: (1) Estimation of model parameters.(2) Selection of optimal model order, n. (3) Model validation to check for the assumptions made in the analysis. (4) Evaluation of sensor characteristics from the model.

Chapter 1 is an introduction to the noise analysis approach described in this report. Some basic notions about probability and useful definitions are summarized in Chapter 2. Chapter 3 provides an overview of time series models for stationary random signals. Model parameter estimation, selection of optimal order and model validation methods are described in Chapter 4. Estimation of response time characteristics from the autoregressive model is detailed in Chapter 5, which also includes results for a simulated fifth order system. In Chapter 6, we present the details of the methodology applied to RTD noise signal from an operating power plant. The limitations of the method are discussed. Results from

tests at four pressurized water reactors are summarized. Summary and concluding remarks are given in Chapter 7.

The limited bandwidth of the noise signal and in some cases the absence of any distinct break frequency in the power spectrum results in either an incorrect (too large) value for the time constant or failure of the method. Based on the analysis and comparison with the loop current step response testing method it is concluded that the noise analysis technique may be used as a tool for monitoring degradation of sensor characteristics, rather than for predicting quantitative values of the time constants.

CHAPTER 1

INTRODUCTION

Random noise techniques in measurements on nuclear reactor systems are developed to measure the dynamic behavior or as a tool for system surveillance with a minimum of interference during normal operation.

The microscopic output noise can be considered as "the response of the system to an input representing the statistical nature of the underlying process." (See Uhrig [U1]). The coolant temperature fluctuations during normal operation of a reactor are caused by random variations in neutron flux and random variations in the heat transfer taking place in the core and steam generator.

Noise signals may be interpreted by modeling the response using a time series model. If $\{y_k\}$ is a sequence of measurements of a random process, and if $\{v_k\}$ is a noise sequence causing the $\{y_k\}$, then the input-output relationship can be represented by the linear transformation:

$$y_k = \sum_{i=1}^{\infty} a_i y_{k-i} + \sum_{i=1}^{\infty} b_i v_{k-i} + v_k. \qquad (1.1)$$

This representation assumes that there are no externally controlled perturbations. In other words, the statistics of $\{v_k\}$ is not known a priori. Special cases of the above model have been used for specific applications. The above model may be generalized to represent the dynamics of multivariable systems. Discussion of time series models is given in Chapter 3.

In the present context, the finite order time series modeling procedure is applied to the noise signals recorded from temperature sensors

installed in the hot leg and cold leg sections of nuclear power reactors. The data acquisition is performed during normal operation of the power plant, requiring a minimum of instrumentation. The data can be processed on site with the aid of a minicomputer and the sensor response characteristics may be estimated. The method can thus be standardized as an integral part of the plant monitoring system.

One method that is potentially useful for sensor response time estimation is frequency domain analysis of fluctuating signals. In the frequency domain method, the time constant is estimated from the power spectral density. The noise power spectrum may be obtained using the FFT algorithm. The time constant estimate may be derived either by fitting a transfer function to the power spectral density or by geometric construction. The time series method discussed in this report needs no such approximations, and all the calculations are made numerically.

Chapter 2 contains a brief introduction to the elements of probability and definitions of some useful statistical quantities. We will introduce the time series modeling of noise signals in chapter 3. Chapter 4 describes the estimation of the models, and the associated power spectrum. Determination of optimal model order and the model validation are also given in chapter 4. In chapter 5 the determination of impulse response and step response from the noise model are presented. The techniques, based on the exact solution to a continuous system and recursive estimation from the model directly, are verified by simulating known systems. Evaluation of RTD (Resistance Temperature Device) characteristics such as impulse and step responses, power spectrum, and time constant is presented in chapter 6. The RTD data were obtained from tests at Millstone 2, St. Lucie, Turkey Point and Oconee 1 pressurized water

reactors. Discussion and concluding remarks are summarized in chapter 7. All the results are presented in Appendix C.

1.1 System Identification

Fundamental to the use of time series analysis, is the problem of determining the appropriate model, representative of the given data. The estimation of this empirical model parameters along with the proper choice of the model is called system identification. We use a finite order autoregressive model to represent the noise signal.

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k \qquad (1.1\text{-}1)$$

The determination of the optimal order, n, is a decision making problem. Figure (1.1) describes the steps in system identification. The optimal model can then be used to determine the power spectrum, impulse response, step response, and the system time constant.

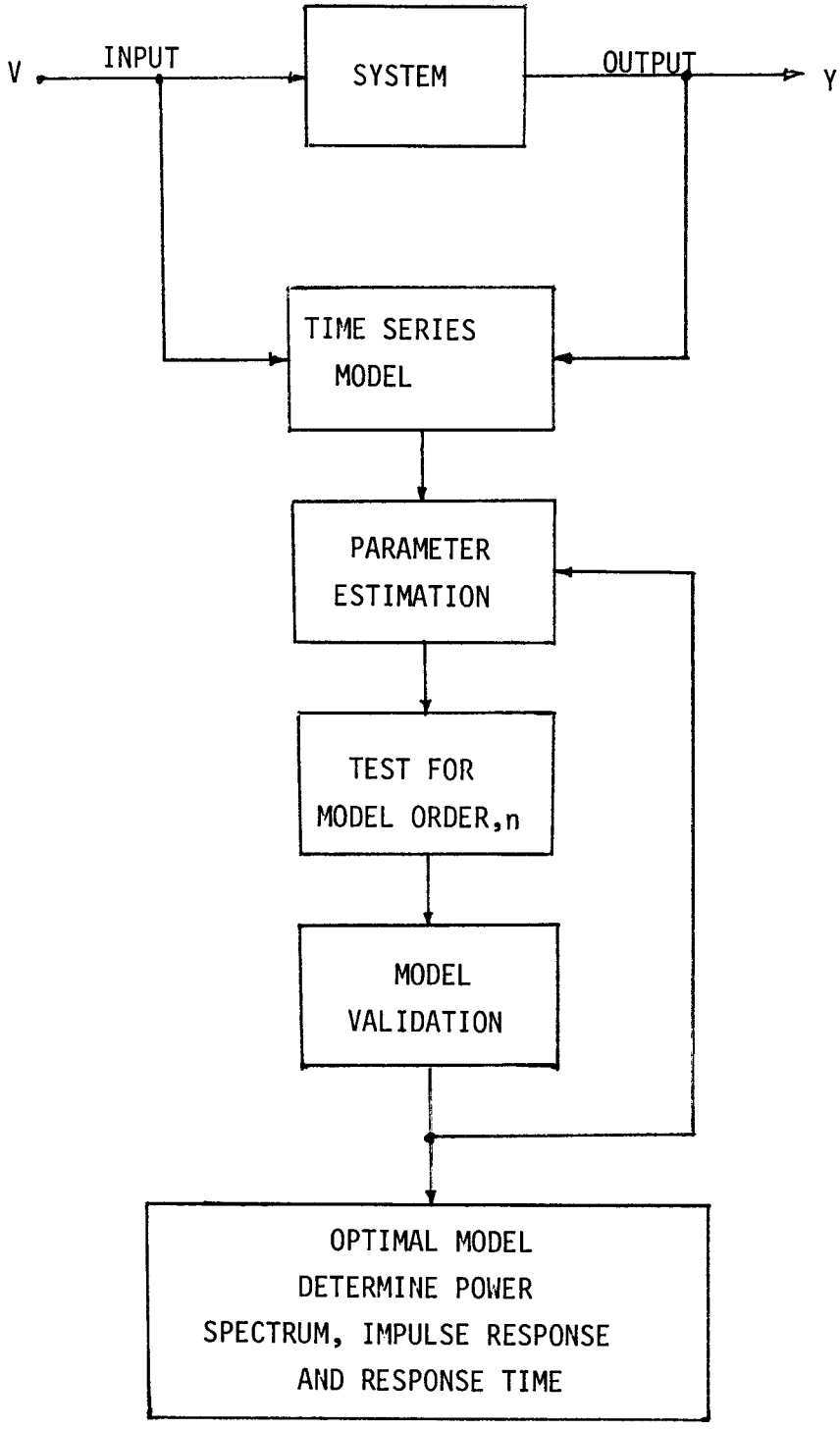


Figure (1.1) Steps in System Identification

CHAPTER 2

ELEMENTS OF PROBABILITY THEORY

In this chapter we shall briefly review the topics on probability pertinent to the discussion to follow. For more detailed treatment the reader is referred to the book by Papoulis [P1] on "Probability, Random Variables, and Stochastic Processes."

2.1 Random Variable, Random Process and Distribution Function

2.1.1 Definition. A Random Variable is a number assigned to the outcome of an experiment and defined by the function $X(\rho)$ where ρ is the outcome of the experiment.

2.1.2 Definition. Let a given experiment be repeated n times. If an event A occurs n_A times, then the probability P(A) that the event A would occur during the experiment is defined by

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n}. \tag{2.1.1}$$

2.1.3 Definition. We thus define a probability space as the triple (Ω, S, P) - where Ω is the space or outcome of the experiment, S is the set of all possible events of Ω (also called the Borel field), and P is the positive number assigned as the probability of a given event.

We have the following axioms for the probability

$P(A) \geq 0$

$P(S) = 1$

$P(0) = 0$, 0 is the null event.

If the events A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B). \tag{2.1.2}$$

2.1.4 Conditional Probability. Given an event R with nonzero probability, $P(R)>0$, we define the "conditional probability of event A given R" by

$$P(A/R) = \frac{P(A \cap R)}{P(R)} . \tag{2.1.3}$$

This can be interpreted as

$$P(A/R) \approx n_{AR}/n_R \tag{2.1.4}$$

i.e., if we discard all trials in which the event R did not occur and retain the sub-sequence of n_R trials in which it occurred, then P(A/R) equals the relative frequency of the occurrence of the event A in that sub-sequence.

2.1.5 Distribution Function. Given a real number, x, we define the event $\{X \leq x\}$ which consists of all outcomes ρ such that $X(\rho) \leq x$.

The distribution function of the random variable X is defined as

$$F_X(x) = P\{X \leq x\} \tag{2.1.5}$$

for any number x from $-\infty$ to $+\infty$.

The distribution function has the following properties

a. $F(-\infty) = 0$, $F(+\infty) = 1$

b. $F(x_1) \leq F(x_2)$ for $x_1 < x_2$, that is, it is a nondecreasing function of x.

c. It is continuous from the right

$$F(x^+) = F(x).$$

2.1.6 Random Process. We are given the probability space (Ω, S, P). To every outcome, ρ, we assign, according to a certain rule a time function

$$X(t,\rho)$$

real or complex. This family of functions $X(t,\rho)$ is called a stochastic process.

We can define $X(t,\rho_i)$ as the time function for specific outcome. For a given time, we can define $X(t_i,\rho)$ as a quantity depending on ρ.

A random process (henceforth denoted by $X(t)$) can be a very irregular process such as the motion of a particle due to its impact with the surrounding medium (Brownian motion). The electromotive force of a generator, although well-defined process, can be treated as a stochastic process

$$X(t) = A \sin(\omega t+\theta) \tag{2.1.6}$$

where the amplitude A, phase θ, and the frequency ω are random variables.

We define the distribution function as

$$F(x,t) = P\{X(t) \leq x\}. \tag{2.1.7}$$

Thus, given two numbers, x and t_1, the function $F(x;t_1)$ equals the probability of the event $\{x(t_1) \leq x\}$ consisting of all outcomes ρ such that, at the specified time t_1, the functions $X(t)$ of the process do not exceed the given number x.

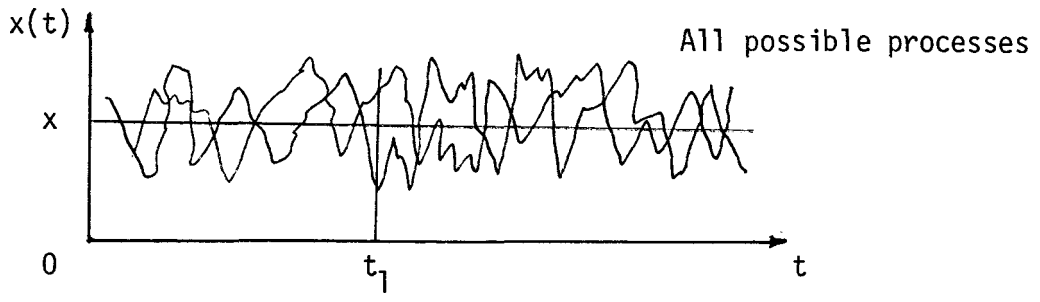


Fig. 2-1

2.2 Probability Density Function

If the derivative of the distribution function $F_X(x)$ with respect to x exists then we define

$$P_X(x) = \frac{d}{dx} F_X(x). \tag{2.2.1}$$

Since the derivative need not exist, we classify the random variable as of continuous type and discrete type.

2.2.1 Random Variables of Continuous Type. We have the following properties for continuous random variables.

a. $p(x) \geq 0$

b. $\int_{-\infty}^{\infty} p(x)\, dx = F(\infty) - F(-\infty) = 1$

c. $F(x) = \int_{-\infty}^{x} p(\xi)\, d\xi$

d. $F(x_2) - F(x_1) = \int_{x_1}^{x_2} p(\xi)\, d\xi$

e. $P\{x_1 \leq X \leq x_2\} = \int_{x_1}^{x_2} p(\xi)\, d\xi$

f. $P\{X = x\} = 0$

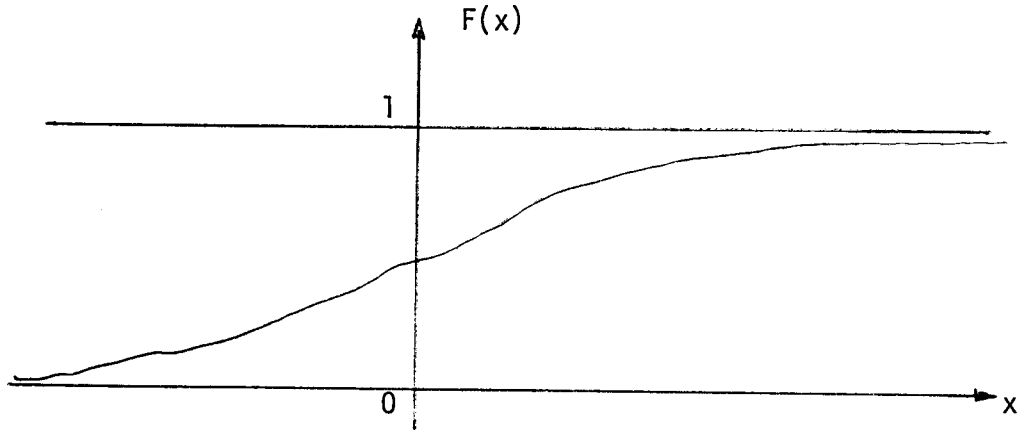


Fig. 2-2

2.2.2 Random Variables of Discrete Type

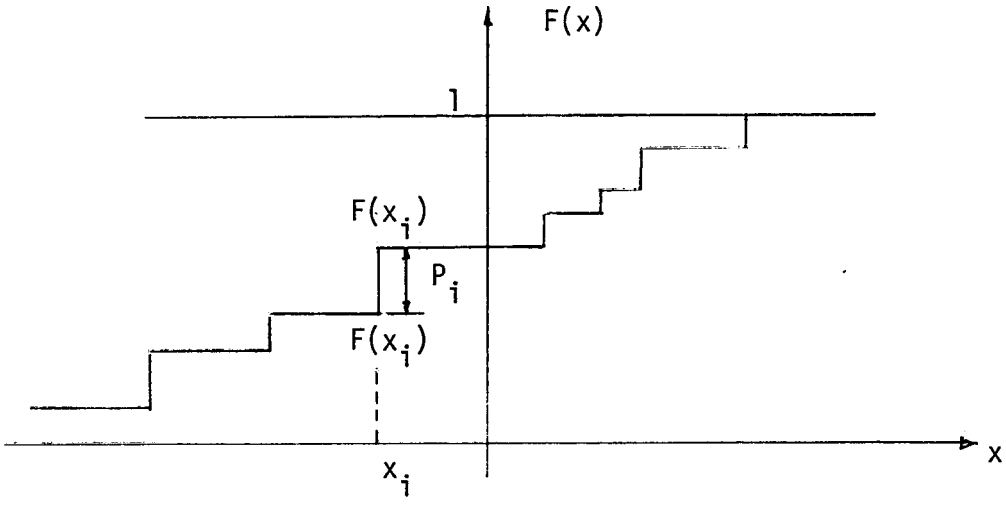


Fig. 2-3

F(x) is of the staircase type with discontinuities at the point x_i (see Figure (2.3)). Let p_i be the jump of F(x) at the point x_i, then

$$P\{x = x_i\} = P_i = F(x_i) - F(x_i^-) \tag{2.2.2}$$

We also have

$$F_x(x) = P\{x \leq x\} = \sum_i P\{x = x_i\}\ ,\ i \text{ such that } x_i \leq x \tag{2.2.3}$$

2.2.3 Random Variables of Mixed Type. In general, a random variable (r.v.) may consist of lattice type as in (2.2.2) and continuous variation as in (2.2.1). Then the r.v. is of mixed type as shown in Figure (2.4)

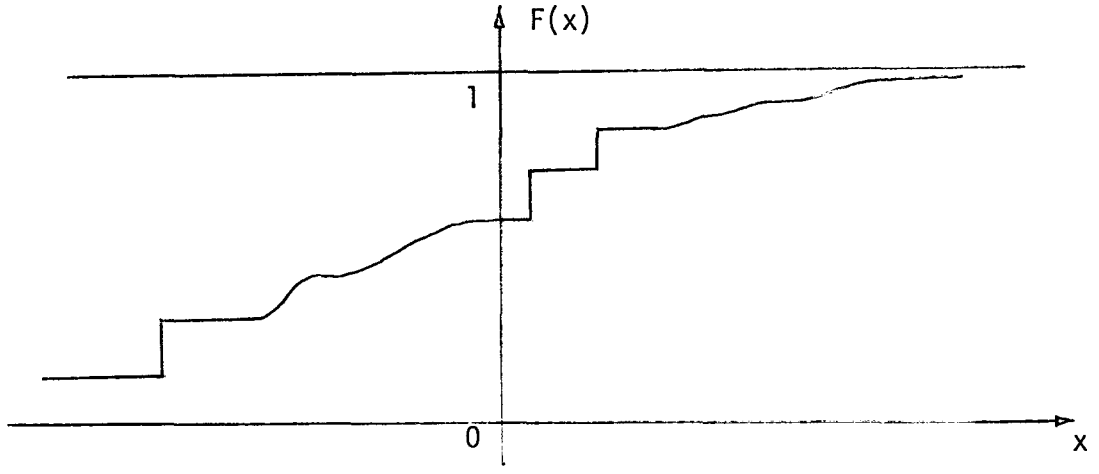


Fig. 2-4

2.3 Examples of Distribution and Density Functions

We define below a few distribution and density functions that appear in common practice.

2.3.1 Gaussian or Normal. A r.v. is normally distributed if its density function is a Gaussian function.

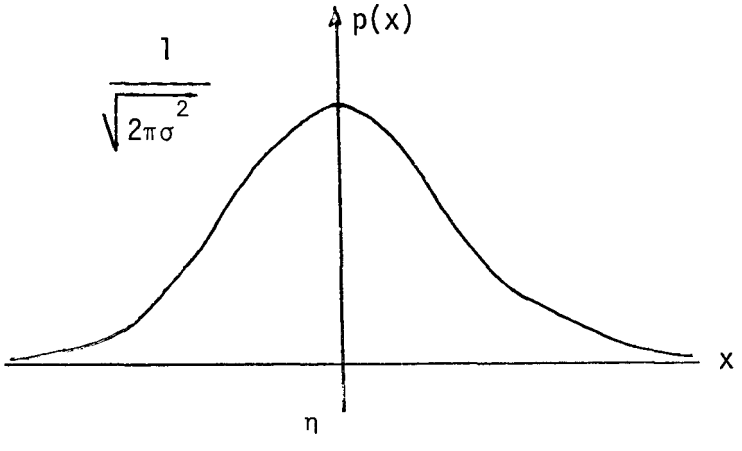


Fig. 2-5

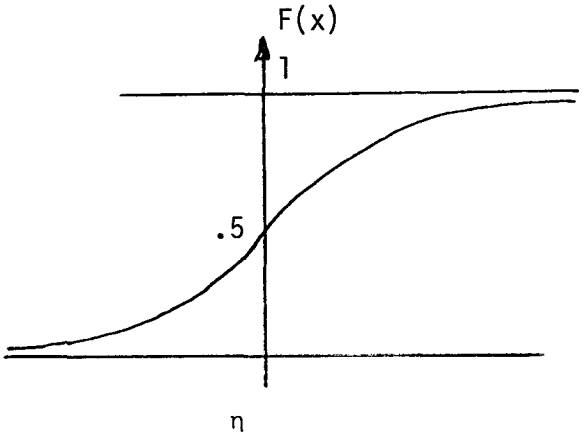


Fig. 2-6

The density is defined as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{ - \frac{(x-\eta)^2}{2\sigma^2} \} \tag{2.3.1}$$

where η = mean value of the Gaussian random variable

σ^2 = variance of the Gaussian random variable.

We define the following

$$\eta = E[x] = \int_{-\infty}^{\infty} x\, p(x)\, dx \tag{2.3.2}$$

$$\sigma^2 = E[(x-\eta)^2] = \int_{-\infty}^{\infty} (x-\eta)^2\, p(x)\, dx \tag{2.3.3}$$

$$= E[x^2] - \eta^2$$

A function closely related to the Gaussian density is the error function defined as

$$\text{erfx} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{ - \frac{y^2}{2} \}\, dy \tag{2.3.4}$$

Thus $$F(x) = \int_{-\infty}^{x} p(y)\, dy = \text{erf}\left(\frac{x-\eta}{\sigma}\right) \tag{2.3.5}$$

and $$F(\eta) = \frac{1}{2} \tag{2.3.6}$$

2.3.2 Cauchy Distribution

$$p(x) = \frac{\alpha/\pi}{\alpha^2 + x^2} \qquad (2.3.7)$$

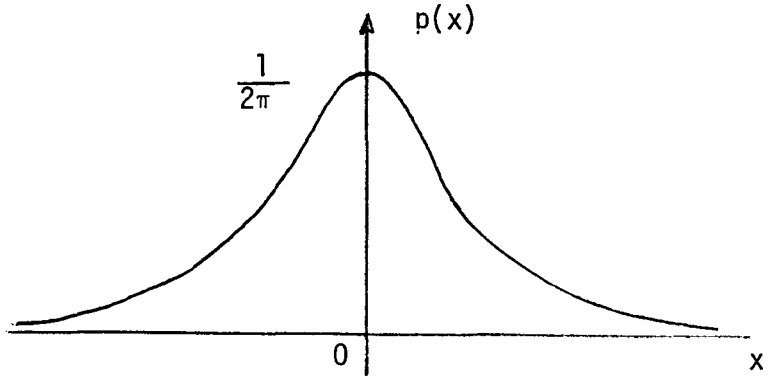


Fig. 2-7

2.3.3 Gamma Distribution

$$p(x) = \frac{c^{b+1}}{\Gamma(b+1)} x^b e^{-ex} U(x) \qquad (2.3.8)$$

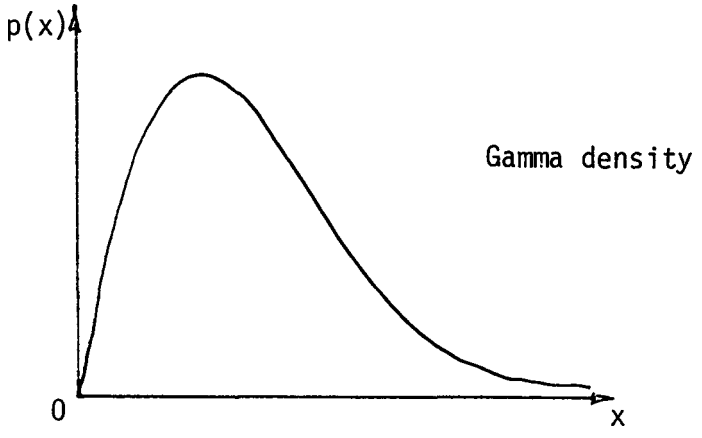


Fig. 2-8

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \quad n > 0 \}$$

2.3.4 Uniform Distribution.

$$p(x) = \frac{1}{b-a} \quad , \quad a \leq x \leq b \tag{2.3.9}$$

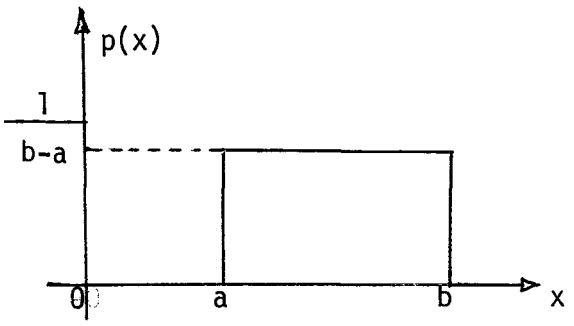


Fig. 2-9

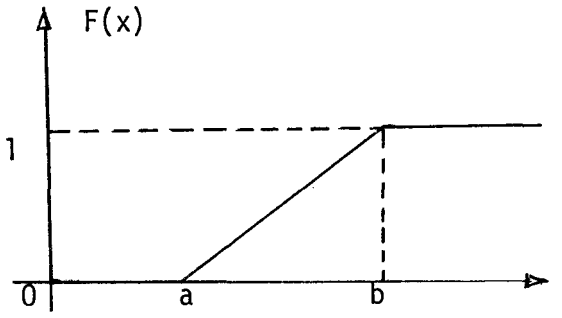


Fig. 2-10

2.3.5 Poisson Distribution. If a r.v. X is of discrete type, taking values at the points 0, 1, 2, . . . with

$$P\{x=k\} = e^{-\lambda} \frac{\lambda^k}{k!} \quad , \quad k=0,1,\ldots; \ \lambda > 0 \tag{2.3.10}$$

then X has a Poisson distribution with parameter $\lambda>0$.

2.3.6 Bionomial. If X is of discrete type taking values at the points k = 0, 1, . . ., n with

$$P\{ x=k\} = \binom{n}{k} p^k \ q^{n-k} \ , \ p + q = 1 \tag{2.3.11}$$

We say that it has a binomial distribution.

Example: Probability that k tossings out of n tossings of a coin are heads.

2.4 Joint Probability Density Function and Multivariable Gaussian Density

Given n random variables $X_1, X_2, \ldots, X_n$, we define the distribution function

$$F(x_1, x_2, \ldots, x_n) = P\{X_1 \leq x_1 \quad , \ldots, X_n \leq x_n\} \tag{2.4.1}$$

and the density function

$$p(x_1, \ldots, x_n) = \frac{\partial F(x_1, \ldots, x_n)}{\partial x_1, \partial x_2 \ldots, \partial x_n} \tag{2.4.2}$$

Denote the vector $(X_1, X_2, \ldots, X_n) \equiv \underline{X}$. The r.v.'s denoted by $\underline{X}$ are jointly normally distributed if their joint probability density function has a Gaussian form. In this case

$$p(x) = \frac{1}{(2\pi)^{n/2} \{\det \quad C\}^{1/2}} \quad \exp\{-1/2\ (\underline{x}-\underline{\eta})^T C^{-1} (x-\eta)\} \tag{2.4.3}$$

$\underline{\eta}$ = vector of mean valves = $(\eta_1, \ldots, \eta_n)$

C = covariance matrix (nxn) whose i-jth element is defined by

$$C_{ij} = E[(x_i-\eta_i)\ (x_j-\eta_j)] \tag{2.4.4}$$

The above definition assumes that the inverse C^{-1} exists. We denote the multivariate Gaussian density by

$$p(x) = G(\underline{\eta},C) \tag{2.4.5}$$

2.4.1 Joint Characteristic Function (CF). We define the Joint CF as

$$\Phi(\omega) = E[\exp(j\underline{w}^T\underline{x})],\ j=\sqrt{-1} \tag{2.4.6}$$

Assuming that $p(\underline{x}) = G(0,C)$ we have

$$\Phi(\omega) = \exp[-1/2\underline{\omega}^T C\underline{\omega}] \tag{2.4.7}$$

Note that the above definition of $\Phi(\omega)$ uses the covariance matrix C instead of its inverse. This is a distinct advantage of the definition $\Phi(\omega)$. Further $\Phi(\underline{\omega})$ is nothing but the Fourier transform of the joint density function. The Gaussian function is the only function whose Fourier transform is also of the same form.

2.4.2 Sums of Random Variables and Central Limit Theorem.

Theorem: If $X_1, X_2, \ldots, X_n$ are jointly Gaussian, then the sum $Y = X_1 + \ldots + X_n$ also has a Gaussian density; that is if $\underline{X} = G(\underline{\eta},C)$ then

$$y = G\left(\sum_{i=1}^{n} \eta_i,\ \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\right) \tag{2.4.8}$$

One of the remarkable properties of the Gaussian distribution is the so-called central limit theorem.

Theorem: If X_i, i = 1, 2, . . ., n are a set of independent and identically distributed random variables, then letting

$$y_n = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{2.4.9}$$

it follows that as $n \to \infty$, Y_n approaches a Gaussian random variable.

Thus if $X_i = (0, \sigma^2)$, i = 1, 2, . . ., n then $Y_n \to G(0, \sigma^2)$ as n becomes large.

The central limit theorem is useful in assumptions made about random processes and in proving certain asymptotic properties of estimation and identification methods.

2.5 Independent, Uncorrelated and Orthogonal Random Variables

Before we discuss these ideas, let us define some terms which are fundamental and appear all the time in future chapters.

Definition. If X and Y are two r.v.'s then the covariance function between X and Y is defined as

$$C_{xy} = E\,[(x-\bar{x})(y-\bar{y})] \tag{2.5.1}$$

Definition. If X(t) and Y(t) are two random processes, we define the following terms

Autocovariance of X: $E[(X(t_1) - \bar{X})\,(X(t_2) - \bar{X})]$ (2.5.2)

Autocorrelation of X: $E[X(t_1)\,X(t_2)]$ (2.5.3)

Cross Covariance of X and Y: $E[(X(t_1) - \bar{X})\,(Y(t_2) - \bar{Y})]$ (2.5.4)

Cross Correlation of X and Y: $E[X(t_1)\,Y(t_2)]$ (2.5.5)

The expectation is defined as follows:

$$E[x(t_1)\, y(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyp(x,t_1,y,t_2)dxdy \tag{2.5.6}$$

2.5.1 Independent r.v.'s. Two random variables, X, Y are said to be independent if

$$p\{X \leq x, Y \leq y\} = p\{X \leq x\}\, p\{Y \leq y\} \tag{2.5.7}$$

$$F_{xy}(x,y) = F_x(x)\, F_y(y) \tag{2.5.8}$$

and $$p_{xy}(x,y) = p_x(x)\, p_y(y) \tag{2.5.9}$$

Thus, we also have

$$E[XY] = E[X]\, E[Y]. \tag{2.5.10}$$

2.5.2 Uncorrelated r.v.'s. Two r.v.s X and Y are said to be uncorrelated if

$$E[(X-\bar{X})\,(Y-\bar{Y})] = 0 \text{ or } E[XY] = E[X]\, E[Y]. \tag{2.5.11}$$

It follows from the definition of independence that if two r.v.s are independent, they are also uncorrelated.

2.5.3 Orthogonal r.v.'s. Two random variables X and Y are said to be orthogonal if

$$E[XY] = 0. \tag{2.5.12}$$

For two vector r.v.s $\underline{X}$ and $\underline{Y}$ we say that they are orthogonal if

$$E[\underline{X}\,\underline{Y}^T] = 0 \text{ matrix.} \tag{2.5.13}$$

These concepts are very important in the estimation problems to derive meaningful algorithms. For most random processes, these conditions are satisfied (or nearly satisfied) and since we are interested in asymptotic properties, such assumptions are not far from truth.

2.6 Stationary Random Processes

2.6.1 Stationary Processes. We say that a stochastic process $X(t)$ is stationary in the strict sense if its statistics are not affected by a shift in the time origin.

For an nth order density function, we must have

$$p(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n) = p(x_1,x_2,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau) \tag{2.6.1}$$

Thus for first order density

$p(x,t) = p(x,t+\tau) \rightarrow$ the first order density is independent of time.

Hence, $E[X(t)] = \eta = \text{constant.}$ (2.6.2)

The second order density becomes

$$p(x_1,x_2;t_1,t_2) = p(x_1,x_2;t_1+\tau,\, t_2+\tau) \Longrightarrow$$

$$p(x_1,x_2;t_1,t_2) = p(x_1,x_2;\tau) \tag{2.6.3}$$

Hence, $E[X(t)\,X(t+\tau)] = R_{xx}(\tau)$. (2.6.4)

2.6.2 Wide Sense Stationary Process. We say that a process X(t) is stationary in the wide sense (or weakly stationary) if

$$E[x(t)] = \text{constant}, \quad E[x(t)\, x\,(t+\tau)] = R_x(\tau) \tag{2.6.5}$$

Note that the wide sense stationarity involves only first and second order moments.

Two processes are jointly stationary in the wide sense if each satisfies (2.6.5) and

$$E[x(t+\tau)y(t)\,] = R_{xy}(\tau) \tag{2.6.6}$$

Remark: If a process X(t) is normal and stationary in the wide sense then it is stationary also in the strict sense (because all moments of X(t) are expressed in terms of the first two moments).

2.6.3 Markov Process. A process X(t) is Markov if the statistics of the future depend only on the present and are independent of the past.

If $t_1 < t_2 < \ldots t_n$, then

$$P\{x(t_n) \leq x_n \,|x(t_{n-1}),\ldots, t_1)\} = P\{x(t_n) \leq x_n|x(t_{n-1})\} \tag{2.6.7}$$

In terms of density function

$$p\{x(t_n) \,|x(t_{n-1}),\ldots,x(t_1)\} = p\,\{x(t_n) \,|x(t_{n-1})\} \tag{2.6.8}$$

We will use this property to express dynamic system models as Markov processes.

2.6.4 Power Spectrum. The power spectrum (or spectral density) $S(\omega)$ of a process $X(t)$ is the Fourier transform of its autocorrelation.

$$S(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} R(\tau)\, d\tau \tag{2.6.9}$$

Conversely

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau}\, d\omega \tag{2.6.10}$$

If $x(t)$ is a real random process then

$$R(\tau) = R(-\tau) \text{ and } S(\omega) = S(-\omega) \tag{2.6.11}$$

Note that the above definition of the spectral density is true only for stationary processes.

2.6.5 White Noise. A random process of primary importance in engineering applications is the so called white noise. This is a stationary process with all the components independent. This of course means they are uncorrelated. In general, it is also assumed that as function of time they have zero mean and the same covariance function.

If $\underline{X}_k$, k = 1, 2, . . . is a white noise process vector with mean zero, we have

$$E[\underline{X}_i \underline{X}_j^t] = R\delta_{ij}, \text{ for all } i \text{ and } j \tag{2.6.12}$$

where δ_{ij} = 1 if i = j

= 0 otherwise.

The autocorrelation function of a white noise process is a delta function, that is

$$E[x(t)\ x(t+\tau)] = N_0\ \delta(\tau) \tag{2.6.13}$$

where N_0 is a constant value. The spectrum is therefore constant for all frequencies.

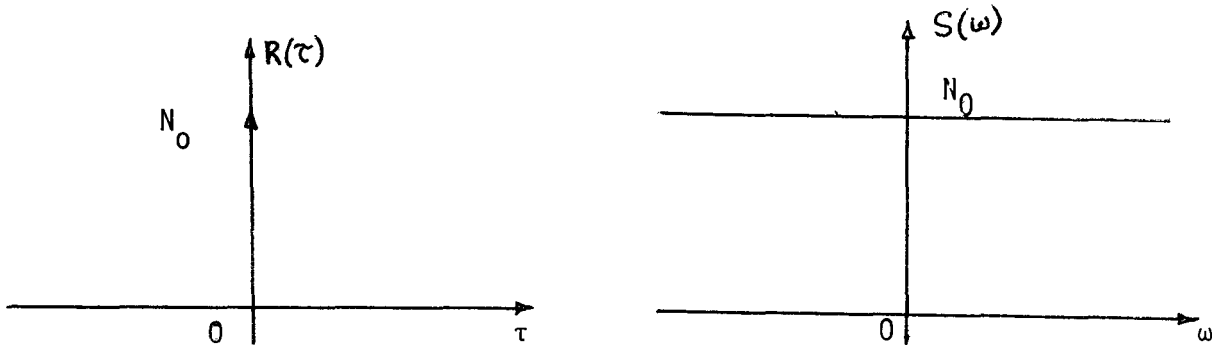


Fig. 2-11 Autocorrelation and Spectrum of White noise

Note that since $R(\tau)$ is not differentiable, the white noise is a process with infinite discontinuities and not physically feasible.

2.7 Ergodic Random Processes

Statistical quantities may be calculated using the average of an ensemble of sample functions. Fig. (2-1) shows such an ensemble. At a given time $t = t_1$, the mean value of the random process $\{x(t)\}$ and its autocorrelation function can be calculated as follows:

$$\eta_x(t_1) = \lim_{N} \frac{1}{N} \sum_{k=1}^{N} x_k(t_1) \tag{2.7-1}$$

$$R_x(t_1, t_1, + \tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x(t_1)\, x_k(t_1 + \tau) \tag{2.6-2}$$

In most cases the above statistics can also be calculated by using just one sample function in the ensemble. Thus the mean value and the autocorrelation function of the k^{th} sample is given by

$$\eta_x(k) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x_k(t)\, dt \tag{2.7-3}$$

$$R_x(\tau, k) = \lim_{T} \frac{1}{T} \int_0^T x_k(t)\, x_k(t + \tau)\, dt \tag{2.7-4}$$

If $\{x(t)\}$ is stationary and $\eta_x(k)$ and $R_x(\tau, k)$ do not differ when computed over different sample functions, the random process is said to be ergodic. Thus if a process is ergodic the ensemble average is equal to the time average. Note that only stationary processes can be ergodic.

The concept of ergodicity can be used to approximate certain statistics such as the mean and correlation functions by replacing the ensemble average by time average, or approximating statistical quantities by summation over time.

2.7-1 Estimation of mean and autocovariance functions for stationary process.

In problems of data processing using digital computers it is only possible to have measurements of a signal available at discrete time points. Two of the quantities often computed are the mean and auto convariance functions.

Let $\{x_k, k=1,2,\ldots,N\}$ be the sequence of measurements of a random process $\{x_k\}$. Then we define the mean as

$$\eta_x = \frac{1}{N} \sum_{k=1}^{N} x_k \tag{2.7-5}$$

and the autocovariance function as

$$R_x(k) = \frac{1}{N-k} \sum_{i=1}^{N-k} (x_i - \eta_x)(x_{i+k} - \eta_x) \tag{2.7-6}$$

Some authors use autocovariance and autocorrelation to mean the same equation (2.7-6). (2.7-6)

For definitions of other statistical quantities see Bendat and Piersol [B1, Ch.6].

CHAPTER 3

LINEAR STATIONARY TIME SERIES MODELS

A general description of the time series models is presented. Special cases of interest are discussed. There are many classical and recent references in the field of time series analysis. Some of these are monographs and, the others are published in technical journals and conference proceedings. The references listed at the end are those that are appropriate to the present discussion.

3.1 General Linear Processes:

Let $\{\underline{y}_k\}_k$ and $\{\underline{u}_{ik}\}_k$, i = 1,2,...,p, denote jointly stationary ergodic progesses. Let $\{\underline{v}_k\}$ be a sequence of white noise process. Then the process $\{\underline{y}_k\}$ may be represented in general, in the form

$$\underline{y}_k = \sum_{i=1}^{\infty} A_i \underline{y}_{k-i} + \sum_{i=1}^{\infty} B_{1i} \underline{u}_{1,k-i} + \ldots + \sum_{i=1}^{\infty} B_{pi} u_{p,k-i} + \sum_{i=0}^{\infty} C_i \underline{v}_{k-i} \qquad (3.1\text{-}1)$$

$\{A_i, B_{ki}, k=1,2,\ldots,p, C_i, i \geq 1\}$ are matrices with constant coefficients. Let $\underline{y}_k$ be an n x 1 dimensional vector and $\underline{u}_i(\cdot)$ be m dimensional for i=1,2,...,p. Let $\underline{v}$ be m dimensional. $\{\underline{u}_i(\cdot)$, i=1,...,p$\}$ is an input sequence of known statistics. A special case of importance is the univariate signal $\{y_k\}$ produced by the linear aggregation of the white noise $\{v_k\}$ given by

$$y_k = \sum_{i=1}^{\infty} a_i y_{k-i} + \sum_{i=1}^{\infty} b_i v_{k-i} + v_k \qquad (3.1\text{-}2)$$

An infinite representation (3.1-2) is not feasible and often may be incorrect. A parsimonious use of parameters is often derived with finite orders for the autoregressive and moving average terms as follows:

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=1}^{m} b_i v_{k-i} + v_k \qquad (3.1\text{-}3)$$

Such a representation is often called the autoregressive moving average (ARMA) process. A complete description of such models is given in Box and Jenkins [B2].

The use of time series models for forecasting, control and spectral estimation has been the subject of works by Parzen [P2], Mann and Wald [M1], Hannan [H1, H2], Anderson [A1], Quenouille [Q1], Kashyap and Rao [K1], Rosenblatt [R1], Rao [R2], Akaike [A2] and others [A3].

Many special cases of the ARMA process are obtained as follows.

ARMA process: $$y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{i=1}^{m} b_i v_{k-i} + v_k \qquad (3.1\text{-}4)$$

Autogressive process: $$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k \qquad (3.1\text{-}5)$$

Moving Average Process: $$y_k = \sum_{i=1}^{m} b_i v_{k-i} + v_k \qquad (3.1\text{-}6)$$

Regression Models: $$y_k = \sum_{i=1}^{n} a_i \theta_{k-i} + v_k \qquad (3.1\text{-}7)$$

3.2 Stationarity and Invertibility of ARMA Processes

Using the backward shift operator

$$D^i v_k = v_{k-i} \qquad (3.2\text{-}1)$$

the ARMA model may be written as

$$y_k = \left(1 - \sum_{i=1}^{n} a_i D^i\right)^{-1} \left(1 - \sum_{i=1}^{m} b_i D^i\right) v_k \qquad (3.2\text{-}2)$$

The convergence of the series $\Psi(B) = (1 - \sum_{i=1}^{n} a_i D^i)^{-1}$ ensures that the process $\{y_k\}$ has a finite variance. This is equivalent to the condition that roots of the equation

$$1 - \sum_{i=1}^{n} a_i D^i = 0 \qquad (3.2\text{-}3)$$

must be outside the unit circle. Thus when the noise process is stationary the above condition ensures that the process $\{y_k\}$ is also stationary.

The concept of invertibility is concerned with recovering $\{v_k\}$ from the semi-infinite history of observations $\{y_k\}$. Writing (3.2-2) as

$$v_k = (1 - \sum_{i=1}^{m} b_i D^i)^{-1} (1 - \sum_{i=1}^{n} a_i D^i)\, y_k \qquad (3.2\text{-}4)$$

the linear process is invertible if the infinite process expansion of $\pi(B) = (1 - \sum_{i=1}^{m} b_i D^i)^{-1}$ converges. This is equivalent to the condition that the roots of the equation

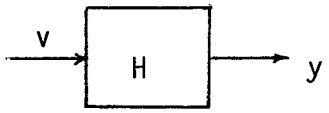
$$1 - \sum_{i=1}^{m} b_i D^i = 0 \qquad (3.2\text{-}5)$$

lie outside the unit circle. The conditions of invertibility are independent of the conditions of stationarity of the time series. Both stationarity and invertibility are necessary conditions for the consistency of parameter estimates.

3.3 Estimation of the power spectrum

Using the estimated values of the ARMA parameters an estimate of the power spectrum can be obtained. The ARMA process can be treated as a linear transformation with the Fourier transform of the filter given by

$$H(f) = \frac{1 - \sum_{k=1}^{m} b_k e^{-j2\pi fkT}}{1 - \sum_{k=1}^{n} a_k e^{-j2\pi fkT}}, \quad |f| \leq \frac{1}{2T} \qquad (3.3\text{-}1)$$



Then the output spectrum $S_{yy}(f)$ is given by $S_{yy}(f) = |H(f)|^2 \; S_{vv}(f)$ (3.3-2)

If $\{v_k\}$ is a white noise sequence $S_{vv}(f)$ has a constant value in $|f| \leq \frac{1}{2T}$ and $|H(f)|^2$ itself represents the power spectrum of the process $\{y_k\}$.

Equation (3.3-2) can also be stated conversely using the Paley - Wiener condition [P3].

Theorem: If the spectrum $S_{yy}(f)$ satisfies the Paley-Wiener condition

$$\int_{-\infty}^{\infty} \frac{\ell n \; | \; S_{yy}(f)|}{1 + f^2} \; df \; < \; \infty \qquad (3.3\text{-}3)$$

then the process $\{y_k\}$ can be generated by passing white noise through the filter whose Fourier transform $H(f)$ is such that

$$C \; |H(f)|^2 \; = \; |S_{yy}(f)|, \quad \text{where C is a constant.}$$

In the discussion of autoregressive process we will obtain an estimate of the error in the power spectrum as obtained from (3.3-1)

3.4 Use of Autoregressive Models for Random Noise Signals

The choice of one type of model over the other depends on the type of signal to be modeled. Any one of the models of equations (3.1-3) - (3.1-7) may be suitable for a given time series.

Let the time series $\{y_k\}$ be a purely random,weakly stationary process. Pure randomness implies that that are no super-imposed frequency components like a sine wave or any periodic signal, or a DC component. If a DC component is present, this is removed by subtracted the mean value of the time series from the original data. Thus under these conditions the $\{y_k\}$ is generated by a sequence of noise pulses such that $\{y_k\}$ has a representation

$$y_k = \sum_{i=1}^{\infty} b_i v_{k-i} + v_k \qquad (3.4\text{-}1)$$

where $\{v_k\}$ is a white noise sequence with

$$E[v_k] = 0, \quad E[v_j v_k] = \sigma_v^2 \, \delta_{jk}, \quad \forall j,k \qquad (3.4\text{-}2)$$

We notice a mapping here between the spaces $\{v_i, i \leq k\}$ and $\{y_i, i \leq k\}$. Thus y_k can be approximated by

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k \qquad (3.4\text{-}3)$$

arbitrarily closely with increasing n. We can consider

$\sum_{i=1}^{n} a_i y_{k-i}$ as the projection of y_k onto the space (or manifold) of

$\{y_{k-i}, i=1,\ldots n\}$

For stationarity of the process,the roots of the equation

$$1 - \sum_{i=1}^{n} a_i D^i = 0 \qquad (3.4\text{-}5)$$

must lie outside the unit circle. These observations provide the rationale for the use and application of autoregressive models both for power spectrum estimation and evaluation of dynamic response characteristics.

CHAPTER 4

ESTIMATION OF OPTIMAL AUTOREGRESSIVE PROCESSES AND MODEL VALIDATION

Since the form of the empirical model may not be known *a priori* we assume that these are represented by models described in Chapter 3, with constant coefficients. There are cases where the functional form of the predictors is changed so that a more efficient representation is obtained. Such autoregressive (AR) processes are referred to as generalized AR processes.

$$y_k = \sum_{i=1}^{n} a_i f_i (y_{k-i}) + v_k \qquad (4.1)$$

The function $f_i(\cdot)$ may be a logarithmic, squared or any other function such that (4.1) is asymptotically stable (see [K1], ch.3). In this chapter we restrict ourselves to the constant coefficient AR processes of the form

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k \qquad (4.2)$$

We will obtain the estimates of $\{a_i, i=1,2,\ldots,n\}$, their properties and compare them to the least squares and maximum likihood estimators. Selection of an optimal class and the validation of the selected model is discussed in detail. Estimation of the power spectrum and its error bounds are given.

4.1 Estimation of AR Parameters

AR model estimation both for prediction and for spectral estimation has been studied by several authors - Yule [Y1], Walker [W1], Hannan [H2], Akaike [A4-A9], Anderson [A10]. We discuss the Yule-Walker equations and their properties as related to maximum likelihood and least squares.

Let the time series under observation be a realization of the auto-regressive process defined by

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k, \quad k = 1,2..... \tag{4.1-1}$$

$\{v_k\}$ is a white noise sequence - v_k are uncorrelated with statisties

$$E[v_k] = 0 \text{ , and } E[v_k^2] = \sigma^2 \text{ for all k.} \tag{4.1-2}$$

Define the autocorrelation function of the stationary process $\{y_k\}$ for lag k, as

$$C_k = E[y_t\, y_{t+k}] \tag{4.1-3}$$

Since $\{y_k\}$ is a real process, C_k is symmetric

$$C_k = C_{-k} \text{ for all k.} \tag{4.1-4}$$

A recurrence relation for the autocorrelation functions of a stationary AR process is found by multiplying equation (4.1-1) by y_{t-k} to obtain

$$y_{t-k}\, y_t = \sum_{i=1}^{n} a_i y_{t-i} y_{t-k} + y_{t-k}\, v_k \tag{4.1-5}$$

Taking the expected value in (4.1-5) and noting that y_{t-k} is independent of v_k we get

$$E[y_{t-k} y_t] = \sum_{i=1}^{n} a_i E[y_{t-i} y_{t-k}] \tag{4.1-6}$$

Using definition (4.1-3) the above equation becomes

$$c_k = \sum_{i=1}^{n} a_i c_{k-i}, \quad k > 0 \tag{4.1-7}$$

The correlations are computed from the observed time series as

$$C_k = \frac{1}{N-k} \sum_{i=1}^{N-k} \tilde{y}_k \tilde{y}_{i+k}, \quad k = 0,1,2,... \tag{4.1-8}$$

$$\tilde{y}_i = y_i - \frac{1}{N} \sum_{k-1}^{N} y_k, \quad i=1,2,... \tag{4.1-9}$$

Where N is the number of observations. Equation (4.1-7) can be written for k = 1,2,...,n giving

$$\begin{aligned} c_1 &= a_1 c_0 + a_2 c_1 + \ldots + a_n c_{n-1} \\ c_2 &= a_1 c_1 + a_2 c_0 + \ldots + a_n c_{n-2} \\ &\cdot \\ &\cdot \\ c_n &= a_1 c_{n-1} + a_2 c_{n-2} + \ldots + a_n c_0 \end{aligned} \tag{4.1-10}$$

Equation (4.1-10) are called the Yule-Walker equations (see [B2], Ch.3). The parameters a_i, i=1,2,...,n are obtained by solving (4.1-10). Rewriting the set of linear equations in matrix form we obtain:

$$\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_0 & \cdots & c_{n-2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \tag{4.1-11}$$

or

$$\underline{c} = P\underline{a} \tag{4.1-12}$$

P is an (nxn) symmetric Toeplitz matrix [G1]. (A matrix T is called a Toeplitz matrix if all the elements along the same diagonal are equal, $t_{ij} = t_{i-j}$). Taking the inverse gives the estimation

$$\hat{\underline{a}} = P^{-1}\underline{c} \tag{4.1-13}$$

Notice that we have been able to determine $\underline{a}$ without the knowledge of the noise variance σ^2.

4.1-1 Estimation of Noise Variance

Multiply both sides of (4.1-1) by y_k and take the expected value giving

$$E[y_k^2] = \sum_{i=1}^{n} a_i E[y_k y_{k-i}] + E[y_k v_k] \tag{4.1-14}$$

Noting that $E[y_k v_k] = E[(\sum_{i=1}^{n} a_i y_{k-i} + v_k)v_k] = E[v_k^2]$ gives

$$c_o = \sum_{i=1}^{n} a_i c_i + E[v_k^2] \tag{4.1-15}$$

$$\text{or} \quad \hat{\sigma}^2 = c_0 - \sum_{i=1}^{n} \hat{a}_i c_i \tag{4.1-16}$$

Notice that the noise variance is obtained as a function of the estimates of AR parameters.

4.1-2 The Least Squares Estimation

The least squares problem of estimating $\underline{a}$ using N observations may be stated as the minimization problem

$$\min_{\underline{a}} \sum_{k=1}^{N} (y_k - \sum_{i=1}^{n} a_i y_{k-i})^2 \tag{4.1-17}$$

The solution to the above problem is given by

$$\hat{\underline{a}} = (A^T A)^{-1} A^T \underline{y} \tag{4.1-18}$$

Where A is (Nxn) matrix defined below.

$$A = \begin{bmatrix} y_0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ y_1 & y_0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_n & y_{n-1} & \cdot & \cdot & \cdot & \cdot & y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_{N-1} & y_{N-2} & \cdot & \cdot & \cdot & \cdot & y_{N-n} \end{bmatrix} \quad (N \times n) \text{ matrix} \tag{4.1-19}$$

Once the $\underline{a}$ is estimated the noise sequence may be estimated as the residual estimate

$$\hat{v}_k = y_k - \sum_{i=1}^{n} \hat{a}_i y_{k-i} \quad , \; k=1,2,\ldots,N. \tag{4.1-20}$$

An estimate of the noise variance is then given by

$$\hat{\sigma}^2 = E[v_k^2] = \frac{1}{N} \sum_{k=1}^{N} \hat{v}_k^2 \tag{4.1-21}$$

By comparing the Yule-Walker equations and (4.1-18) it is evident that the least squares estimate approaches the Yule-Walker estimate as $N \to \infty$.

4.1-3 The Maximum Likelihood Estimation

This method is based on the assumption that a joint probability density function (pdf) of the observations may be constructed. Let $p(\underline{y}^N, a, \sigma^2)$ be such a density function where $\underline{y}^N = \{y_1, \ldots, y_N\}$. In many cases the evaluation of $p(\underline{y}^N, \underline{a})$ is very complicated. Alternately a conditional likelihood function is defined as

$$L = p(\underline{y}^N \mid \underline{a}, \sigma^2, y_0) \tag{4.1-22}$$

Where y_0 is the initial state. The maximum likelihood estimator is defined by

$$\max_{\underline{a}, \sigma^2} p(\underline{y}^N \mid \underline{a}, \sigma^2, y_0) \tag{4.1-23}$$

We search for the value of $(\underline{a}, \sigma^2)$ for which the conditional pdf becomes maximum. In general we assume that $\underline{a}$ belongs to some subset of the parameter space so that the assumptions of stationarity and stability of the systems are satisfied.

Let the uncorrelated noise sequence $\{v_k\}$ be distributed as a Gaussian distribution with mean zero and variance σ^2. $v_k \sim G(o, \sigma^2)$. Then the joint conditional pdf becomes

$$p(\underline{y}^N \mid \underline{a}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{k=1}^{N} \left(y_k - \sum_{i=1}^{n} a_i y_{k-i}\right)^2\right\} \tag{4.1-24}$$

It is convenient to use the natural logarithm of p. Thus we have

$$\ell n L = -\frac{N}{2} \ell n(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{k=1}^{N} \left(y_k - \sum_{i=1}^{n} a_i\, y_{k-i}\right)^2 \tag{4.1-25}$$

It is clear from above that since $\ell n\, L$ is a monotonic function of L, maximizing L is the same as maximizing $\ell n\, L$. This is also equivalent to

minimizing - ℓn L

$$-\ell nL = \frac{N}{2}\ \ell n(2\pi\sigma^2) + \frac{1}{2\sigma^2} \sum_{k=1}^{N} (y_k - \sum_{i=1}^{n} a_i y_{k-i})^2 \qquad (4.1\text{-}26)$$

The problem may be restated as

$$\min_{(\underline{a},\sigma^2)} - \ell nL = \min_{(\underline{a},\sigma^2)} g(y^N, \underline{a}, \sigma^2) \qquad (4.1\text{-}27)$$

To obtain $\{a_i, i=1,2,\ldots,n\}$:

$$\frac{\partial}{\partial a_i}\ g(y^N, \underline{a},\sigma^2) = 0,\ i=1,2,\ldots,n) \qquad (4.1\text{-}28)$$

Equs. (4.1-28) are the same as the least squares solution of section (4.1-2). Once the $\{a_i\}$ are obtained we form

$$g(y^N,\underline{a}, \sigma^2) = \frac{N}{2}\ \ell n(2\pi\sigma^2) + \frac{1}{2\sigma^2} \sum_{k=1}^{N} (y_K - \sum_{i=1}^{n} a_i y_{k-i})^2$$

$$\frac{\partial g}{\partial \sigma} \cdot = \frac{N}{\sigma} - \frac{1}{\sigma^3} \sum_{k=1}^{N} (y_k - \sum_{i=1}^{n} a_i y_{k-i})^2 = 0 \qquad (4.1\text{-}29)$$

$$\sigma^2 = \frac{1}{N} \sum_{k=1}^{N} (y_k - \sum_{i=1}^{n} a_i y_{k-i})^2 \qquad (4.1\text{-}30)$$

The above equation is the same as (4.1-21).

Thus we see a matching of the least squares and the Gaussian conditional maximum likelihood estimators. As pointed out in section (4.1-2) the least square estimation and the Yule-Walker estimation are equivalent for $N \to \infty$. All the three techniques have asympotitically the same property. We say that a given solution is asymptotically efficient if it approaches the maximum likelihood estimator in the Gaussian sense. Thus even without the assumption of a form for the density function we can obtain consistent estimators similar to the CML estimator in the Gaussian case.

4.1-4 Error Convariance of AR Parameters

Using the asymptotic properties of the errors in the estimates of $\underline{a}$ we can derive an expression for the error convariance of $\underline{a}$. The following theorem is proved in [A7,H2].

Theorem 4.1-1 Let $\{y_k\}$ be generated by a stationary AR process described by (4.1-1). Then $\hat{\underline{a}}$ and $\hat{\sigma}^2$ converge to $\underline{a}$ and σ^2 in probability one. Moreover $\sqrt{N}\ (\underline{a} - \hat{\underline{a}})$ has a distribution which converges as $N \to \infty$ to that of a normally distributed vector with zero mean and covariance matrix $\sigma^2\ C_{yy}^{-1}$ where C_{yy} is n x n matrix.

$$\sqrt{N}\ (\underline{a} - \hat{\underline{a}}) \sim G(0, \sigma^2\ C_{yy}^{-1}) \tag{4.1-27}$$

Remark: The covariance matrix may be estimated as

$$E[(\underline{a} - \hat{\underline{a}})\ (\underline{a} - \hat{\underline{a}})^T] = \frac{\hat{\sigma}^2}{N}\ \hat{C}_{yy}^{-1}$$

Where

$$C_{yy} = \begin{bmatrix} C_0 & C_1 & \cdot & \cdot & \cdot & C_{n-1} \\ C_1 & C_0 & \cdot & \cdot & \cdot & C_{n-2} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ C_{n-1} & C_{n-2} & \cdot & \cdot & \cdot & C_0 \end{bmatrix}$$

4.2 Estimation of Power Spectrum

It is not our intention to rely on evaluation of the power spectrum of the process y_k to estimate sensor time constants. However, the AR parameters may be used to find an estimate of $S_{yy}(f)$ for comparison with power spectrum estimates obtained by other methods. The following relation is used:

$$S_{yy}(f) = |H(f)|^2\ S_{vv}(f) \tag{4.2-1}$$

Where $S_{yy}(f)$ is the power spectrum of $\{y_k\}$

$S_{vv}(f)$ is the power spectrum of $\{v_k\}$

$H(f)$ is the linear transformation obtained from the AR model. Thus

$$\hat{S}_{yy}(f) = \frac{\hat{\sigma}^2 T}{\left| 1 - \sum_{k=1}^{n} \hat{a}_k e^{-i2\pi fkT} \right|^2}, \quad f \leq \frac{1}{2T} \tag{4.2-2}$$

where $S_{vv}(f) = \hat{\sigma}^2 T$ and f is in Hertz.

T = Sampling time in seconds

It is possible to obtain an expression for the variance of the power spectrum by using the result of the previous section that the error in $\hat{\underline{a}}$ is distributed asymptotically as a Gaussian function.

Define the following:

$$A(f) = 1 - \sum_{k=1}^{n} \hat{a}_k e^{-i2\pi fkT} \tag{4.2-3}$$

$$\Delta A(f) = - \sum_{k=1}^{n} \Delta\hat{a}_k e^{-i2\pi fkT} \tag{4.2-4}$$

$\{\Delta\hat{a}_k\}$ is the error in the estimate of $\{a_k\}$, discussed in section (4.1-4). The total differential of $\Delta S_{yy}(f)$ of $S_{yy}(f)$ for the differentials $\Delta a_k = \hat{a}_k - a_k$ and $\Delta\sigma^2 = \hat{\sigma}^2 - \sigma^2$ is

$$\Delta S_{yy}(f) = \frac{\partial S_{yy}(f)}{\partial \sigma^2} \Delta\hat{\sigma}^2 + \sum_{k=1}^{n} \frac{\partial S_{yy}(f)}{\partial a_k} \Delta\hat{a}_k \tag{4.2-5}$$

We are interested in obtaining a value for the limit distribution of

$$\frac{\Delta S_{yy}(f)}{S_{yy}(f)} \tag{4.2-6}$$

It is shown by Akaike [A6] that the expected value of (4.2-6) as $N \to \infty$ is given by

$$E_\infty \left(\frac{\Delta S_{yy}}{S_{yy}}\right)^2 = E_\infty \left(\frac{\Delta\sigma^2}{\sigma^2}\right)^2 + E_\infty \left(\frac{\Delta |A(f)|^2}{|A(f)|^2}\right)^2 \qquad (4.2\text{-}7)$$

This result is obtained by using the earlier results (see[A6], theorem 2) that the limit distribution of $\sqrt{N}\ \Delta a$ and $\sqrt{N}\ \Delta\sigma^2$ are mutually independent.

In (4.2-7) $\Delta|A(f)|^2$ is given by

$$\Delta |A(f)|^2 = A(f)\ \overline{\Delta A(f)} + \overline{A(f)}\ \Delta A(f) \qquad (4.2\text{-}8)$$

Where —— denotes complex conjugate.

The first term in (4.2-7) is evaluated as

$$E_\infty \left(\frac{\Delta\sigma^2}{\sigma^2}\right)^2 = \frac{1}{N}\left(\frac{m_4}{\sigma^4} - 1\right) \qquad (4.2\text{-}9)$$

Where $m_4 = E[v_k^4]$. For a Gaussian noise $m_4 = 3\sigma^4$.

The following are the steps in computing the variance of the power spectrum from AR model.

1. Define $$A(f) = 1 - \sum_{k=1}^{n} \hat{a}_k e^{-i2\pi fkT}$$

$$\Delta A(f) = -\sum_{k=1}^{n} \Delta\hat{a}_k e^{-i2\pi fkT}$$

$\{\Delta\hat{a}_k\}_{k=1}^n$ are obtained from the error covariance matrix defined as follows

$$\begin{bmatrix} \Delta a_1 \\ \Delta a_2 \\ \vdots \\ \Delta a_n \end{bmatrix} = \frac{\sigma^2}{N} \text{ diag } \begin{bmatrix} C_0 & C_1 & \cdots & C_{n-1} \\ C_1 & C_0 & \cdots & C_{n-2} \\ \vdots & & & \\ C_{n-1} & C_{n-2} & \cdots & C_0 \end{bmatrix}^{-1}$$

2. Calculate $\Delta \left|A(f)\right|^2 = A(f)\ \overline{\Delta A(f)} + \overline{A(f)}\ \Delta A(f)$

3. Compute

$$\mathrm{Var}\left(\frac{\Delta S_{yy}(f)}{S_{yy}(f)}\right) = \frac{1}{N}\left(\frac{m_4}{\sigma^4} - 1\right) + \left\{\frac{\Delta\left|A(f)\right|^2}{\left|A(f)\right|^2}\right\}^2$$

$$\Delta Syy(f) = \left\{\frac{1}{N}\left(\frac{m_4}{\sigma^4} - 1\right) + \left[\frac{\Delta\left|A(f)\right|^2}{\left|A(f)\right|^2}\right]^2\right\}^{1/2} S_{yy}(f)$$

where

$$S_{yy}(f) = \frac{\sigma^2 T}{\left|1 - \sum_{k=1}^{n} a_k e^{-i2\pi tkT}\right|^2}$$

4.3 Selection of Optimal Autoregressive Models

A model that is chosen based on a given criterion should also pass all the validation tests (discussed in sec.4.4). As we have mentioned earlier an optimal model of an AR process may not be the best model for a given data. But we assume that this is indeed the case and restrict ourselves to the selection of the best model in this class. Many common criteria such as least squares may not lead to good models. A criterion must be sensitive to changes in the model order; the prediction capability of the model must be reflected by the criterion function. Bayes' minimum probability of error criterion is the most versatile of the methods described below.

It is important to realize that the selection of a best model is a decision making process. There may be cases when it is difficult to choose a model based on the absolute optimum of the criterion function. We suggest that in such a situation all the tests described below may be applied to be satisfied simultaneously. We want to avoid using rather arbitrary quantities such as significance levels to minimize the subjectivity of

decision. The hypothesis testing method, which uses these quantities will be described since it compares models of different orders based on a significance of the change in model order. The methods for order selection are:

(a) Likelihood Approach (Akaike [A2], Kashyap [K1])

(b) Final Prediction Error (FPE) (Akaike [A7])

(c) Hypothesis Testing (Wilks [W1], Kashyap [K1]).

(d) Bayesian Probability Criterion (Kashyap [K4])

4.3.1 The Likelihood Approach

This method is based on computing the maximum value of the log likelihood function for a given order n. Once the conditional maximum likelihood (CML) estimate $\hat{\underline{\theta}} = (\hat{\underline{a}}, \hat{\underline{\theta}})$ is obtained then an average value of the log likelihood function given $\hat{\underline{\theta}}$ is derived. The following theorem gives this value.

Theorem 4.3-1. Let $\hat{\underline{\theta}}$ be the CML estimate of $\underline{\theta}$, based on the observations $\underline{y}^N$. Then

$$E[\ell n\ p(\underline{y}^N \mid \underline{\theta})] = \ell n\ p(\underline{y}^N \mid \hat{\underline{\theta}}) - (n+1) \qquad (4.3\text{-}1)$$

For a proof of this theorem see [K1 , p184].

We note that $\ell n\ p(\underline{y}^N \mid \underline{\theta})$ would have been the correct log likelihood value if $\underline{y}^N$ had come from a model characterized by $\hat{\underline{\theta}}$. Since $\hat{\underline{\theta}}$ is only an estimate of $\underline{\theta}$ this additional ignorance about $\underline{\theta}_o$ manifests itself in a reduction in the likelihood by a quantity (n+1). The optimal model is chosen such that the expression in (4.3-1) is a maximum. This decision rule was first proposed by Akaike [A2] based on considerations different from that given in [K1].

Decision Rule 1: For a given model of order n estimate the CML parameters $\underline{\theta}$. Calculate

$$L = \ell n\ p\ (\underline{y}^N \mid \underline{\theta}) - (n + 1) \tag{4.3-2}$$

Choose the model for which L is the maximum.

REMARK: Calculation of L requires the knowledge of the distribution of the noise sequence $\{v_k\}$. In many cases the assumption of $\{v_k\}$ to have a Gaussian distribution will be close to reality. As discussed in section 4.1, the least squares and the Yule-walker equations will give the same consistent estimates as the Gaussian maximum likelihood estimate. Thus the assumption is not very restrictive.

Calculation of L: With the above assumption we first calculate

$$\hat{v}_k = y_k - \sum_{i=1}^{n} \hat{a}_i y_{k-i}, \quad k=1,2,\ldots,N \tag{4.3-3}$$

$\hat{v}_k$ is also called the prediction error, residual or the innovations process. With the above definition

$$p(\underline{y}^N \mid \underline{\hat{\theta}}) = \frac{1}{(2\pi\hat{\sigma}^2)^{N/2}} \exp\{-\frac{1}{2\hat{\sigma}^2} \sum_{k=1}^{N} \hat{v}_k^2\} \tag{4.3-4}$$

$$L = -\frac{N}{2} \ell n\ (2\pi\hat{\sigma}^2) - \frac{1}{2\hat{\sigma}^2} \sum_{k=1}^{N} \hat{v}_k^2 - (n + 1)$$

By taking the negative of L

$$-L = \{\frac{N}{2} \ell n\ (2\pi) + 1\} + \frac{N}{2} \ell n\ \hat{\sigma}^2 + \frac{1}{2\hat{\sigma}^2} \sum_{k=1}^{n} \hat{v}_K^2 + n$$

Since the first term is a constant, not a function of model order we can rewrite the above as

$$L_1 = \frac{N}{2} \ell n\ (\hat{\sigma}^2) + \frac{1}{2\hat{\sigma}^2} \sum_{k=1}^{n} \hat{v}_k^2 + n \tag{4.3-5}$$

n = number of independently variable parameters. If the estimate of σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{k=1}^{n} \hat{v}_k^2 , \tag{4.3-6}$$

substitution in (4.3-5) gives

$$L_1 = \frac{N}{2} \ln (\hat{\sigma}^2) + \frac{N}{2} + n$$

Once again neglecting $\frac{N}{2}$ and multiplying by 2 gives

$$\boxed{AIC = N \ln (\hat{\sigma}^2) + 2n} \tag{4.3-7}$$

We call AIC as the criterion such that

$$\min \quad AIC = N \ln (\hat{\sigma}^2) + 2n^* \tag{4.3-8}$$

where n* is the optimum model order.

4.3-2 Final Prediction Error (FPE) Criterion

The final prediction error is defined as

$$FPE = E[(y_N - \hat{y}_N)^2] \tag{4.3-9}$$

where

$$\hat{y}_N = \sum_{i=1}^{n} \hat{a}_i y_{N-i} \tag{4.3-10}$$

The $\hat{a}_i$'s are determined using $\{y_k, k=1,2,\ldots,N\}$. This was proposed by Akaike in 1970 [A7] as a method of choosing an optimal order n such that the FPE is a minimum. The evaluation of FPE assumes that as N increases, the dependency of $\hat{a}(N)$ will be completely vanishing. In other words the estimates of $\underline{a}$ are independent of the present or recent values of the data and that $\hat{y}_N$ depends on the statistics of $\{y_{N-i}, i=1,\ldots n\}$. Based on this and the estimates of $\underline{a}$ using the

Yule-Walker equation Akaike derived an expression for the FPE as

$$(FPE)_n = \frac{N+n}{N-n} S(n) \tag{4.3-11}$$

where

$$S(n) = c_0 - \sum_{i=1}^{n} \hat{a}_i c_i \tag{4.3-11}$$

and $$c_i = \frac{1}{N-i} \sum_{k=1}^{n-i} y_k y_{k+i} \tag{4.3-12}$$

If the noise statistic has a Gaussian distribution then it can be shown that the $(FPE)_n$ asymptotically approaches the likelihood criterion given by AIC of equation (4.3-7).

Theorem 4.3-2 If the noise statistic of $\{v_k\}$ has a Gaussian distribution then

$$\lim_{N \to \infty} \ell n\ (FPE)_n = \frac{1}{N}\ (AIC) \tag{4.3-13}$$

Proof: $(FPE)_n = \frac{N+n}{N-n}\ S(n)$

As $N \to \infty$, $S(n) \to \hat{\sigma}^2$, a consistent estimate of σ^2.

$$\ell n\ (FPE)_n = \ell n \hat{\sigma}^2 + \ell n\ (1 + \tfrac{n}{N}) - \ell n\ (1 - \tfrac{n}{N})$$

$\ell n\ (1 + \frac{n}{N}) \simeq \frac{n}{N}$ and $\ell n\ (1 - \frac{n}{N}) \doteq -\frac{n}{N}$

$$\ell n\ (FPE)_n = \ell n \hat{\sigma}^2 + \frac{2n}{N}\ .$$

$\frac{1}{N}\ (AIC) = \ell n \hat{\sigma}^2 + \frac{2n}{N}$ from eqn. (4.3-7)

$\Longrightarrow$

$$\lim_{N \to \infty} \ell n\ (FPE)_n = \frac{1}{N}\ (AIC)$$

4.3-3 Hypothesis Testing

This procedure, also called the likelihood ratio test, is used for small samples in testing the various hypotheses concerning the parameters of a normal distribution. This is treated by several authors. We refer to the work by Wilks [W1], Chapter 13.

We apply this theory to compare two classes of models, one with n_1 parameters and the other with n_2 parameters, $n_2 > n_1$. Based on the observation $\{y_1,\ldots,y_N\}$,we define a quantity $\eta_d(\underline{y}^N)$ as a statistic which is a function of $\underline{y}^N$.

$$\text{Decision rule } D: \eta_d(\underline{y}^N) \quad \begin{cases} \leq \eta_0 \rightarrow \text{accept } n_1 \\ > \eta_0 \rightarrow \text{accept } n_2 \end{cases} \tag{4.3-14}$$

Where η_0 is a threshold. The value of η_0 is fixed based on the property of the statistic $\eta_d(\underline{y}^N)$.

Define the likelihood ratio as

$$\eta(\underline{y}^N) = \frac{\max p_2(\underline{y}^N \mid \underline{\theta})}{\max p_1(\underline{y}^N) \mid \underline{\theta})} \tag{4.3-15}$$

where p_1 and p_2 are conditional probability density functions with n_1 and n_2 parameters.

Then the likelihood ratio test is given by

$$D: \quad \begin{array}{l} \eta(\underline{y}^N) \leq \eta_0 \rightarrow \text{accept } n_1 \\ \eta(\underline{y}^N) > \eta_0 \rightarrow \text{accept } n_2 \end{array} \tag{4.3-16}$$

Where η_0 is a suitable threshold.

The probabilities of errors associated with the above decision are classified as follows.

Probability of type I error

= Prob $\{\eta(\underline{y}^N) > \eta_0 \mid \underline{y}^N$ belongs to model $n_1\}$

Probability of type II error

= Prob $\{\eta(\underline{y}^N) \leq \eta_0 \mid \underline{y}^N$ belongs to model $n_2\}$

Decision Rule: The following decision rule is developed by Wilks. It it stated in the following form so that the decision rule under n_1 has a standard distribution.

Let N = No. of observations

n_1= No. of AR parameters for model 1.

n_2= No. of AR parameters for model 2.

Let $V(\underline{\theta},n_1)$ = Residual sum of squares for model 1.

$V(\underline{\theta}\ n_2)$ = Residual sum of squares for model 2.

Define

$$F = \frac{N - n_2}{n_2 - n_1} \quad \frac{V(\underline{\theta},\ n_1) - V(\underline{\theta},\ n_2)}{V(\underline{\theta},\ n_2)} \tag{4.3-18}$$

Using a theorem due to Cochran (see W[1], p212) $\{V(\underline{\theta},n_1) - V(\underline{\theta},n_2)\}$ and $V(\underline{\theta},n_2)$ are independently distributed according to χ^2- distributions with (n_2-n_1) and $(N-n_2)$ degrees of freedom. In order to derive this the noise statistics are assumed to be normal. It then follows that the function F has a F-distribution with (n_2-n_1) and $(N-n_2)$ parameters-$F(n_2-n_1, N-n_2)$. The probability distribution of F is independent of the parameters $(\underline{a},\sigma^2)$. The F-distribution is tabulated and we can choose a threshold η_0 to yield a particular value of the error probability of type I. But we cannot determine the type II error probability. The type I error probability is chosen as 0.05 or 0.02 etc. The ambiguity of this choice suggests that the decision rule based on AIC or FPE be used. The value of F, however, gives an estimate of the type I error probability when using

the likelihood criterion or final prediction error criterion.

Remark: We want the number F in (4.3-18) to be less than some threshold. We choose this threshold by computing the $100\alpha\%$ point F_α and see if $F < F_\alpha$. We do not want F to be greater than F_α. Set the limit by letting $P(F>F\alpha) = \alpha$ = area under the F- distribution curve between F_α and ∞. Then the error probability of type I is prechosen as α. See figure (4.1). For description of χ^2 and F-distributions see Appendix 4A.

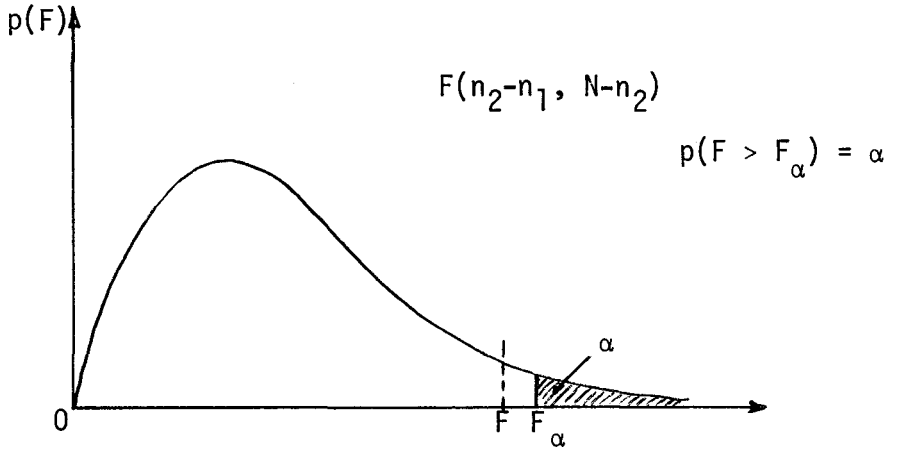


Figure 4.1

For a given value of α, if $F<F\alpha$ then we can stop the estimation procedure and choose $n_2(>n_1)$ as the optimal order. If n_2 and n_1 are successive orders then $n_2-n_1=1$ and we have to compute $F(1,N-n_2)$.

4.3.4 Bayesian Comparison Criterion

The Bayes' criterion is derived such that the probability of error in selecting an optimal model is minimized [K4]. The decision rule states that if there are n models with parameters $(n_1, n_2, \ldots, n_m)$ then choose the model whose a posteriori probability $P(n_i|\underline{y}^N)$ is a maximum. This criterion has the property of asymptotic consistency and gives a quantitative

explanation of the "principle of parsimony" defined in the construction of empirical models. Moreover the effect of the assumption of prior probabilities is negligible for large N. The decision rule is transitive and the probability of error of the optimal decision rule is given by

$$\text{Probability of Error} = 1 - \max P(n_i|\underline{y}^N) \tag{4.3-19}$$

The decision rule is given by

$$D(\underline{y}^N) = \text{argument} \max_{i\in\{1,2,\ldots,m\}} P(n_i|\underline{y}^N) \tag{4.3-20}$$

That is, choose n such that the a posteriori probability is a maximum. For autoregressive processes with Gaussian distributions, the criterion is given by

$$\text{BPC} = N\ell n\sigma_v^2 - n\ell n\ N\text{-}n\{\ell n(\frac{\sigma_y^2}{\sigma_v^2}) - 1\} \tag{4.3-21}$$

where σ_v^2 is the variance of $\{v_k\}$, σ_y^2 is the variance of $\{y_k\}$. Choose n such that BPC is a maximum.

4.3.5 Discussion Among the four model selection methods presented above the Bayesian approach is very versatile, requires less ambiguity and gives good results in practice (see discussion in reference [K4]). In fitting AR models the model order from the likelihood criterion may be larger than is necessary to pass validation tests.

The final prediction error in the AR case gives results close to the likelihood method. Both the tests may be implemented to reinforce the validity of the model.

The hypothesis testing approach is valid for large N. Only then the assumption of F-distribution is valid. There are instances where the first two methods may not produce an absolute minimum of the criterion function. After reaching an elbow shape the values of AIC or FPE may oscillate. See Figure 4.2. If this happens the F-test could be used to make a valid judgement about the model order without increasing the model order to an arbitrarily large number. Since this approach compares the change in error from one to the next model the F-test can be used in conjunction with the likelihood or prediction error tests.

4.4 Validation of Empirical Models.

After an optimal model is obtained, the model must be tested for the assumptions made regarding the statistics of the observations. It is important to realize that the best model obtained above may not always adequately represent the data. We have not taken into account any periodic or sinusoidal variations in the signal. Also additional input variables may not have been considered.

The first validation test is to check the whiteness of the noise sequence $\{v_k\}$. One may use the residual sequence and compute the auto-correlation of the residuals to calculate an index and compare this against a desired level of significance. Since we are interested in checking if the model is satisfactory, the arbitrariness of the significance level will not introduce any error. The output data itself may be tested for spectral density and correlogram.

In the following we present several validation tests. Simultaneous verification of these tests will exhibit any deficiency in the model. For details see [K1, B2].

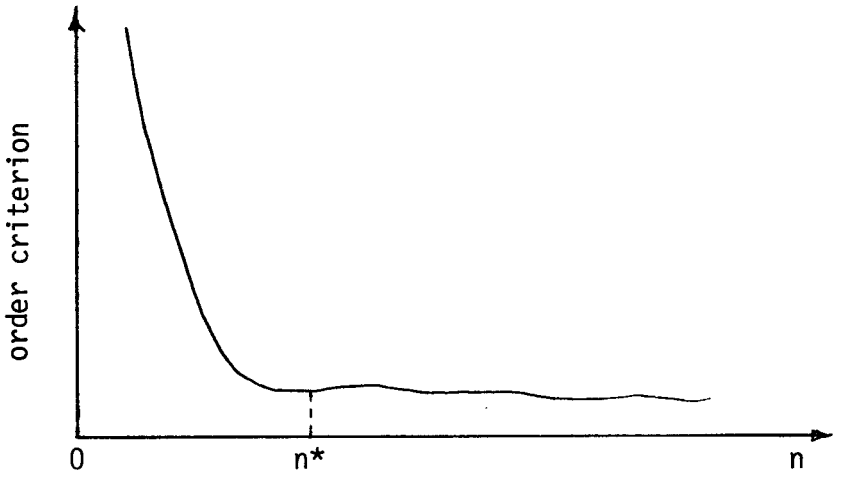


Figure 4.2

4.4.1 Autocorrelation Check Using Prediction Error

The major assumption in the model is that the background noise is a white noise sequence. Such a noise process has the property that the adjacent points of the series are not correlated. The correlation function must have the characteristics of an impulse function. The spectrum of the noise sequence must be flat within the band, limited by the folding frequency.

The residual or prediction error estimate is calculated as

$$\hat{v}_k = y_k - \sum_{i=1}^{n} \hat{a}_i y_{k-i}, \quad k=1,2,\ldots,N \tag{4.4-1}$$

If the parameters $\underline{a}$ of the AR model are known exactly, then it is shown by Anderson [A11] that the estimated autocorrelation

$$\rho_k(v) = C_k(v)/C_o(v) \tag{4.4-2}$$

of the noise sequence would be uncorrelated and distributed approximately normally about zero with variance $\frac{1}{N}$ and hence with a standard deviation of $1/\sqrt{N}$. When we only have an estimate $\hat{\underline{a}}$, the residuals can be determined as in (4.4-1). The autocorrelation of $\{\hat{v}_k\}$ is then determined and plotted. It is shown by Box and Pierce [B3] that this value may underestimate the error in $\rho_k(\hat{v})$ at low lags, but can be employed as a good estimate of error at moderate or high lags. The whitness of the noise is then checked by comparing a given number of $\rho_k(\hat{v})$ against a 95% confidence level given by $\pm$ $1.96/\sqrt{N}$. If less than 5% of the autocorrelation functions are outside this limit, then the whiteness is assumed with 95% confidence.

4.4.2 "Portmanteau" Lack of Fit Test (Box and Jenkins, [B2])

This is a goodness of fit test. Since taking the $C_K(v)$ individually and checking their boundedness is similar to testing a random sequence,

an indication is often needed of whether the first few correlation functions taken as a whole indicate inadequacy of the model.

Let C_k, k=0,1,2,... be the autocorrelations of $\{v_k\}$. The estimate of the residual spectrum $S(\omega)$ is

$$S(\omega) = \sum_{k=-\infty}^{\infty} C_k e^{-ik\omega} \tag{4.4-3}$$

If $\{v_k\}$ is white then $S(\omega) = S(0)$ $\forall\omega$ and the mean square deviation of $S(\omega)$ from $S(0)$ is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (S(\omega) - S(0))^2 \, d\omega = \sum_{k=1}^{\infty} C_k^2 \tag{4.4-4}$$

We can test the whiteness of $\{v_k\}$ by evaluating (4.4-4). In practice the series must be truncated and C_k replaced by their computed values. It is shown by Box and Pierce [B3] that

$$Q = N \sum_{k=1}^{M} \rho_k^2 (\hat{v}) \tag{4.4-5}$$

where

$$\rho_k (\hat{v}) = \sum_{i=1}^{N-k} \hat{v}_i \hat{v}_{i+k} \Big/ \sum_{i=1}^{N} \hat{v}_i^2$$

is approximately distributed as $\chi^2(M-n)$. If the model is inadequate the average value of Q will be inflated. This test is a form of hypothesis testing where the value of Q is compared with a value of χ^2 with (M-n) degrees of freedom, chosen according to an accepted value of the error probability. M is chosen between 0.01N and 0.1N depending on the size of N.

4.4-3 Bandwidth of Residual Power Spectrum

If the noise sequence is a white noise process, the spectrum will be flat in the band $-\frac{1}{2T} \leq f \leq \frac{1}{2T}$ Hertz. The flatness of the spectrum can be checked by calculating the estimate of $S_{vv}(f)$ as follows:

a. Determine $\hat{v}_k = y_k - \sum_{i=1}^{n} \hat{a}_i y_{k-i}$, k=1,2,...M.

b. Determine $C_k(\hat{v}) = \frac{1}{N} \sum_{i=1}^{N-k} \hat{v}_i \hat{v}_{i+k}$, k= 1,2,...,M.

c. Calculate the Fourier transform of $C_k(\hat{v})$ as follows:

$$S_{vv}(f) = \sum_{k=-M}^{M} C_k e^{-i2\pi fkT} , \quad |f| \leq \frac{1}{2T} \tag{4.4-6}$$

Since C_k is symmetric (4.4-6) becomes

$$S_{vv}(f) = C_o + 2 \sum_{k=1}^{M} C_k \cos 2\pi fkT, \quad |f| \leq \frac{1}{2T} \tag{4.4-7}$$

$S_{vv}(f)$ may be calculated and plotted. Satisfaction of all the above three diagnostic checks assures the appropriateness of the model and provides the needed confidence about the statistical assumptions on the driving noise.

4.4.4 Comparison of Spectral Estimates

The estimate of the power spectrum obtained from the AR model (see section 4.2) can be compared with the power spectrum obtained by Fourier transforming the given data. The latter can be obtained by using Fast Fourier Techniques (FFT) and an appropriate window function such as Kaiser-Bessel, Hamming or Parzen windows.

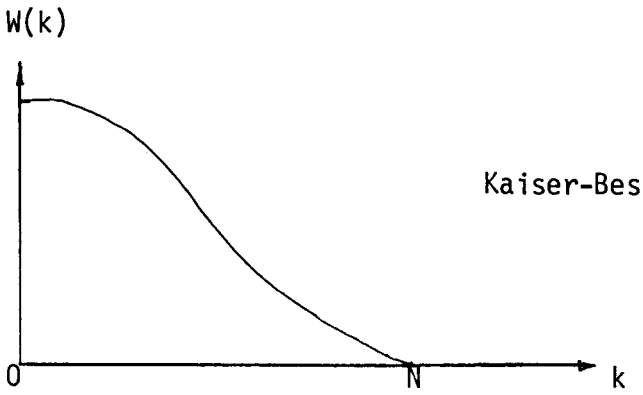
The Kaiser-Bessel window is defined as follows:

$$w(k) = \begin{cases} \dfrac{I_o[\beta\{1 - (\frac{k}{N})^2\}^{1/2}]}{I_o(\beta)} & , |k| \leq N \\ 0 & , |k| > N \end{cases} \tag{4.4-8}$$

$I_o(x) = \sum_{k=o}^{\infty} \left[\dfrac{(\frac{x}{2})^k}{k!} \right]^2$ is the Bessel Function of order zero.

β= window parameter. Values of β=3.384 to B= 7.865 are commonly used. β= 5.658 is recommended. For definitions of other window functions see Jenkins and Watts [J1]. Figure (4.3) shows the shape of the Kaiser-Bessel

window.



Kaiser-Bessel Window

Figure 4.3

The time series is multiplied by the corresponding window function. to smooth out the effects of data truncation. The stability of the AR spectral estimate is given by the asymptotic variance

$$|\Delta S_{yy}(f)|^2 = \{\frac{1}{N}(\frac{m_4}{\sigma^4} - 1) + [\frac{\Delta|A(f)|^2}{|A(f)|^2}]^2\} S_{yy}(f)^2 \qquad (4.4\text{-}9)$$

(Comparison to classical spectral estimator is discussed by Kaveh and Cooper [K3])

4.4.5 Test for Normality: The assumption of normality of the noise statistics is not a very serious one. For most of the analysis this assumption is not at all necessary. However, it would be informative to see if the data has a normal distribution.

One simple test is to plot the estimate of the residual sequence $\{\hat{v}_k\}$ and see if 95% of these lie within $\pm 2\hat{\sigma}$ where $\hat{\sigma}^2$ is the estimated variance of the noise. A test such as this is once again a hypothesis testing scheme by assuming a level of significance in the curve of normal probability density.

Another test which evaluates the goodness of assumed density is called the Kolmogorov-Smirnov goodness-of-fit test. This is also a

hypothesis test and gives the answer by evaluating a single number. This stastistical test is described below.

Kolmogorov-Smirnov (K-S) Test:

Let $p(v)$ be the probability density function of the noise process. Let us postulate that this has a normal distribution.

$$p(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v^2}{2\sigma^2}} \tag{4.4-10}$$

Let $\underline{V}^N = \{v_1 v_2,\ldots,v_N\}$ be the sequence of estimated values of residuals. Define

$$\delta_i(v) = \begin{cases} 1 & \text{if } v_i \leq v \\ 0 & \text{otherwise} \end{cases} \tag{4.4-11}$$

Then the empirical distribution $S_N(v)$ of the sample $\underline{V}^N$ is given by

$$S_N(v) = \sum_{i=1}^{N} \frac{\delta_i(v)}{N} \tag{4.4-12}$$

The postulated distribution function is given by

$$F(v) = \int_{-\infty}^{v} p(v)\, dv \tag{4.4-13}$$

It can be shown that if $F(v)$ is indeed the correct distribution then $S_N(v)$ converges to $F(v)$ in probability (see Gibbons [G1]).

The K-S statistic (for a given sample) is defined as

$$D_N = \sup_v \left| S_N(v) - F(v) \right| \tag{4.4-14}$$

This D_N statistic can be shown to be independent of the continuous distribution $F(v)$. It is called distribution-free statistic.

The probability $P(D_N \leq Z)$ has been derived in Gibbons. For a given probability of error of the first kind, denoting

$$P(D_N > D_{N\alpha}) = \alpha \tag{4.4-15}$$

we can either calculate or refer to a table of K-S statistic, and determine D_N. Then, if $D_N > D_{N\alpha}$ reject the hypothesis that $S_N(v)$ has the distribution postulated as $F(v)$. If $D_N \leq D_{N,\alpha}$ then the postulated distribution $F(v)$ is acceptable at a level of significance of $100(1-\alpha)\%$. For finite $N \leq 50$ the values of $D_{N\alpha}$ are tabulated in Gibbson for $\alpha = 0.01$ and $\alpha = 0.05$. For large N, an asymptotic expression is derived by Kolmogorov as follows.

$$\lim_{N \to \infty} P\left(D_N > \frac{Z_\alpha}{\sqrt{N}}\right) = 1 - L(Z\alpha) \tag{4.4-16}$$

where

$$L(z_\alpha) = 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp(-2i^2 z_\alpha^2) \tag{4.4-17}$$

The values of this probability for values of z_α are tabulated by Smirnov [S2].

The above approximation is close enough for $N > 50$.

4.4-6 Discussion: This concludes the summary of the techniques. We have presented the five important tests one can carry out with straight forward calculations and use of proper tables. Appendix 4A gives more definitions of various distributions mentioned in this chapter, together with some useful tables. The knowledge of the distribution of noise processes is useful if the analyst needs to generate synthetic data simulating the actual process for any future analysis. The reader is encouraged to refer to the books and papers referenced in this chapter and extend the methodology presented

here to suit his/her special needs. It is important to realize that the empirical model building, though based on sound theoretical analysis, requires a good understanding of the problem under study and needs modification if necessary.

APPENDIX 4A

In this appendix we will outline the probability density functions of important distributions that appear in statistical inference.

4A.1 Normal or Gaussian Distribution

If x is a normal random variable with mean μ_x, variance σ_x^2 - denoted by $x \sim N(\mu_x, \sigma_x^2)$ then

$$Z = \frac{x - \mu_x}{\sigma_x} \sim N(0,1) \tag{4A-1}$$

It is desirable to denote the value of z which corresponds to a specific probability $P(z) = 1-\alpha$

$$P(z_\alpha) = \int_{-\infty}^{Z_\alpha} p(z)\, dz = P\{z \leq z\alpha\} = 1-\alpha \tag{4A-2}$$

$$\int_{Z_\alpha}^{\infty} P(z)\, dx = P\{Z > Z_\alpha\} = \alpha \tag{4A-3}$$

The value of Z_α that satisfies the above is called the $100\alpha\%$ point. Table (4A-1) gives values of Z_α as a function of α.

4A-2 Chi-Squared Distribution

Let z_i, i=1,2,...,n be such that $z_i \sim N(0,1)$ Define the new random variable

$$\chi_n^2 = z_1^2 + z_2^2 + \ldots + z_n^2 \tag{4A-4}$$

The random variable χ_n^2 is the Chi-squared variable with n degrees of freedom; degrees of freedom , represents the number of independent or free variables entering into the expression. The density function of χ_n^2 is given by

$$p(\chi^2_n \mid n) = \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \left(\chi^2\right)^{\frac{n}{2}-1} e^{-\chi^2/2}, \quad \chi^2 > 0 \qquad (4A\text{-}5)$$

Where $\Gamma(\frac{n}{2})$ is the gamma function defined as

$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x}\, dx, \quad m>0 \qquad (4A\text{-}6)$$

The 100 α % point of the χ^2 distribution with n degrees of freedom is denoted by $\chi^2_{n,\alpha}$

$$\int_{\chi^2_{n,\alpha}}^\infty p\,(\chi^2)\, d\chi^2 = P\{\chi^2_n > \chi^2_{n,\alpha}\} = \alpha \qquad (4A\text{-}7)$$

Table (4A-2) gives values of $\chi^2_{n,\alpha}$.

4A.3 Student t Distribution

Let y have a χ^2_n distribution with n degrees of freedom and $z \sim N(0,1)$. Define

$$t_n = \frac{z}{\sqrt{y/n}} \qquad (4A\text{-}8)$$

The random variable t_n is the Student t variable with n degrees of freedom and its probability density function has the form

$$p(t_n) = \frac{\Gamma\left\{\frac{n+1}{2}\right\}}{\sqrt{n\pi}\;\Gamma(\frac{n}{2})} \left[1 + \frac{t^2_n}{n}\right]^{-(\frac{n+1}{2})} \qquad (4A\text{-}9)$$

The 100α% point of the t-distribution will be denoted by $t_{n,\alpha}$, that is

$$\int_{t_{n,\alpha}}^\infty p(t_n)\, dt = P\{t_n > t_{n,\alpha}\} = \alpha \qquad (4A\text{-}10)$$

Values of $t_{n,\alpha}$ are listed in table (4A-3).

4A.4 The F-Distribution

Let y_1 and y_2 be two independent random variables such that

$y_1 \sim (\chi^2, n_1)$, $y_2 \sim (\chi^2, n_2)$. Define

$$F_{n_1,n_2} = \frac{y_1/n_1}{y_2/n_2} = \frac{y_1 n_2}{y_2 n_1} \tag{4A-11}$$

The random variable F_{n_1,n_2} has an F-distribution with n_1 and n_2 degrees of freedom and its probability density function is given by

$$p(F_{n_1 n_2}) = \frac{\Gamma\left\{\frac{n_1 + n_2}{2}\right\} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} F^{\left\{\frac{n_1}{2} - 1\right\}}}{\Gamma\left\{\frac{n_1}{2}\right\} \Gamma\left\{\frac{n_2}{2}\right\} \left[1 + \frac{n_1}{n_2} F\right]^{n_1 + n_2/2}}, \quad F \geq 0 \tag{4A-12}$$

The $100\alpha\%$ point of F-distribution will be denoted by $F_{n_1,n_2;\alpha}$

$$\int_{F_{n_1,n_2;\alpha}}^{\infty} p(F)\, dF = P\{F_{n_1,n_2} > F_{n_1,n_2;\alpha}\} = \alpha \tag{4A-13}$$

Table (4A-4) shows the values of $F_{n_1 n_2;\alpha}$.

4A.5 The Kolmogorv-Smirnov statistic

Define

$$D_n = \sup |S_n(x) - F(x)| \tag{4A-14}$$

where $F(x)$ is any continuous distribution function.

$$P\{D_n < \frac{1}{2n} + v\} = \begin{cases} 0 & \text{for } v \leq 0 \\ \int_{\left(\frac{1}{2n} - v\right)}^{\left(\frac{1}{2n} + v\right)} \int_{\left(\frac{3}{2n} - v\right)}^{\left(\frac{3}{2n} + v\right)} \cdots \int_{\left(\frac{2n-1}{2n} - v\right)}^{\left(\frac{2n-1}{2n} + v\right)} f(u_1, u_2, \ldots, u_n)\, du_n \ldots du_1 & \text{for } 0 < v < \frac{2n-1}{2n} \\ 1 & \text{for } v \geq \frac{2n-1}{2n} \end{cases}$$

where

$$f(u_1, u_2, \ldots u_n) = \begin{cases} n! & \text{for } 0 < u_1 < u_2 < \ldots < u_n < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4A-15}$$

The values of $D_{n,\alpha}$ such that $P\{D_n > D_{n,\alpha}\}$ are tabulated in table (4A-5a).

The asymptotic expression is given by

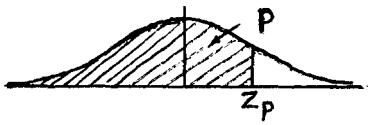
$$\lim_{n \to \infty} P\{D_n \leq \frac{z}{n}\} = L(z) \qquad ,z \geq 0$$

where

$$L(z) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k-1} \exp(-2k^2 z^2) \tag{4A-16}$$

The asymptotic expression for $P\{D_n > \frac{z_\alpha}{\sqrt{n}}\}$ and the corresponding values of z_α are listed in table (4A-5b).

TABLE 4A-1 CUMULATIVE NORMAL DISTRIBUTION - VALUES OF P



Values of P corresponding to z_p for the normal curve.

z is the standard normal variable. The value of P for $-z_p$ equals one minus the value of P for $+z_p$.

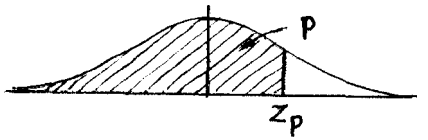
e.g., the P for -1.62 equals 1 -.9474 = .0526.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

TABLE 4A-1 (Continued)

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	..9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	..9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

TABLE 4A-1 CUMULATIVE NORMAL DISTRIBUTION-VALUES OF z_p



Values of z_p corresponding to P for the normal curve.

z is the standard normal variable.

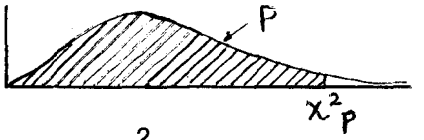
P	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.00	-----	-2.38	-2.05	-1.88	-1.75	-1.64	-1.55	-1.48	-1.41	-1.34
.10	-1.28	-1.23	-1.18	-1.13	-1.08	-1.04	-0.99	-0.95	-0.92	-0.88
.20	-0.84	-0.81	-0.77	-0.74	-0.71	-0.67	-0.64	-0.61	-0.58	-0.55
.30	-0.52	-0.50	-0.47	-0.44	-0.41	-0.39	-0.36	-0.33	-0.31	-0.28
.40	-0.25	-0.23	-0.20	-0.18	-0.15	-0.13	-0.10	-0.08	-0.05	-0.03
.50	0.00	0.03	0.05	0.08	0.10	0.13	0.15	0.18	0.20	0.23
.60	0.25	0.28	0.31	0.33	0.36	0.39	0.41	0.44	0.47	0.50
.70	0.52	0.55	0.58	0.61	0.64	0.67	0.71	0.74	0.77	0.81
.80	0.84	0.88	0.92	0.95	0.99	1.04	1.08	1.13	1.18	1.23
.90	1.28	1.34	1.41	1.48	1.55	1.64	1.75	1.88	2.05	2.33

Special Values

P	.001	.005	.010	.025	.050	.100
z_p	-3.090	-2.576	-2.326	-1.960	-1.645	-1.282

P	.999	.995	.990	.975	.950	.900
z_p	3.090	2.576	2.326	1.960	1.645	1.282

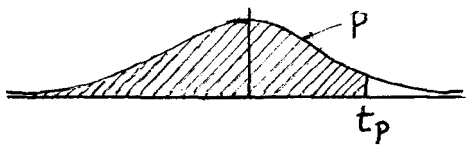
TABLE 4A-2 PERCENTILES OF THE χ^2 DISTRIBUTION



Values of χ^2_p corresponding to P

df	$\chi^2_{.005}$	$\chi^2_{.01}$	$\chi^2_{.025}$	$\chi^2_{.05}$	$\chi^2_{.10}$	$\chi^2_{.90}$	$\chi^2_{.95}$	$\chi^2_{.975}$	$\chi^2_{.99}$	$\chi^2_{.995}$
1	.000039	.00016	.00098	.0039	.0158	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.1026	.2107	4.61	5.99	7.38	9.21	10.60
3	.0717	.115	.216	.352	.584	6.25	7.81	9.35	11.34	12.84
4	.207	.297	.484	.711	1.064	7.78	9.49	11.14	13.28	14.86
5	.412	.554	.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	.676	.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
120	83.85	86.92	91.58	95.70	100.62	140.23	146.57	152.21	158.95	163.64

TABLE 4A-3 PERCENTILES OF THE t DISTRIBUTION



df	$t_{.60}$	$t_{.70}$	$t_{.80}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$
1	.325	.727	1.376	3.078	6.314	12.706	31.821	63.657
2	.289	.617	1.061	1.886	2.920	4.303	6.965	9.925
3	.277	.584	.978	1.638	2.353	3.182	4.541	5.841
4	.271	.569	.941	1.533	2.132	2.776	3.747	4.604
5	.267	.559	.920	1.476	2.015	2.571	3.365	4.032
6	.265	.553	.906	1.440	1.943	2.447	3.143	3.707
7	.263	.549	.896	1.415	1.895	2.365	2.998	3.499
8	.262	.546	.889	1.397	1.860	2.306	2.896	3.355
9	.261	.543	.883	1.383	1.833	2.262	2.821	3.250
10	.260	.542	.879	1.372	1.812	2.228	2.764	3.169
11	.260	.540	.876	1.363	1.796	2.201	2.718	3.106
12	.259	.539	.873	1.356	1.782	2.179	2.681	3.055
13	.259	.538	.870	1.350	1.771	2.160	2.650	3.012
14	.258	.537	.868	1.345	1.761	2.145	2.624	2.977
15	.258	.536	.866	1.341	1.753	2.131	2.602	2.947
16	.258	.535	.865	1.337	1.746	2.120	2.583	2.921
17	.257	.534	.863	1.333	1.740	2.110	2.567	2.898
18	.257	.534	.862	1.330	1.734	2.101	2.552	2.878
19	.257	.533	.861	1.328	1.729	2.093	2.539	2.861
20	.257	.533	.860	1.325	1.725	2.086	2.528	2.845
21	.257	.532	.859	1.323	1.721	2.080	2.518	2.831
22	.256	.532	.858	1.321	1.717	2.074	2.508	2.819
23	.256	.532	.858	1.319	1.714	2.069	2.500	2.807
24	.256	.531	.857	1.318	1.711	2.064	2.492	2.797
25	.256	.531	.856	1.316	1.708	2.060	2.485	2.787
26	.256	.531	.856	1.315	1.706	2.056	2.479	2.779
27	.256	.531	.855	1.314	1.703	2.052	2.473	2.771
28	.256	.531	.855	1.313	1.701	2.048	2.467	2.763
29	.256	.530	.854	1.311	1.699	2.045	2.462	2.756
30	.256	.530	.854	1.310	1.697	2.042	2.457	2.750
40	.255	.529	.851	1.303	.684	2.021	2.423	2.704
60	.254	.527	.848	1.296	1.671	2.000	2.390	2.660
120	.254	.526	.845	1.289	1.658	1.980	2.358	2.617
∞	.253	.524	.842	1.282	1.645	1.960	2.326	2.576

TABLE 4A-4. PERCENTILES OF F - DISTRIBUTION

See Table A-5, pp T-6 - T-9 of Experimental Statistics, National Bureau of Standards Handbook 91, 1963.

TABLE 4A-5a

95%-POINTS $\varepsilon_{N,.99}$ AND 99%-POINTS $\varepsilon_{n,.99}$

FOR KOLMOGOROV'S STATISTIC

(1) N	(2) $\varepsilon_{N,.95}$	(3) $\varepsilon_{N,.99}$	(4) $\bar{\varepsilon}_{N,.95}$	(5) $\bar{\varepsilon}_{N,.99}$	(6) $\frac{\bar{\varepsilon}_{N,.95}}{\varepsilon_{N,.95}}$	(7) $\frac{\bar{\varepsilon}_{N,.99}}{\varepsilon_{N,.99}}$
2	.8419	.9293	.9612	1.1509	1.142	1.238
3	.7076	.8290	.7841	.9397	1.108	1.134
4	.6239	.7341	.6791	.8138	1.088	1.109
5	.5633	.6685	.6074	.7279	1.078	1.089
10	.4087	.4864	.4295	.5147	1.051	1.058
15	.3375	.4042	.3507	.4202	1.039	1.040
20	.2939	.3524	.3037	.3639	1.033	1.033
25	.2639	.3165	.2716	.3255	1.029	1.028
30	.2417	.2898	.2480	.2972	1.026	1.025
40	.2101	.2521	.2147	.2574	1.022	1.021
50	.1884	.2260	.1921	.2302	1.019	1.018
60	.1723	.2067	.1753	.2101	1.018	1.016
70	.1597	.1917	.1623	.1945	1.016	1.015
80	.1496	.1795	.1518	.1820	1.015	1.014
90	.1412		.1432		1.014	
100	.1340		.1358		1.013	

TABLE 4A-5b

Asymptotic Approximation to $D_{n,\alpha} = z_\alpha / \sqrt{n}$

$P(D_n > Z_\alpha / \sqrt{n})$	0.20	0.15	0.10	0.05	0.01
Z_α	1.07	1.14	1.22	1.36	1.63

CHAPTER 5

ESTIMATION OF RESPONSE CHARACTERISTICS

The impulse and step responses of the dynamic system modeled by the fitted autoregressive process are derived. The time constant is estimated from the step response. The methods are tested by simulating known systems of order two and five to include a wide range of dynamics.

5.1 A First Order System

The standard definition of the time constant is given for a first order system. Consider the following system:

$$\dot{x} + ax = u(t) \tag{5.1-1}$$

The unit step response of (5.1-1) with $x(0) = 0$ is given by

$$x(t) = \frac{1}{a}(1-e^{-at}) \tag{5.1-2}$$

Letting $t = \frac{1}{a}$ gives

$$x(\tfrac{1}{a}) = \frac{1}{a}(1-\tfrac{1}{e}) = \frac{0.632}{a} \tag{5.1-3}$$

$$x(\infty) = \frac{1}{a}, \text{ steady state value.} \tag{5.1-4}$$

When the time $t = \frac{1}{a}$, the value of $x(t)$ attains 0.632 of the steady state value. $\tau = \frac{1}{a}$ sec, which is the time required for the step response of a stable first order system to attain 0.632 of its steady state value, is generally referred to as the time constant of the system.

We observe that a closed form expression similar to (5.1-3) cannot be obtained for a system of order greater than one. Still, we can define τ as the time at which the response of the system to a step input will achieve 0.632 of steady state value. Such a point on the response curve can be determined numerically.

5.2 Impulse, Step and Ramp Responses From the Autoregressive Model

Consider the AR process determined for a given noise measurement

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k, \quad k=1,2,... \tag{5.2-1}$$

The dynamics of the process is represented by the AR parameters. The dynamic information (transient and steady state) is given by the poles of the equivalent z-transform. Without evaluating the poles the impulse response is determined. The step response is then obtained by integrating the impulse response.

5.2.1 Type I Respones: In Section 3.4 it was pointed out that the AR process can be considered as the approximation to an infinite order moving average process for increasing values of n. When the system is stable it follows that

$$y_k = \sum_{i=1}^{n} a_i y_{k-i} + v_k = \sum_{i=1}^{\infty} b_i v_{k-i} + v_k \tag{5.2-2}$$

It then follows that the coefficients b_i represent the impulse response of the system such that $v_k = 0$ for $k \geq 1$ and $v_0 =$ constant. From this observation, we can derive the impulse response from AR process by recursively computing y_k as a function of previous y when $v_k = 0$, $k \geq 1$ and $v_0 =$ constant. The impulse response is calculated using

$$y_k^I = \sum_{i=1}^{n} a_i y_{k-i}^I \quad , \quad y_0^I = \text{constant}. \qquad y_{-\ell}^I = 0, \quad \ell > 0 \tag{5.2-3}$$

For systems with more than one pole the impulse response has the value $y_0^I = 0$. Hence in actual calculations the impulse response will be close to the real case if we let $y_0^I = 0$ and $y_1^I =$ constant and then evaluate $\{ y_k^I \}$ using (5.2-3). We will refer to the impulse response obtained

this way as the Type I response.

5.2.2 Type II Response: Here the impulse response is derived by analogy to an nth order continuous system.

Consider an nth order continuous system given by

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_{n-1} \frac{dx}{dt} + a_n x(t) = u(t) \qquad (5.2\text{-}4)$$

The unit impulse response of (5.2-4) is

$$x_I(t) = L^{-1} \left\{ \frac{1}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \right\} \qquad (5.2\text{-}5)$$

Now if we take the Laplace transform of (5.2-4), letting $u(t) = 0$ and all the initial conditions equal to zero except $x^{(n-1)}(0) = \frac{d^{n-1} x(0)}{dt^{n-1}}$, we get

$$x(s) = \frac{x^{(n-1)}(0)}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \qquad (5.2\text{-}6)$$

The response to an initial condition of $x^{(n-1)}(0) = 1$ is

$$x_{IC}(t) = L^{-1} \left\{ \frac{x^{(n-1)}(0) = 1}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \right\} \qquad (5.2\text{-}7)$$

(5.2-5) and (5.2-7) are the same. In one case the impulse response is obtained using a unit impulse input and in the other a nonzero initial condition on the (n-1)th derivative is used.

A method which approximates the continuous case is obtained by using a differencing scheme. Denote by $\{d_k^1, d_k^2, \ldots, d_k^{n-1}\}$ the first (n-1) th derivatives of $\{y_k\}$. These derivatives are approximated as follows.

$$d_k^1 = y_k^I - y_{k-1}^I$$

$$d_k^2 = d_k^1 - d_{k-1}^1$$

.

.

.

$$d_k^{n-1} = d_k^{n-2} - d_{k-1}^{n-2} \qquad (5.2\text{-}8)$$

By letting $d^{n-1} \neq 0$ and all lower order differences equal to zero, the impulse reponse will be evaluated recursively. A new expression in terms of a_i, i=1,2,...,n and $\{y_k^I, d_k^1, d_k^2, \ldots, d_k^{n-1}\}$ is derived. The resulting response has the form

$$y_k^I = A_1 y_{k-1}^I + A_2 d_{k-1}^1 + A_3 d_{k-1}^2 + \ldots + A_n d_{k-1}^{n-1} \qquad (5.2\text{-}9)$$

The coefficients A_i are functions of AR parameters such that the relationships (5.2-8) are satisfied. The result of this procedure is called TYPE II response. Values of A_i for different model order are given in table (5.1).

Example: Computation of A_i for a fourth order system is illustrated below.

Consider $$y_k = \sum_{i=1}^{4} a_i\, y_{k-i} \qquad (5.2\text{-}10)$$

Define the following:

$$d_k^1 = y_k - y_{k-1} \qquad (a)$$

$$d_k^2 = d_k^1 - d_{k-1}^1 \qquad (b)$$

$$d_k^3 = d_k^2 - d_{k-1}^2 \qquad (c)$$

From (a) $y_{k-2} = y_{k-1} - d_{k-1}^1$ (d)

from (b) $d_{k-1}^2 = d_{k-1}^1 - d_{k-2}^1 = d_{k-1}^1 - (y_{k-2} - y_{k-3})$

Using (d) in above equation $y_{k-3} = y_{k-1} - 2d_{k-1}^1 + d_{k-1}^2$ (e)

From (c) $d_{k-1}^3 = d_{k-1}^2 - d_{k-2}^2$

$$= d^2_{k-1} - (d^1_{k-2} - d^1_{k-3})$$

$$= d^2_{k-1} - (y_{k-2} - y_{k-3}) + y_{k-3} - y_{k-4}$$

$$= d^2_{k-1} - (y_{k-1} - d^1_{k-1}) + 2(y_{k-1} - 2d^1_{k-1} + d^2_{k-1}) - y_{k-4}$$

or $$y_{k-4} = y_{k-1} - 3d^1_{k-1} + 3d^2_{k-1} - d^3_{k-1} \qquad (f)$$

Using (d-f) in (5.2-10) gives

$$y_k = a_1 y_{k-1} + A_2 d^1_{k-1} + A_3 d^2_{k-1} + A_4 d^3_{k-1} \qquad (5.2\text{-}11)$$

where $a_1 = a_1 + a_2 + a_3 + a_4$, $A_2 = -(a_2 + 2a_3 + a_4)$,

$A_3 = a_3 + 3a_4$, $A_4 = -a_4$.

5.2.3 Computation of Step Response

Once the impulse response is determined as outlined in (5.2-1) and (5.2.2), the step response is derived by integrating the derived impulse response.

$$x_S(t) = \int_0^t x_I(\tau)\, d\tau \qquad (5.2\text{-}12)$$

A simple trapezoidal integration scheme is used to evaluate the step response since the integrand is available only at sample points.

5.2.4 Ramp Response and Delay Time Evaluation

It is often useful to know how the RTD responds to a ramp change in the fluid temperature. The steady state error in the ramp response and the delay time in attaining the same response level as the ramp input can be computed by integrating the step response to obtain the ramp response. Thus

$$x_R(t) = \int_0^t x_S(\tau)\, d\tau \qquad (5.2\text{-}13)$$

The RTD delay time is then calculated by estimating the lag time between the ramp response and the input ramp whose slope is the same as that of ramp response.

5.3 Verification of the Methods Using Known Systems.

The methods developed above are verified using known systems. We will consider a second order system with inequal poles. For more examples see reference [R3].

Consider the continuous system

$$G(S) = \frac{1}{(S + 1)(S + 2)} \qquad (5.3\text{-}1)$$

Synthetic data with white noise as the input is generated and an optimal AR model is fitted.

The analysis is based on a sample size of N = 4000 with a sampling interval of $\Delta t = 0.05$ sec.

Results: A 4th order AR model is fitted:

$$y_k = 2.1218y_{k-1} - 1.4159y_{k-2} + 0.34375\, y_{k-3} \cdot$$

$$-0.05322y_{k-4} + v_k \qquad (5.3\text{-}2)$$

Fig. (5.3-1) shows a sample of the time series. Fig. (5.3-2) is a plot of the calculated autocorrelation function. The estimated power spectrum from the AR model is shown in fig. (5.3-3). Figures (5.3-4) and (5.3-5) are the estimated impulse and step response of the model. The system step response is shown in fig. (5.3-6). The comparison of time constant is as follows:

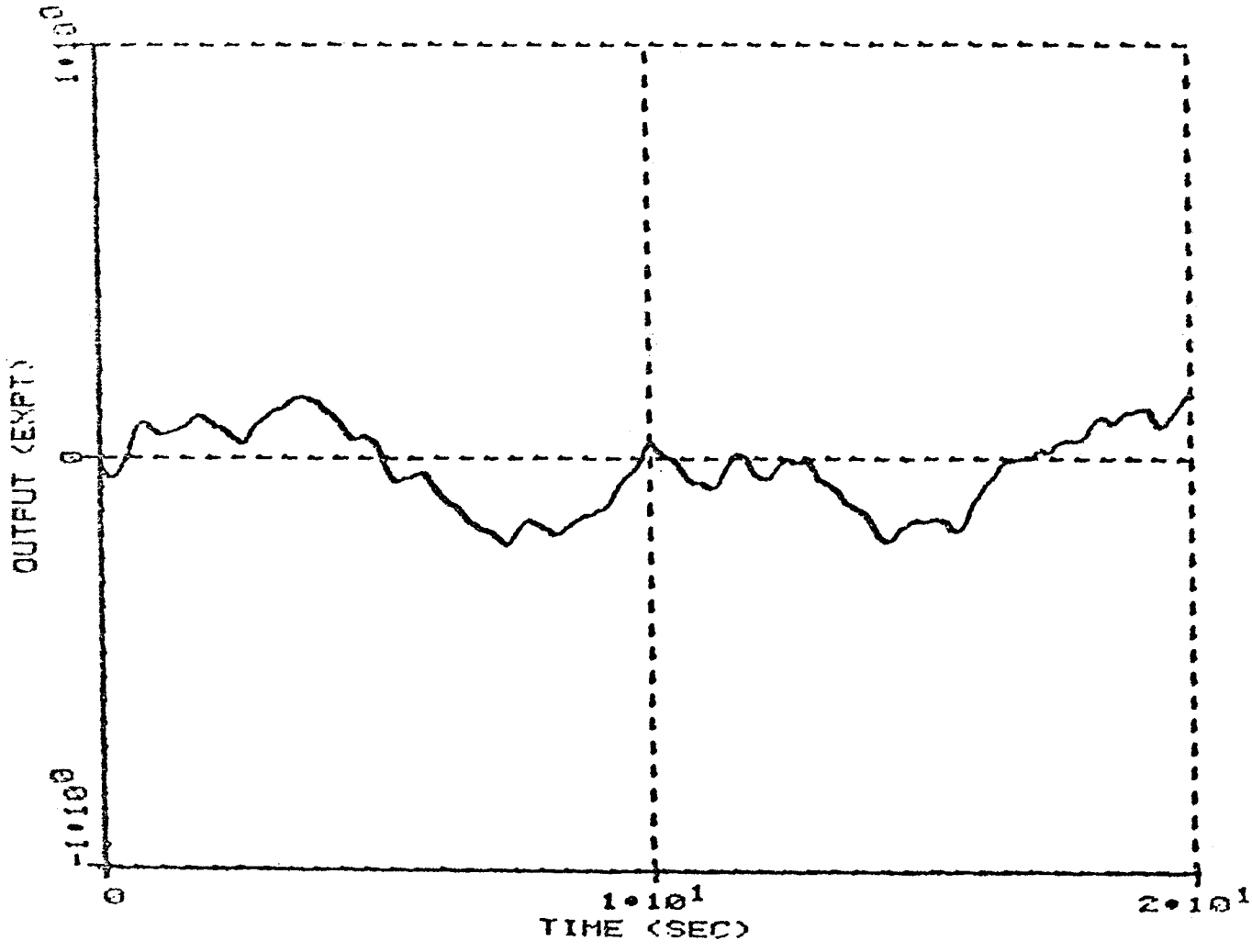


Figure 5.3-1 Synthetic Noise Signal for a second order system with transfer function

$$G(S) = \frac{1}{(S+1)(S+2)}$$

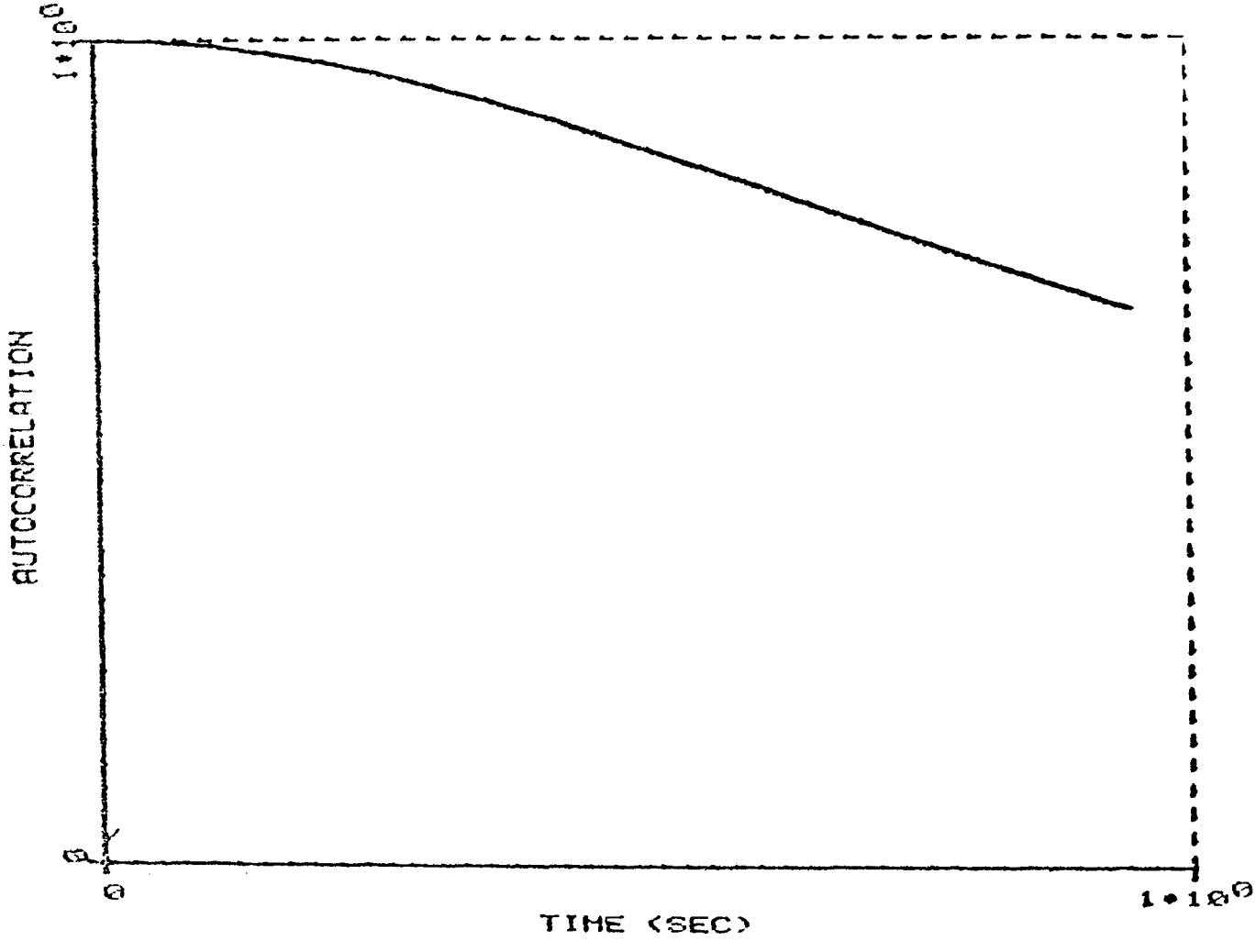


Figure 5.3-2 Autocorrelation Function for data of figure 5.3-1.

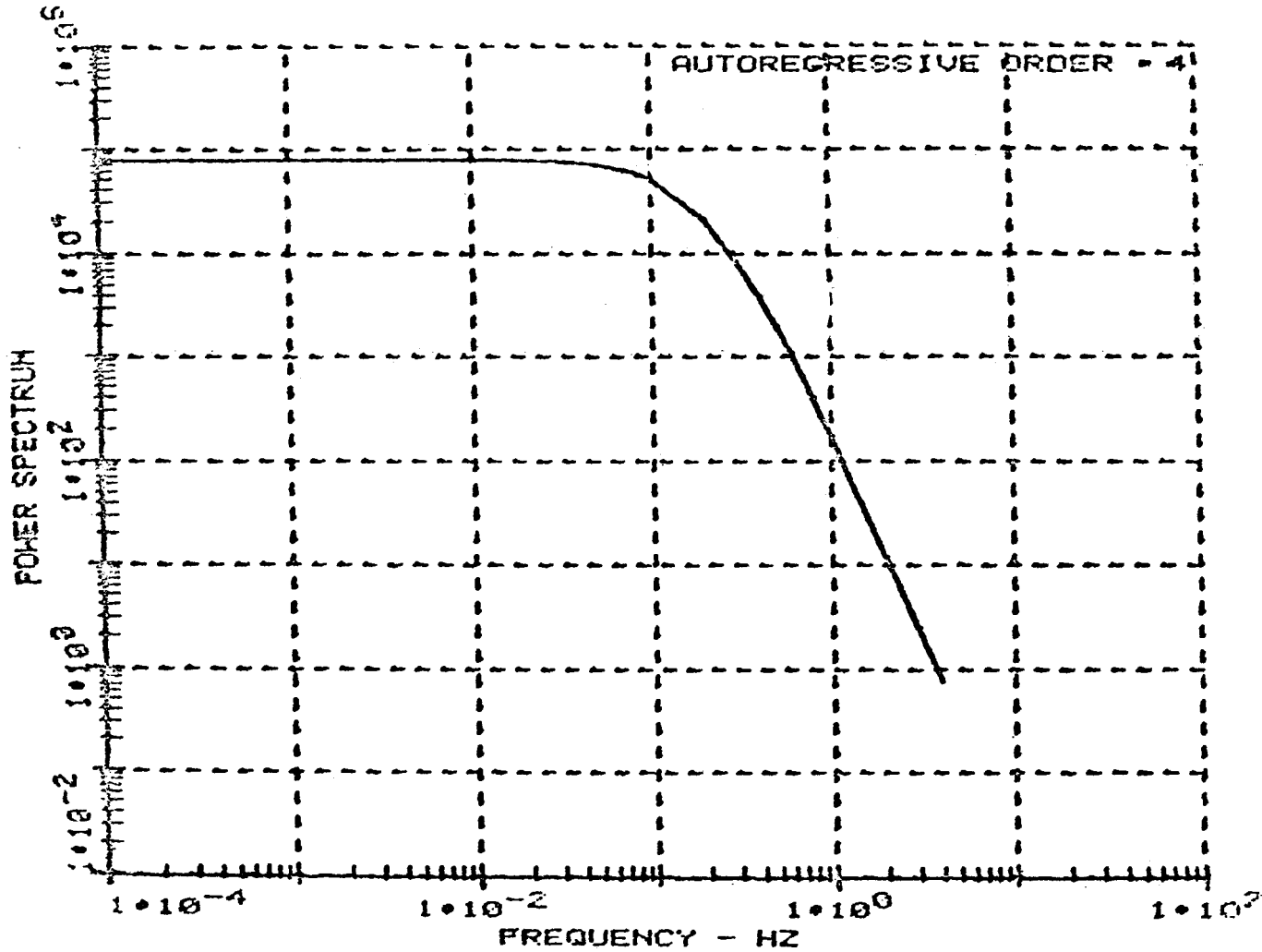


Figure 5.3-3 Power Spectrum from AR (4) Model for Signal of figure 5.3-1

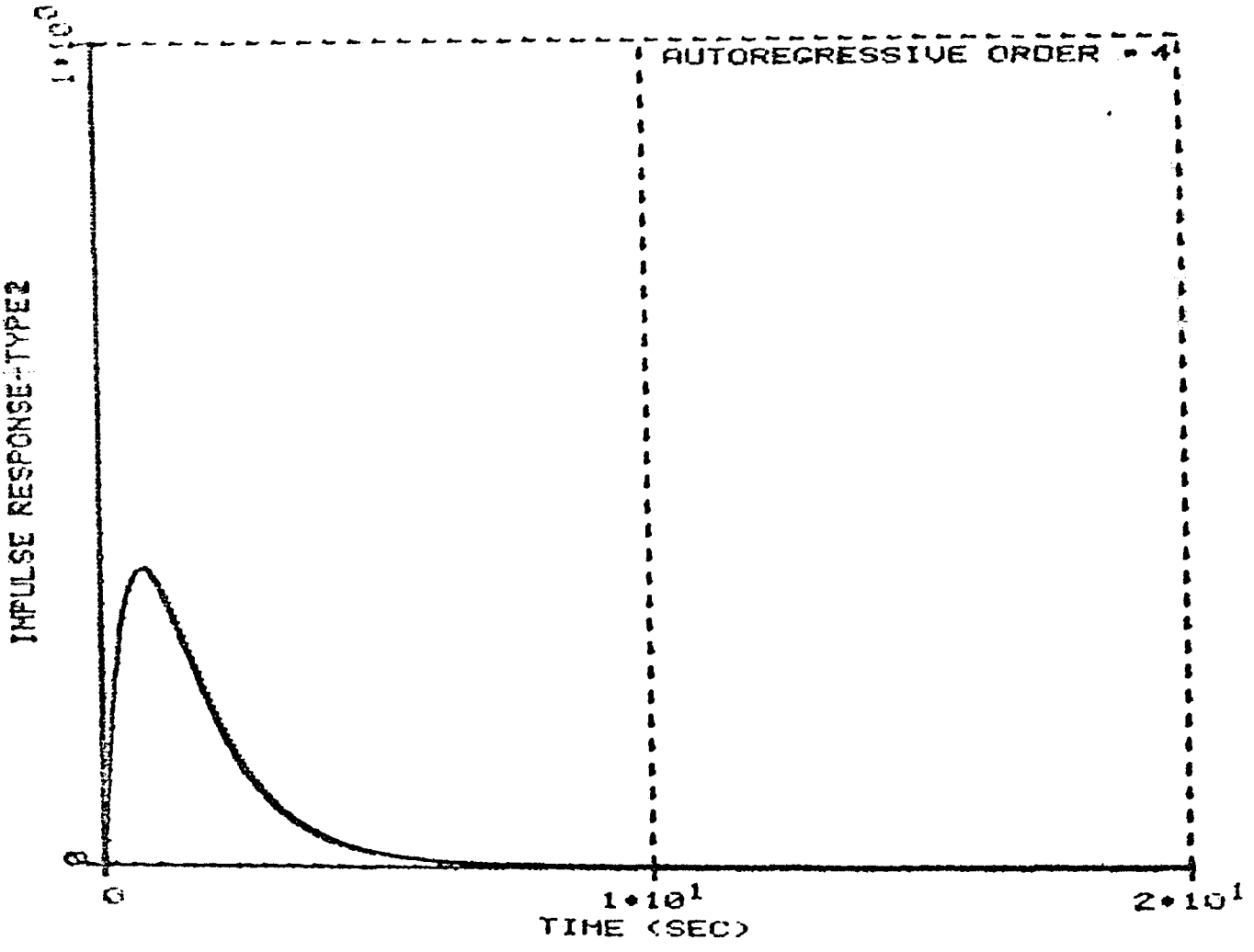


Figure 5.3-4 TYPE 2 Impulse Response from AR (4) Model.

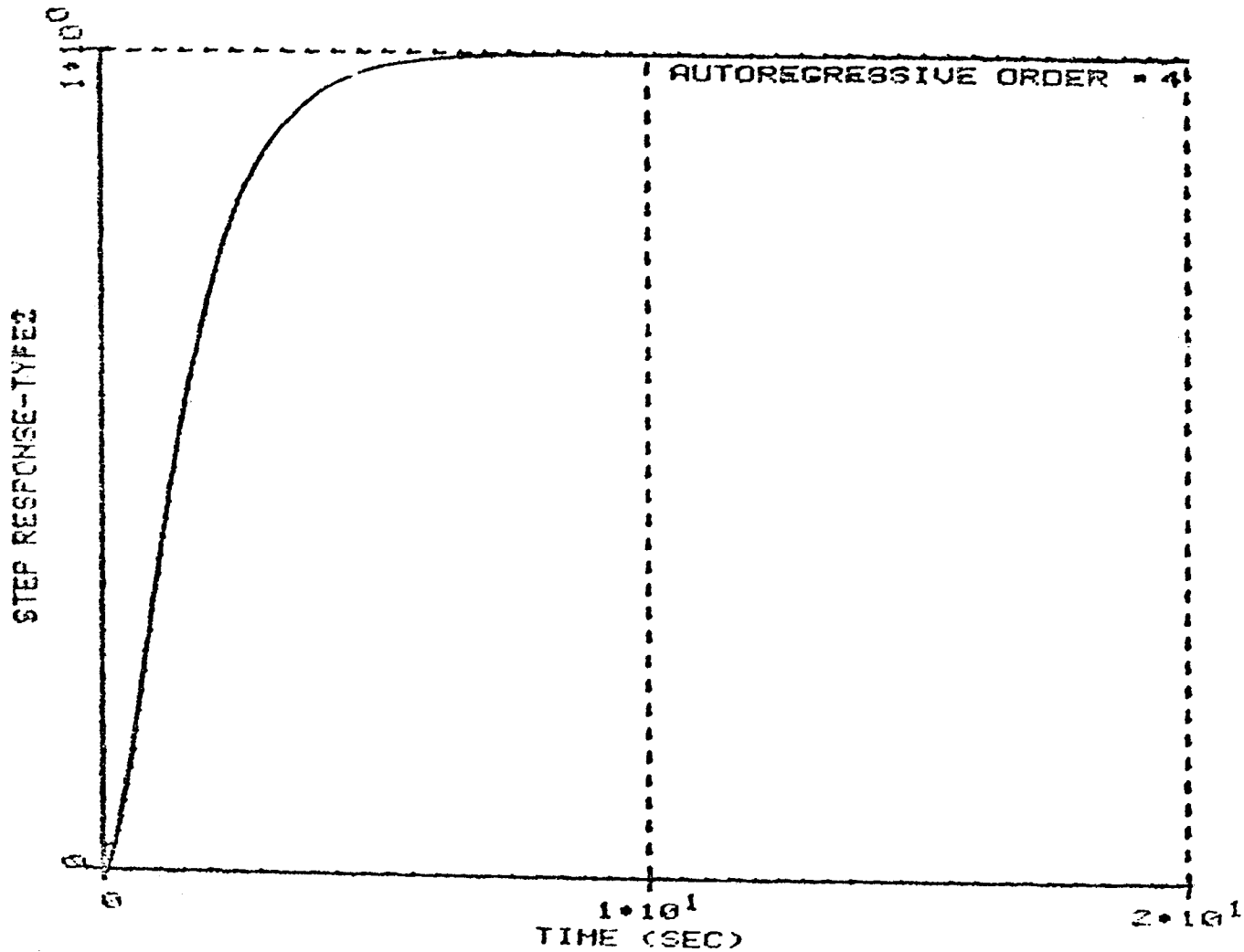


Figure 5.3-5 TYPE 2 Step Response from AR (4) Model.

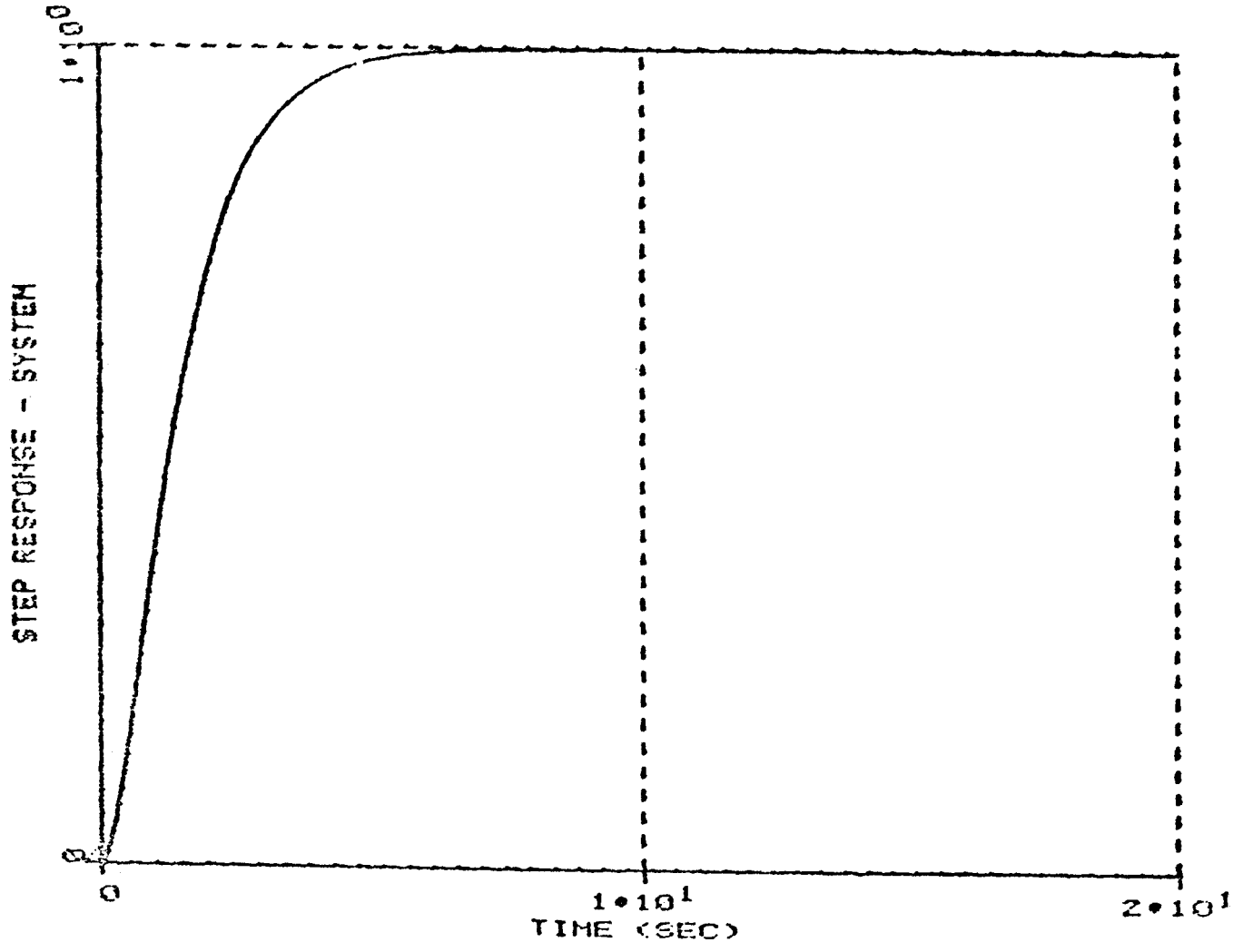


Figure 5.3-6 System Step Response.

The true time constant, $\tau = 1.5848$ sec.

Type I estimate, $\tau_1 = 1.6353$ sec.

Type II estimate, $\tau_2 = 1.6465$ sec.

Note that both estimates essentially give the same value for τ. This example and others show that the impluse response can be successfully obtained by fitting a finite order AR process to the noise data, instead of the large dimensional moving average model. The AR model estimation requires solution to a low order system and the uncertainty of nonlinear optimization as in a moving average case is not present.

TABLE 5.1

In this table are given the values of A_i, i=1,2,...,n as a function of the autoregressive parameters a_i, i=1,2,...,n. $\{A_i\}$ are used to determine the TYPE 2 impulse response from the equation

$$y_I(k) = A_1 y_I(k-1) + \sum_{i=1}^{n-1} A_{i+1} \quad X_i(k-1) \quad , \; n \geq 2$$

$$y_I(k) = A_1 y_I(k-1) \quad , \quad n=1$$

$\underline{n = 1}$

$A_1 = a_1$.

$\underline{n = 2}$

$A_1 = a_1 + a_2$, $A_2 = -a_2$.

$\underline{n = 3}$

$A_1 = a_1 + a_2 + a_3$, $A_2 = -(a_2 + 2a_3)$, $A_3 = a_3$.

$\underline{n = 4}$

$A_1 = a_1 + a_2 + a_3 + a_4$, $A_2 = -(a_2 + 2a_3 + 3a_4)$,

$A_3 = a_3 + 3a_4$, $A_4 = -a_4$,

$\underline{n = 5}$

$A_1 = \sum_{i=1}^{5} a_i$, $A_2 = -\sum_{i=1}^{4} i a_{i+1}$, $A_3 = a_3 + 3a_4 + 6a_5$

$A_4 = -(a_4 + 4a_5)$, $A_5 = a_5$

$\underline{n = 6}$

$A_1 = \sum_{i=1}^{6} a_i$, $A_2 = -\sum_{i=1}^{4} i\, a_{i+1}$, $A_3 = a_3 + 3a_4 + 6a_5 + 10a_6$,

$A_4 = -(a_4 + 4a_5 + 10a_6)$, $A_5 = a_5 + 5a_6$, $A_6 = -a_6$.

n = 7

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
A_1	1	1	1	1	1	1	1
A_2		-1	-2	-3	-4	-5	-6
A_3			1	3	6	10	15
A_4				-1	-4	-10	-20
A_5					1	5	15
A_6						-1	-6
A_7							1

n = 8

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
A_1	1	1	1	1	1	1	1	1
A_2		-1	-2	-3	-4	-5	-6	-7
A_3			1	3	6	10	15	21
A_4				-1	-4	-10	-20	-35
A_5					1	5	15	35
A_6						-1	-6	-21
A_7							1	7
A_8								-1

n = 9

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
A_1	1	1	1	1	1	1	1	1	1
A_2		-1	-2	-3	-4	-5	-6	-7	-8
A_3			1	3	6	10	15	21	28
A_4				-1	-4	-10	-20	-35	-56
A_5					1	5	15	35	70
A_6						-1	-6	-21	-56
A_7							1	7	28
A_8								-1	-8
A_9									1

n = 10

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
A_1	1	1	1	1	1	1	1	1	1	1
A_2		-1	-2	-3	-4	-5	-6	-7	-8	-9
A_3			1	3	6	10	15	21	28	36
A_4				-1	-4	-10	-20	-35	-56	-84
A_5					1	5	15	35	70	126
A_6						-1	-6	-21	-56	-126
A_7							1	7	28	84
A_8								-1	-8	-36
A_9									1	9
A_{10}										-1

n = 11

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
A_1	1	1	1	1	1	1	1	1	1	1	1
A_2		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
A_3			1	3	6	10	15	21	28	36	45
A_4				-1	-4	-10	-20	-35	-56	-84	-120
A_5					1	5	15	35	70	126	210
A_6						-1	-6	-21	-56	-126	-252
A_7							1	7	28	84	210
A_8								-1	-8	-36	-120
A_9									1	9	45
A_{10}										-1	-10
A_{11}											1

n = 12

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
A_1	1	1	1	1	1	1	1	1	1	1	1	1
A_2		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11
A_3			1	3	6	10	15	21	28	36	45	55
A_4				-1	-4	-10	-20	-35	-56	-84	-120	-165
A_5					1	5	15	35	70	126	210	330
A_6						-1	-6	-21	-56	-126	-252	-462
A_7							1	7	28	84	210	462
A_8								-1	-8	-36	-120	-330
A_9									1	9	45	165
A_{10}										-1	-10	-55
A_{11}											1	11
A_{12}												-1

CHAPTER 6

APPLICATION OF NOISE ANALYSIS APPROACH TO RTD RESPONSE TIME MEASUREMENT

Autoregressive modeling is applied to RTD data from four reactors. The results are summarized in table 6.2. The time constants from noise analysis are compared with those available from LCSR tests. Appendix C shows the plots of autocorrelation function, AR power spectrum, sensor step response and residual (white noise) spectrum for the sensors listed in table 6.2. The detailed procedure for estimating and verifying the empirical noise model and determination of sensor parameters is given below. A discussion of the noise results, limitations and applicability for quantitative sensor analysis is presented in section 6.3. Based on the available reactor data it has been concluded that the noise analysis may be used as a tool for monitoring changes in sensor characteristics and not for estimating an accurate value for the time constant.

6.1 Estimation of Optimal AR Model

Figure (6.1) shows a portion of the Millstone 2 hotleg temperature during normal operation at 50% power. Autoregressive models of order up to 12 are fitted using Yule-Walker equations (4.1-11). The data is processed in blocks of N = 2000 and a total sample size = 32000. Sampling time = 0.125 sec. Estimates of parameters $\underline{a}$, noise variance σ_v^2, information criterion of Akaike, the final prediction error (FPE), the F-test index and Bayes Probability Criterion (BPC) are calculated for each model order, n. Table (6.1) is a listing of the output showing model parameters and their standard deviations. We have also shown the Bayes' probability criterion

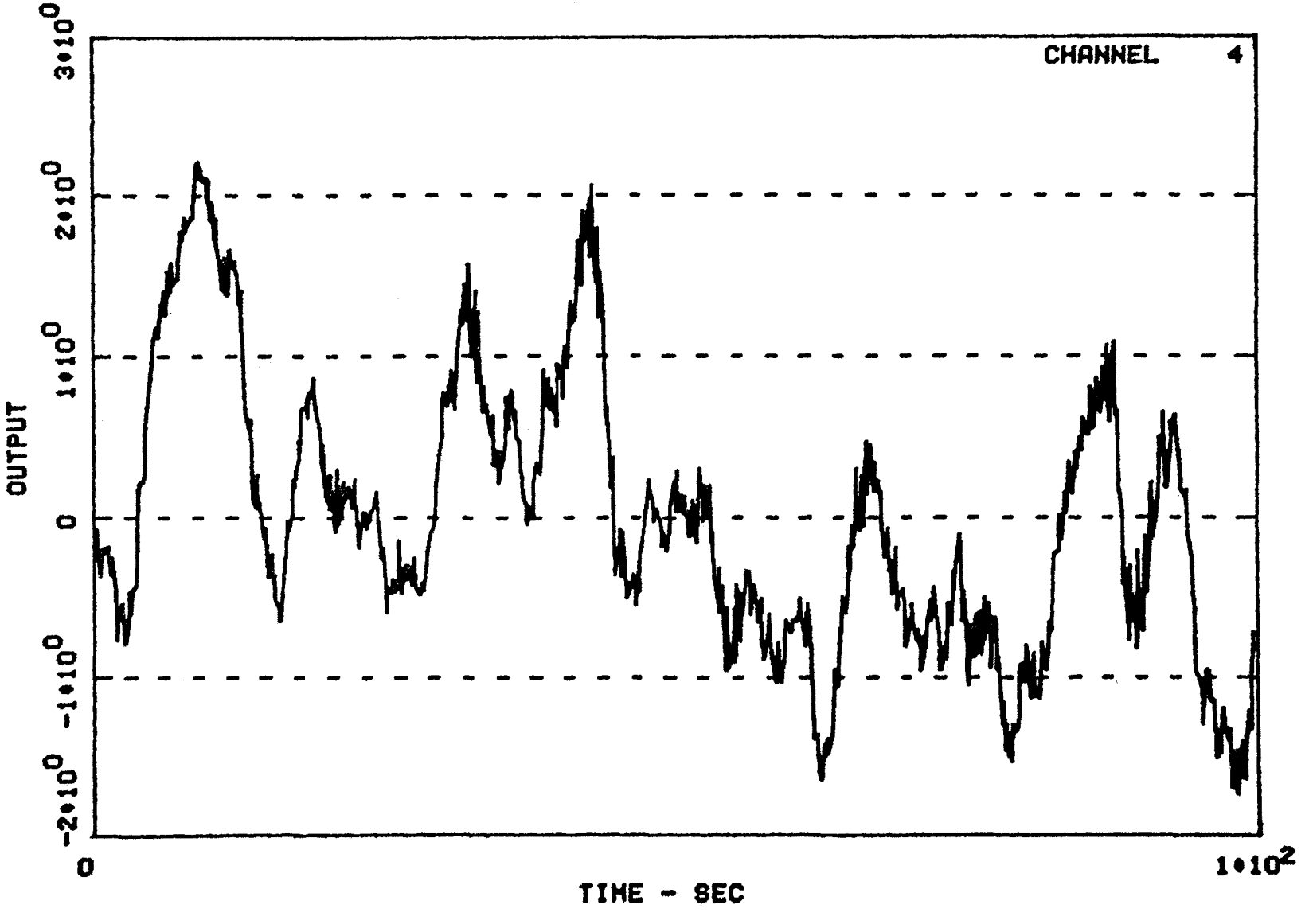


Figure 6.1 A sample of Millstone RTD data.

TABLE 6.1

COMPUTER LISTING OF RESULTS FOR MILLSTONE 2 RTD

A sample computer analysis listing of RTD noise signal evaluation is given below:

The model order is varied for n = 6 to n = 12

Definition of Variables

N = No. of points in each data block.

NSKIP = No. of points skipped at the beginning of data file.

ISKIP = Every ISKIPth point is selected from the data file.

DELTAT = Sampling time (second).

The estimation is based on 16 blocks of data.

INFO CRIT = Akaike Information Criterion; equation (4.3-7).

FPE = Final Prediction Error; equation (4.3-11).

STD DEV1 = Standard deviation of v_k ; equation (4.1-21).

STD DEV2 = Standard deviation of v_k ; equation (4.1-16).

F-TEST INDEX = Hypothesis testing criterion for model order; equation (4.3-18).

BAYES PROBABILITY CRITERION = Criterion for model order; equation (4.3-21).

PHI(I) = ith autoregressive parameter in the equation

$$y_k = \sum_{i=1}^{n} a_i \, y_{k-i} + v_k.$$

EPS(I) = Standard deviation of the ith parameter a_i.

RTD TIME CONSTANT = Obtained for step response which is calculated from (5.2-12).

DELAY IN RAMP RESPONSE = Obtained from ramp response which is calculated from (5.2-13).

PORTMANTEAU TEST INDEX, PORT = Equation (4.4-5) (for whiteness test).

TABLE 6.1

```
     MILLSTONE 2 RTD--MILSTN.DT8--CH 4--16 BLOCKS OF N=2000


     NOISE ANALYSIS OF RTD DATA : N = 2000
     NSKIP =  512  ISKIP =    1  DELTAT (SEC) = 0.12500E+00
     NO. OF BLOCKS TO AVERAGE =  16


**************************************************
     MEAN OF BLOCK  1 = -0.15556E+01
     MEAN OF BLOCK  2 = -0.13772E+01
     MEAN OF BLOCK  3 = -0.15350E+01
     MEAN OF BLOCK  4 = -0.22332E+01
     MEAN OF BLOCK  5 = -0.20990E+01
     MEAN OF BLOCK  6 = -0.23858E+01
     MEAN OF BLOCK  7 = -0.21155E+01
     MEAN OF BLOCK  8 = -0.21671E+01
     MEAN OF BLOCK  9 = -0.15587E+01
     MEAN OF BLOCK 10 = -0.40639E+00
     MEAN OF BLOCK 11 = -0.60427E+00
     MEAN OF BLOCK 12 = -0.13062E-01
     MEAN OF BLOCK 13 = -0.26416E+00
     MEAN OF BLOCK 14 = -0.56598E+00
     MEAN OF BLOCK 15 = -0.93297E+00
     MEAN OF BLOCK 16 = -0.60872E+00
**************************************************

     AUTOREGRESSIVE ORDER =  6
     INFO CRIT =-0.12044988E+06  FPE = 0.23027E-01
     STD DEV1 = 0.15139E+00  STD DEV2 = 0.15172E+00
     F - TEST INDEX = 0.00000E+00
     BAYES PROBABILITY CRITERION = 0.12030374E+06

         PHI(I)         EPS(I)

      0.11787E+01    0.54842E-02
     -0.60383E+00    0.84699E-02
      0.69818E+00    0.89302E-02
     -0.32748E+00    0.89302E-02
      0.22491E+00    0.84699E-02
     -0.18325E+00    0.54842E-02


     RTD TIME CONSTANT =  0.77361E+01 SEC


**************************************************
```

```
AUTOREGRESSIVE ORDER =  7
INFO CRIT =-0.12045824E+06  FPE = 0.22976E-01
STD DEV1 = 0.15122E+00  STD DEV2 = 0.15154E+00
F - TEST INDEX = 0.70848E+02
BAYES PROBABILITY CRITERION = 0.12028798E+06

     PHI(I)         EPS(I)

 0.11695E+01    0.55725E-02
-0.59261E+00    0.85528E-02
 0.68182E+00    0.91050E-02
-0.29260E+00    0.97318E-02
 0.19475E+00    0.91051E-02
-0.12437E+00    0.85528E-02
-0.49946E-01    0.55725E-02

RTD TIME CONSTANT =  0.69863E+01 SEC
```

****

```
AUTOREGRESSIVE ORDER =  8
INFO CRIT =-0.12082015E+06  FPE = 0.22748E-01
STD DEV1 = 0.15022E+00  STD DEV2 = 0.15079E+00
F - TEST INDEX = 0.42708E+03
BAYES PROBABILITY CRITERION = 0.12062716E+06

     PHI(I)         EPS(I)

 0.11639E+01    0.55425E-02
-0.60648E+00    0.85240E-02
 0.70356E+00    0.91088E-02
-0.32528E+00    0.98023E-02
 0.27084E+00    0.98023E-02
-0.19053E+00    0.91088E-02
 0.80555E-01    0.85240E-02
-0.11161E+00    0.55425E-02

RTD TIME CONSTANT =  0.55399E+01 SEC
```

****

```
AUTOREGRESSIVE ORDER =  9
INFO CRIT =-0.12082763E+06  FPE = 0.22635E-01
STD DEV1 = 0.15006E+00  STD DEV2 = 0.15041E+00
F - TEST INDEX = 0.70201E+02
BAYES PROBABILITY CRITERION = 0.12061082E+06

    PHI(I)         EPS(I)

 0.11587E+01    0.55712E-02
-0.60270E+00    0.85264E-02
 0.69462E+00    0.91605E-02
-0.31255E+00    0.99071E-02
 0.25560E+00    0.99578E-02
-0.15757E+00    0.99071E-02
 0.52168E-01    0.91605E-02
-0.57129E-01    0.85264E-02
-0.46783E-01    0.55712E-02

RTD TIME CONSTANT =  0.50787E+01 SEC
```

****

```
AUTOREGRESSIVE ORDER = 10
INFO CRIT =-0.12086235E+06  FPE = 0.22597E-01
STD DEV1 = 0.14983E+00  STD DEV2 = 0.15028E+00
F - TEST INDEX = 0.97532E+02
BAYES PROBABILITY CRITERION = 0.12062191E+06

    PHI(I)         EPS(I)

 0.11562E+01    0.55687E-02
-0.60582E+00    0.85193E-02
 0.69749E+00    0.91511E-02
-0.32119E+00    0.99308E-02
 0.26957E+00    0.10044E-01
-0.17467E+00    0.10044E-01
 0.90097E-01    0.99308E-02
-0.90046E-01    0.91511E-02
 0.16472E-01    0.85193E-02
-0.54577E-01    0.55687E-02

RTD TIME CONSTANT =  0.45927E+01 SEC
```

****

```
AUTOREGRESSIVE ORDER = 11
INFO CRIT =-0.12079948E+06  FPE = 0.22566E-01
STD DEV1 = 0.14983E+00  STD DEV2 = 0.15017E+00
F - TEST INDEX =-0.11879E+00
BAYES PROBABILITY CRITERION = 0.12053500E+06

    PHI(I)         EPS(I)

 0.11553E+01    0.55771E-02
-0.60554E+00    0.85198E-02
 0.69598E+00    0.91649E-02
-0.31967E+00    0.99435E-02
 0.26663E+00    0.10091E-01
-0.17012E+00    0.10156E-01
 0.84719E-01    0.10091E-01
-0.78323E-01    0.99435E-02
 0.62637E-02    0.91648E-02
-0.35168E-01    0.85198E-02
-0.16792E-01    0.55770E-02

RTD TIME CONSTANT =  0.44701E+01 SEC
```

****

```
AUTOREGRESSIVE ORDER = 12
INFO CRIT =-0.12075052E+06  FPE = 0.22640E-01
STD DEV1 = 0.14979E+00  STD DEV2 = 0.15041E+00
F - TEST INDEX = 0.13783E+02
BAYES PROBABILITY CRITERION = 0.12046208E+06

    PHI(I)         EPS(I)

 0.11552E+01    0.55766E-02
-0.60573E+00    0.85202E-02
 0.69598E+00    0.91629E-02
-0.32006E+00    0.99510E-02
 0.26705E+00    0.10100E-01
-0.17102E+00    0.10198E-01
 0.86108E-01    0.10198E-01
-0.80021E-01    0.10100E-01
 0.99583E-02    0.99509E-02
-0.38374E-01    0.91629E-02
-0.10681E-01    0.85202E-02
-0.53024E-02    0.55766E-02

RTD TIME CONSTANT =  0.44140E+01 SEC
```

****

RAMP RESPONSE FOR AR ORDER = 10
DELAY IN RAMP RESPONSE = 0.43988E+01 SEC

****

PORTMANTEAU TEST INDEX FOR NDEG = 60
PORT = 0.67064E+02
99% CONFIDENCE LEVEL FOR RESIDUAL AUTOCORRELATION = 0.40737E-01

RESIDUAL AUTOCORRELATION BASED ON N = 4000

0.37623E-01	-0.72310E-02	0.23475E-01	-0.67929E-02	-0.23210E-02
-0.21279E-01	0.11021E-01	0.18612E-01	-0.20291E-02	-0.12740E-01
0.42026E-03	-0.13483E-01	-0.55146E-02	0.25894E-01	-0.25492E-01
0.31263E-01	-0.26491E-02	-0.52537E-02	0.15740E-01	-0.25358E-01
-0.41141E-02	-0.37438E-02	-0.13102E-01	0.38132E-02	0.29543E-02
-0.16818E-01	0.21862E-01	-0.71339E-02	-0.45102E-02	-0.13508E-01
-0.14016E-02	-0.17417E-01	0.28137E-01	0.46544E-02	-0.14584E-01
0.18179E-01	-0.31852E-01	0.76785E-02	-0.21804E-01	0.39708E-01
-0.99162E-02	0.61767E-02	0.13765E-01	-0.14974E-01	0.12477E-01
0.18343E-01	0.11861E-01	0.25518E-01	-0.93206E-02	-0.11053E-01
-0.53864E-02	0.15651E-01	-0.14731E-02	0.10945E-01	-0.18793E-01
0.77100E-02	0.13687E-01	-0.59785E-02	0.29022E-01	-0.17673E-01

which is given by [K4]

$$BPC = -N \ln \sigma_v^2 - n \ln \left(\frac{\sigma_y^2}{\sigma_v^2}\right) - n \ln N + n \qquad (6.1\text{-}1)$$

where

N = no. of data points

n = model order

σ_v^2 = variance of noise process v_k

σ_y^2 = variance of signal y_k

The model order is chosen such that BPC is a maximum. From the computer listing the optimal order is n = 10. We will not choose the model order based on F-test to avoid the ambiguity of choice.

The AR(10) model

$$y_k = \sum_{i=1}^{10} a_i \, y_{k-i} + v_k \qquad (6.1\text{-}2)$$

has the following set of parameters

$a_1 = 1.1562$

$a_2 = -0.6058$

$a_3 = 0.6975$

$a_4 = -0.3212$

$a_5 = 0.2696$

$a_6 = -0.1747$

$a_7 = 0.0901$

$a_8 = -0.09005$

$a_9 = 0.0165$

$a_{10} = -0.0546$

Figure (6.2) shows the computed autocorrelation function and (6.3) is the power spectrum computed from the AR (10) model. The break frequency obtained as the intersection between low frequency and high frequency asymptotes is 0.05 Hertz. The power spectrum from direct Fourier transform is shown in figure (6.4). Notice that both curves give the same value for the break frequency. Thus, the time series model provides an alternative method of estimating the power spectrum. Frequency response testing in nuclear reactors is described in reference [K3].

The impulse and step responses by Type 1 method derived from the AR model are shown in figures (6.5) and (6.6). The estimated time constant is

$$\tau = 4.59 \text{ sec.}$$

6.2 Model Validation

The important assumption made in the analysis is that the driving function is a white noise sequence. We have described three tests in section (4.4). If $\{v_k\}$ is uncorrelated then the autocorrelation function must be an impulse function. Figure (6.9) is a plot of the residual autocorrelation function, $C_k(\hat{v})$ where

$$\hat{v}_k = y_k - \sum_{i=1}^{n} \hat{a}_i y_{k-i} \qquad (6.2\text{-}1)$$

The normalized values of correlation function are shown. The variation in this correlation function must be within $\pm \frac{1.96}{\sqrt{N}} = 0.031$ for $N = 4000$. For small lags this bound underestimates the correlation values. But for lags greater than, say, the model order the 95% level is satisfied. The "Portmanteau" lack of fit test is applied by calculating the index

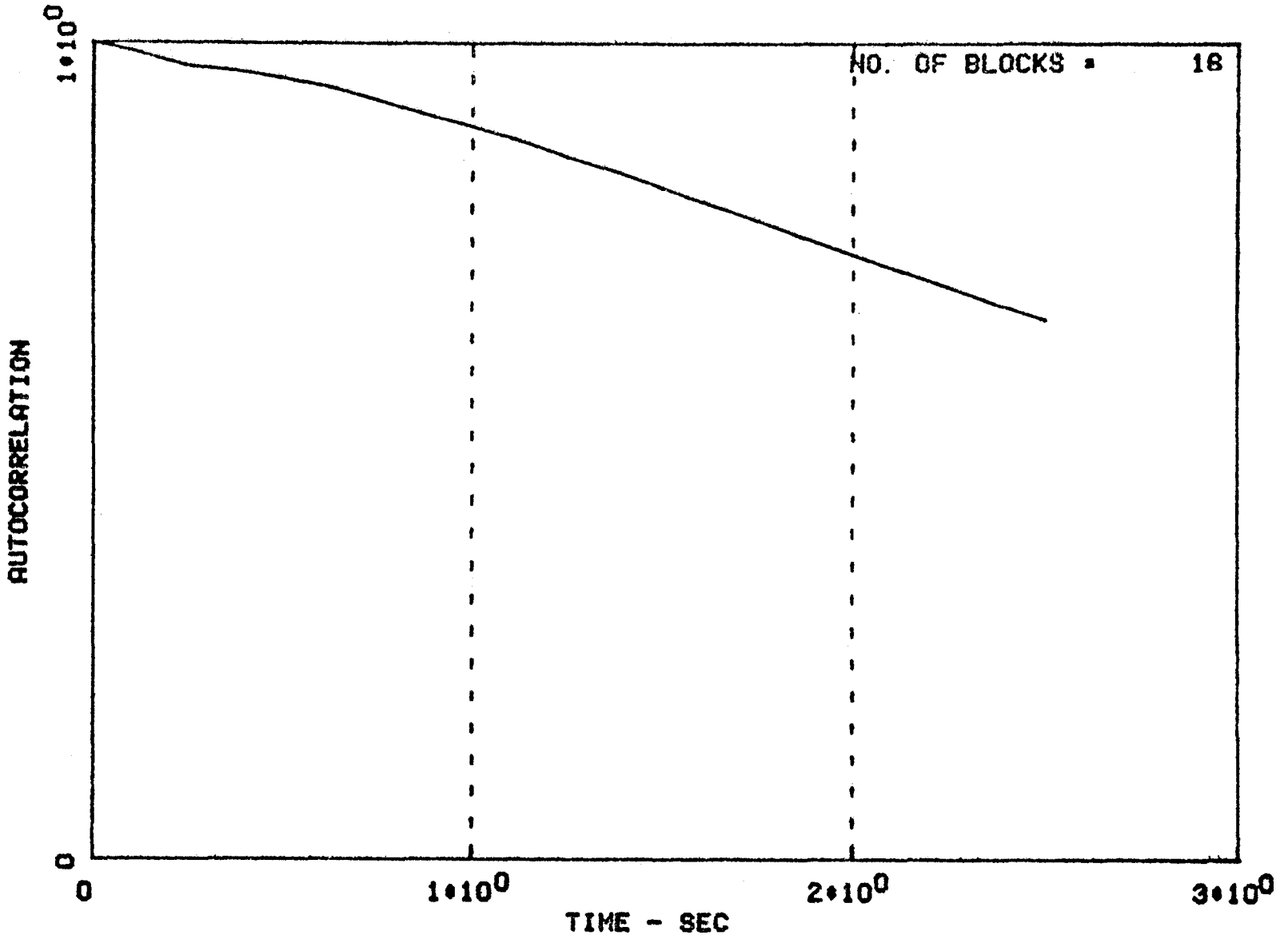


Figure 6.2 Autocorrelation Function of RTD data (Millstone).

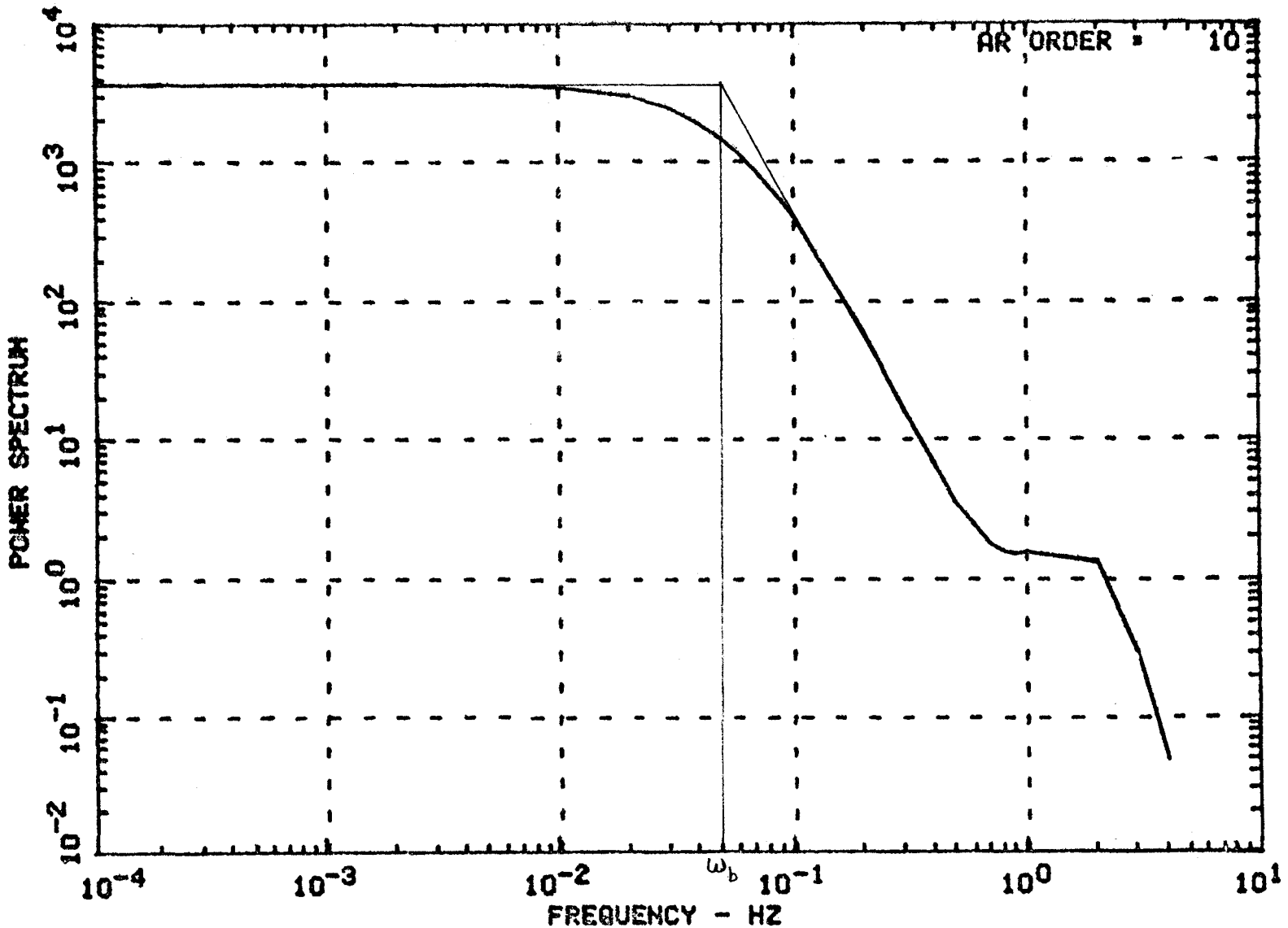


Figure 6.3 Autoregressive Power Spectrum (Millstone).

MILLSTONE 2-HOTLEG TEMPERATURE SENSOR

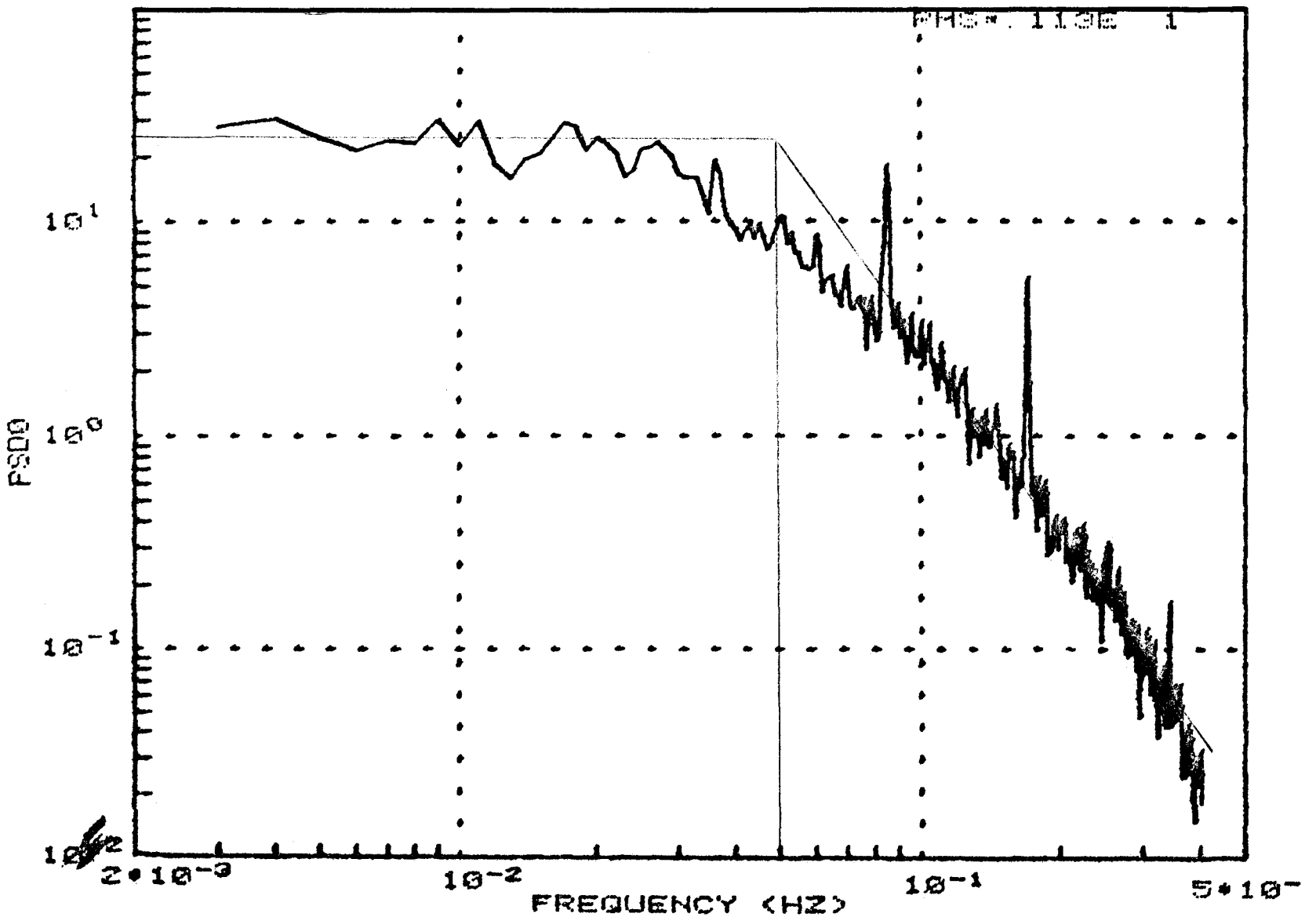


Figure 6.4 Power Spectrum Using FFT Analysis (Millstone).

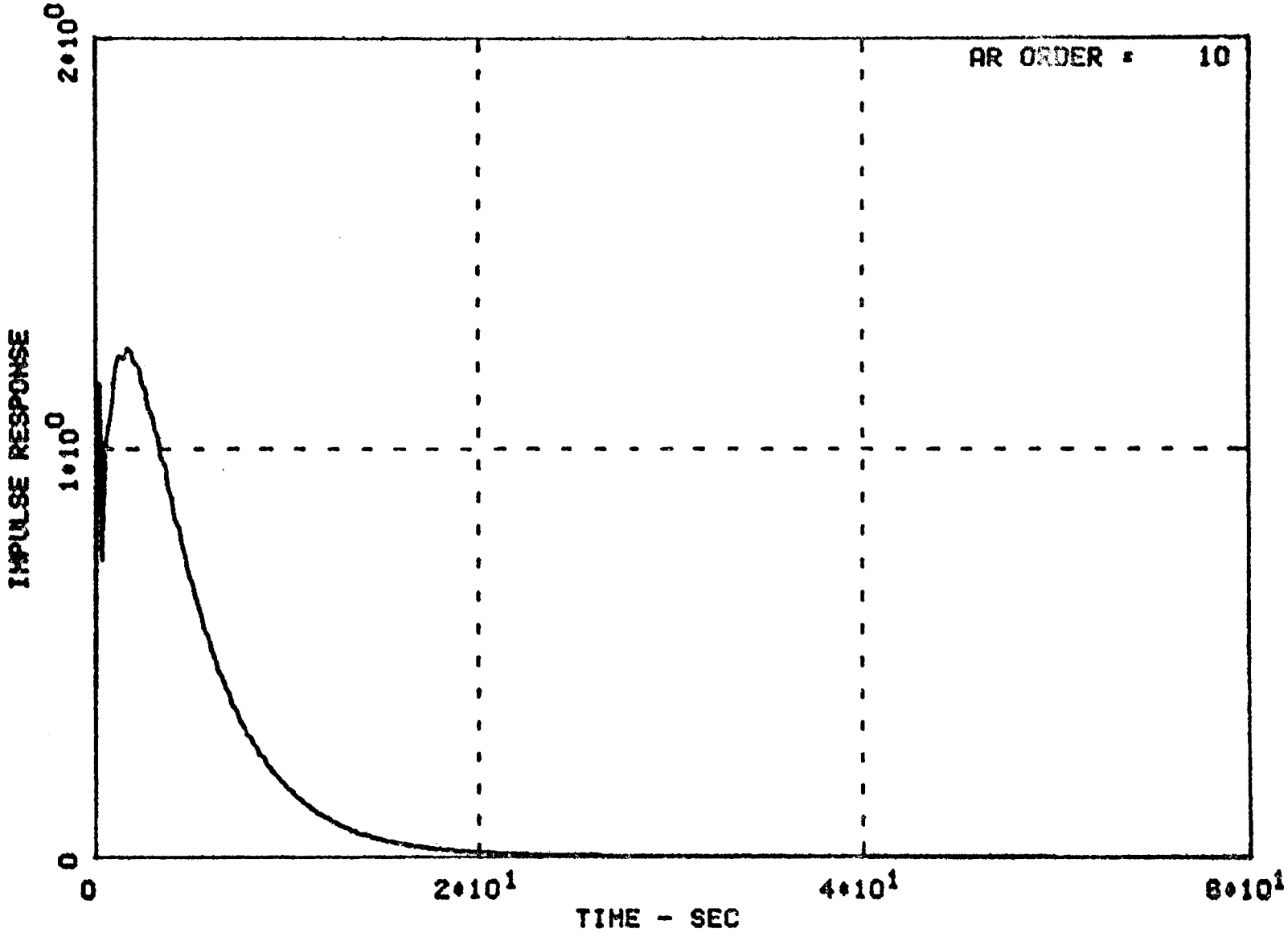


Figure 6.5 Impulse Response from AR Model (Millstone).

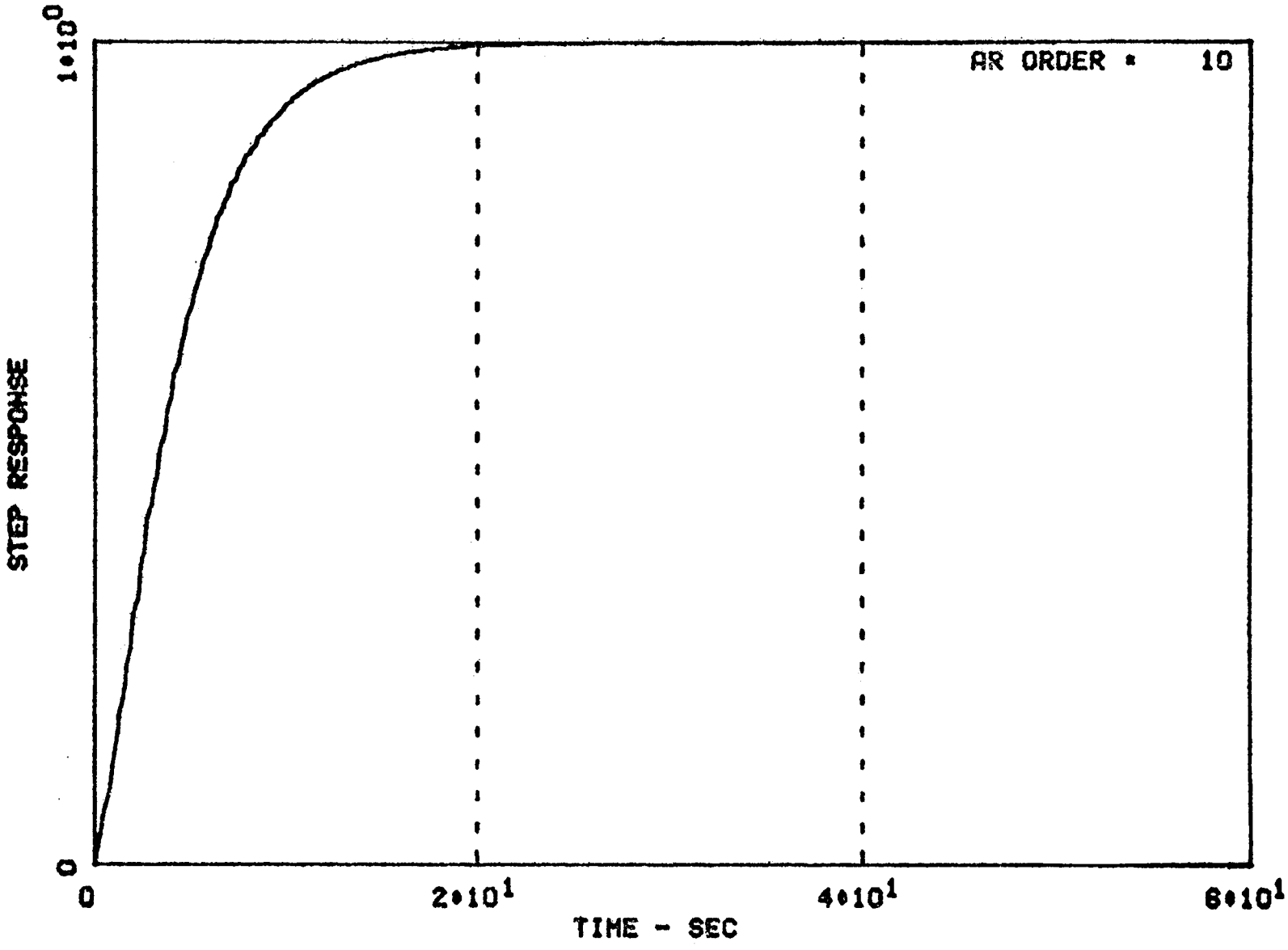


Figure 6.6 Step Response from AR Model (Millstone).

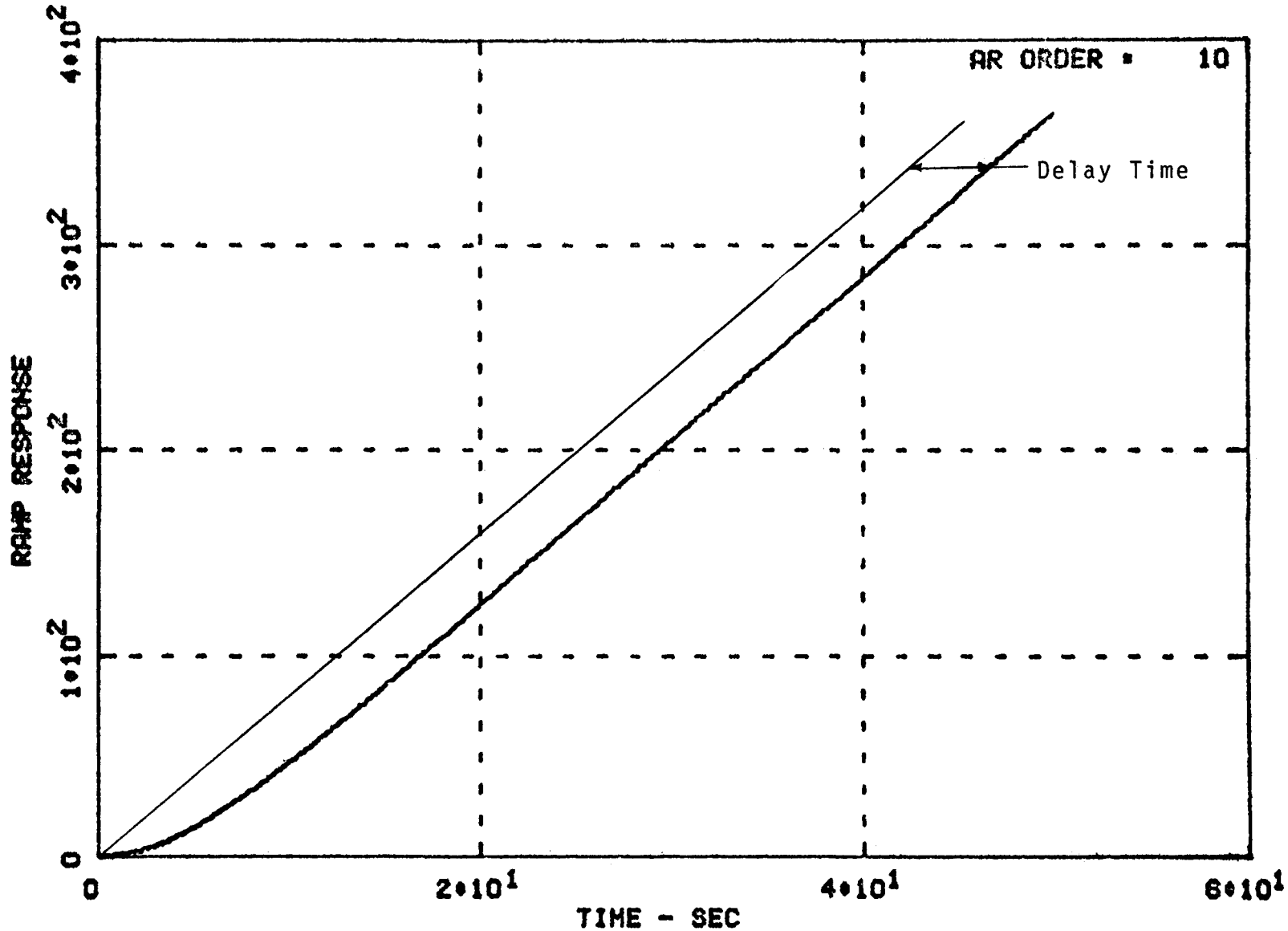


Figure 6.7 RTD Response to a Ramp Input (Millstone).

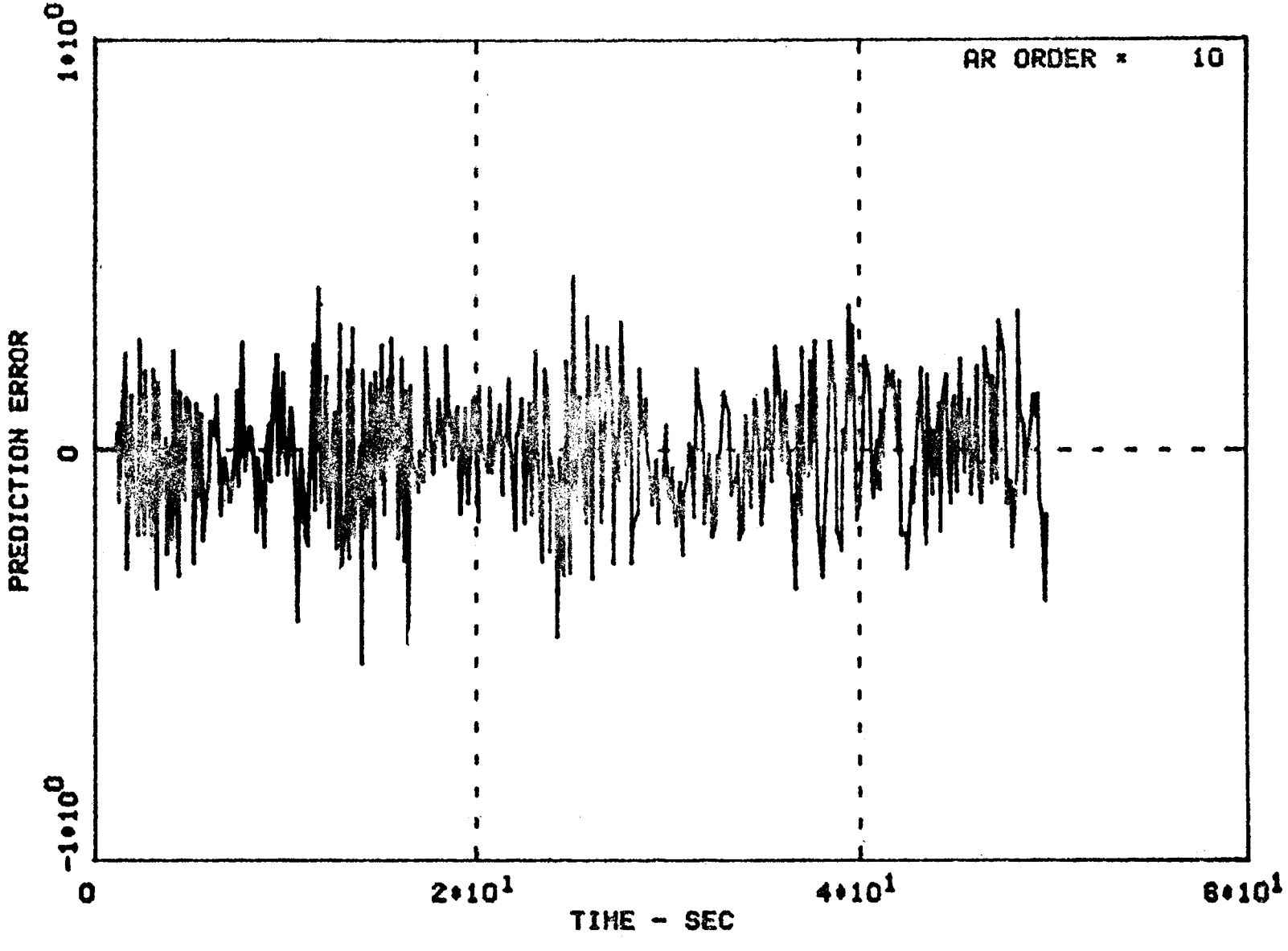


Figure 6.8 Residual Noise Sequence (Millstone).

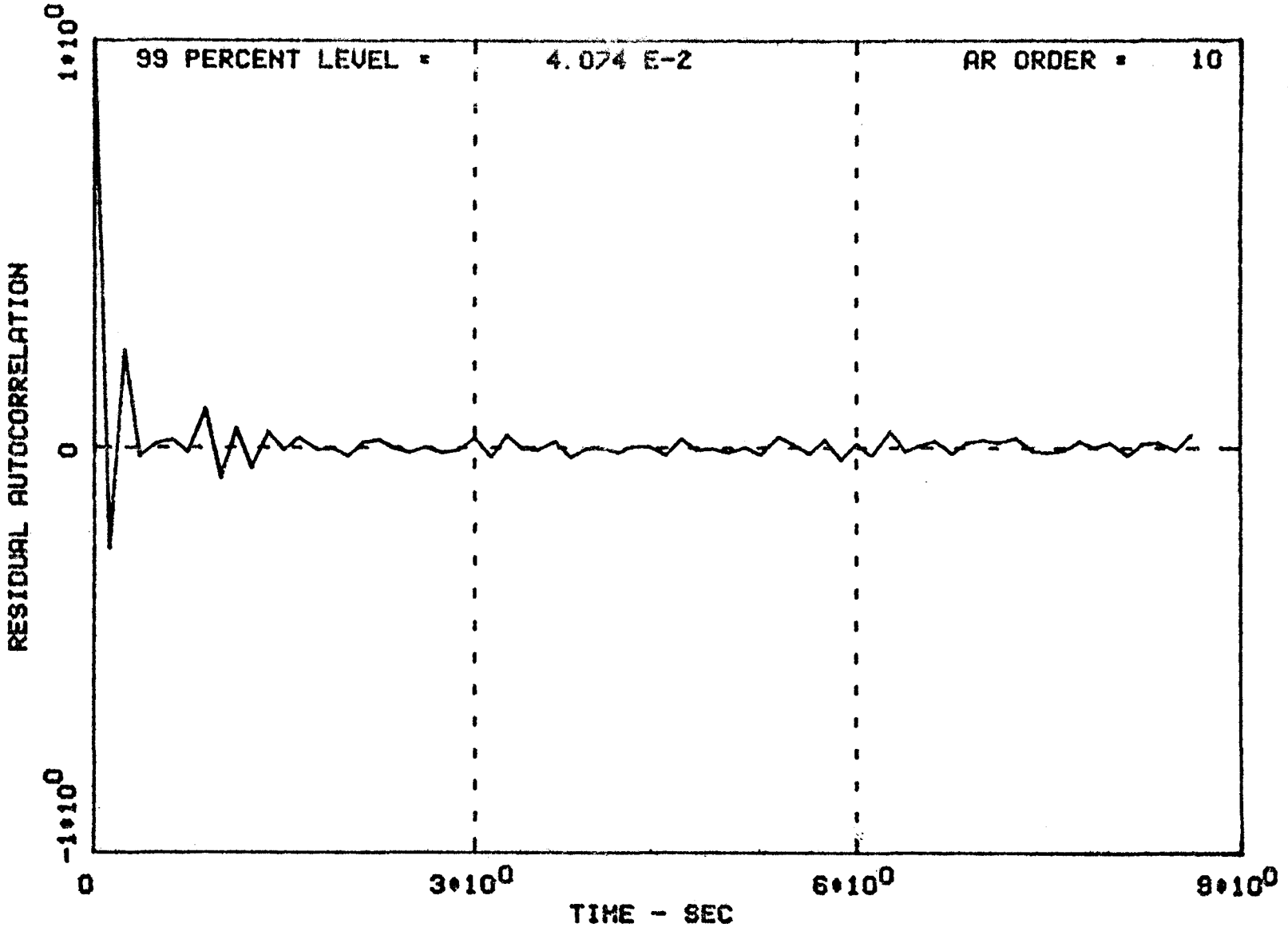


Figure 6.9 Residual Correlation Function (Millstone).

$$Q = N \sum_{k=1}^{M} \rho_k^2 (\hat{v}) \qquad (6.2\text{-}2)$$

If the assumption is satisfied then Q is distributed as a χ^2 function with M degrees of freedom, with N = 4000, M = 60, Q = 67.06. From table (4A-2) for degrees of freedom M = 60 and a 95% significance level

$$\chi^2_{.95} = 79.08$$

Thsu, Q(69.06) $<\chi^2_{.95}$ (=79.08). The final check is performed by the graphical plotting of the spectrum of the residual sequence. This is shown in figure (6.10). The flatness of the spectrum is a further evidence of model adequacy.

6.3 Discussion

Table (6.2) is a summary of RTD noise test results from four PWR's -- Millstone 2, St. Lucie, Oconee 1, and Turkey Point. The model order, time constant and ramp delay time are listed for each sensor. Note that in several cases the method failed to construct an autoregressive model. These cases are indicated as "failed". The following observations are made from reactor test results.

A. Failure of the Noise Model to fit the Data:

In several cases the AR modeling strategy was unable to predict a noise model. These were encountered in attempting to analyze noise data from coldleg sensors. A Fourier analysis of such data showed that either the power spectrum of the signal showed peaks at several frequencies as in figure (6.11) or, the spectrum had no break throughout the low and high frequency range as in figure (6.12). The log-log plot shows a linear trend

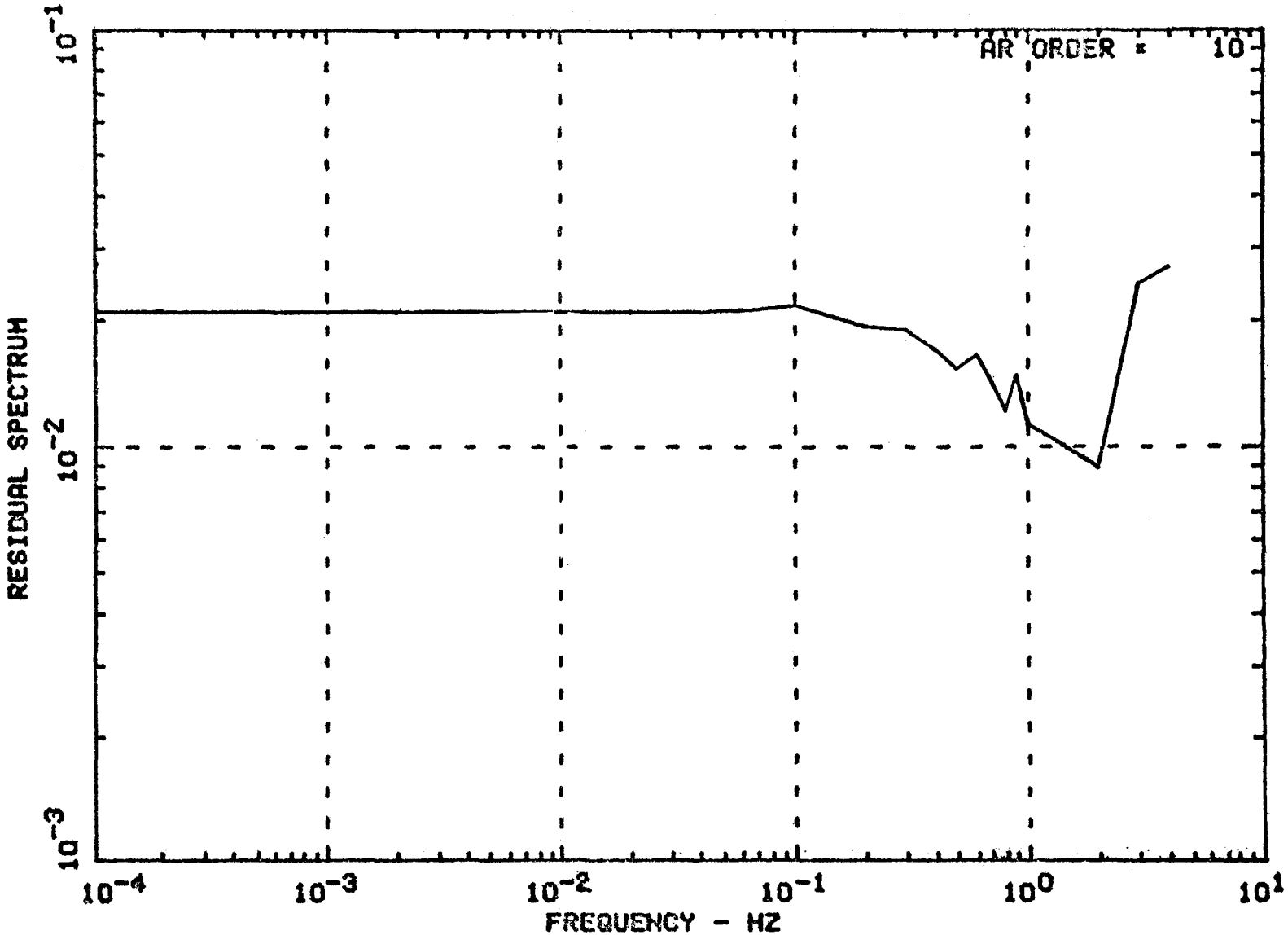


Figure 6.10 Residual Power Spectrum (Millstone).

TABLE 6.2

Summary of RTD Noise Tests

	Data File Name	Model Order	Time Constant (second)	Delay Time (second)
Millstone 2				
Sensor 1	MILSTN. DT8	10	4.59	4.40
Sensor 2		Failed	----	----
Sensor 3		Failed	----	----
St. Lucie				
Sensor 1	STLUCE. T63	6	4.84	4.81
Sensor 2		6	8.23	8.14
Sensor 3		Failed	----	----
Sensor 1	STLUCE. T07	8	6.90	6.76
Sensor 2		10	8.49	8.37
Sensor 3		Failed	----	----
Oconee 1				
Sensor 1	OCONEE. T55	4	6.77	6.74
Sensor 2		4	3.61	3.53
Sensor 3		Failed	----	----
Turkey Point	TURKEY .T35			
Sensor 1		7	0.382	0.353
Sensor 2		10	0.389	0.347

Number of data samples used = 32000.

Sampling time for all data except TURKEY. T35 is 0.125 sec.

Sampling time for TURKEY. T35 is 0.02 sec.

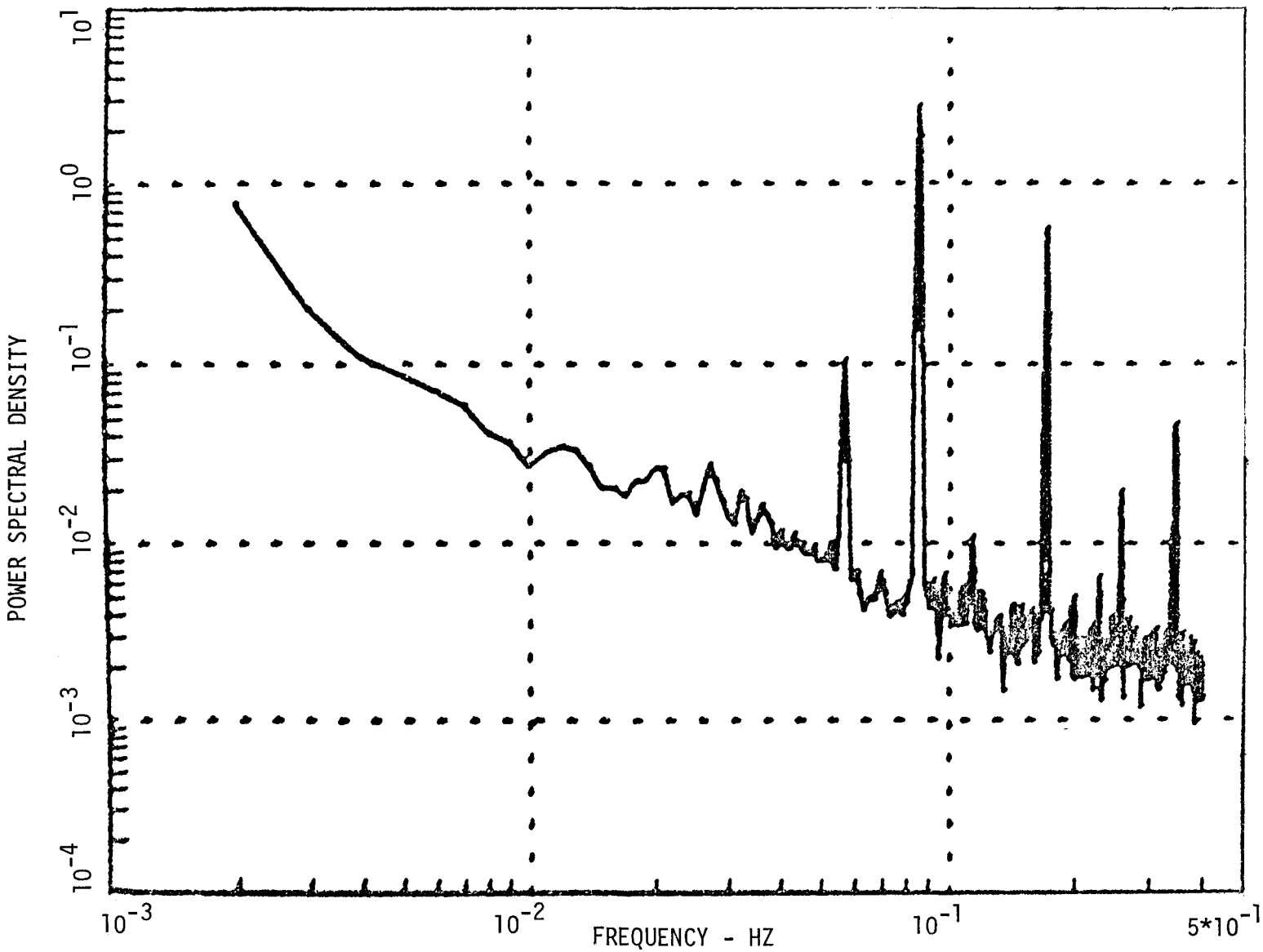


Figure 6.11 Millstone 2 RTD -- Power Spectrum of Coldleg Temperature Noise Signal (MILSTN.DT8, Sensor 2).

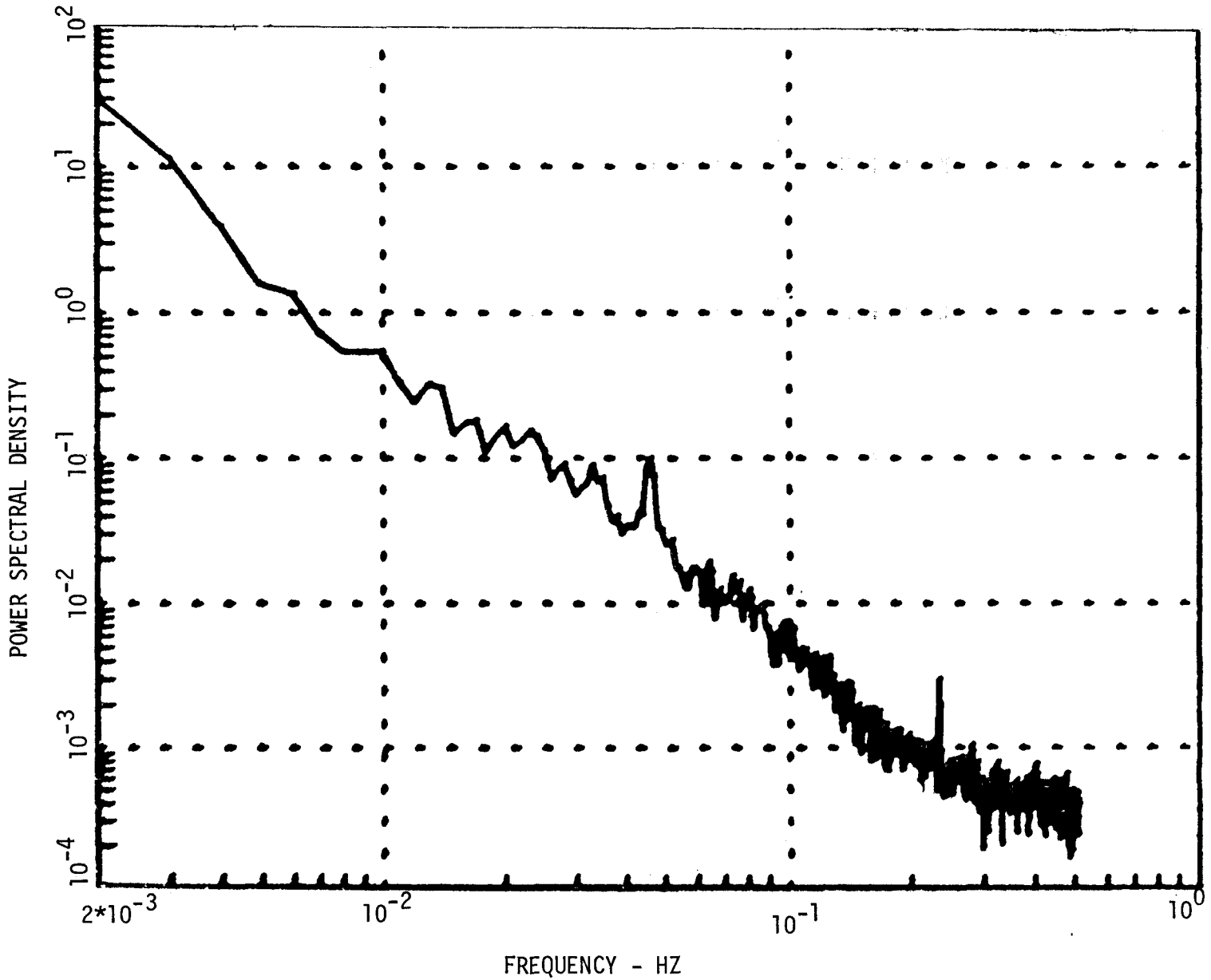


Figure 6.12 St. Lucie RTD -- Power Spectrum of Coldleg Temperature Noise Signal (STLUCE.T63, Sensor 3).

in the noise spectra. Such behavior in the spectrum does not satisfy the assumptions made in the analysis and hence results in a failure.

B. Variation in Time Constants

It is evident from the results of table 6.2 that for St. Lucie and Oconee 1 there is wide variation in time constant among the sensors. It is observed that the time constants for the hotleg sensors are smaller than those for the coldleg sensors. There is no explanation available for this at this time. It may be possible that the coldleg temperature variation has a smaller bandwidth compared to the hotleg temperature noise.

It has been further observed that the time constant estimates for both of the Turkey Point sensors have practically the same value (see table 6.2). This consistent estimate shows that the noise properties of the temperature signal are similar in the hotleg and coldleg. The sensors are installed in a bypass loop thus causing uniformity in the noise properties.

C. Comparison of Noise Analysis and LCSR Test

Table 6.3 gives a comparison between the noise analysis and LCSR results for the available common data. From the St. Lucie test it is seen that the noise analysis, in general, gives a higher value for the time constant. This conservative estimate is probably due to bandlimited noise. If the bandwidth of driving noise is close to or smaller than the sensor bandwidth, this causes a distortion in the output signal spectrum, in effect moving the break frequency towards a lower value.

TABLE 6.3

Comparison of Noise Analysis and LCSR Results RTD Tests at St. Lucie

Plant ID No.	Time Constant from Noise Analysis (second)	Time Constant from LCSR Test (second)
1111Y	8.23	3.52
1121X	6.90	4.94
1121Y	8.49	5.80

Figures C-6 and C-10 are AR power spectra for two St. Lucie sensors with time constants estimated to be 4.84 sec. and 8.23 sec. The sensor with a smaller time constant has a larger break frequency compared to the break frequency of the higher time constant sensor. Thus the signal bandwidth is a limiting factor in estimating an accurate value of the time constant.

Such limitations are not prevalent in an LCSR test and hence the quantitative estimate is closer to the actual value.

From the above observations it has been concluded that the noise analysis method can be used as a tool for monitoring changes in the sensor characteristics. The application of noise analysis for quantitative time constant evaluation is invalid in some cases because essential conditions for validity of the method are not satisfied in the operating plant.

CHAPTER 7

SUMMARY AND CONCLUDING REMARKS

In the previous chapters we have described the use and application of time series models to normal fluctuating signals from temperature sensors. The application is not just limited to these noise signals. Any signal that has a random variation characterized by uncorrelated noise may be modeled by AR processes. For systems when this is not satisfied a more generalized model must be constructed.

7.1 Summary

In chapter 2 a brief description of elements of probability theory is given. The topics are central to the discussion of this report. Chapter 3 provides an overview of time series models, generally called the auto-regressive moving-average (ARMA) process. The model can also be used to estimate the spectral density of the signal.

The important topics of model estimation for AR processes, optimal model order determination and validation of fitted model are discussed in chapter 4. It is suggested that the optimal model be determined using the Bayes' maximum a posteriori probability (BPC) criterion. The validation of the assumption of white driving noise is carried out by computing the residual autocorrelation and applying proper tests.

Estimation of sensor characteristics such as impulse, step and ramp responses is described in chapter 5. From these the sensor time constant and ramp delay time are derived. The entire procedure is numerical in nature without involving any geometric approximation. Chapter 6 describes in detail the procedure for one RTD. Results from tests at Millstone,

St. Lucie, Oconee and Turkey Point reactors are summarized in chapter 6 and appendix C.

The AR parameter estimation requires matrix inversion. As the model order increases, the inversion of a large matrix may be necessary. To avoid this procedure, a recursive method of estimating successively the higher order parameter set as a function of lower order parameters is given in appendix A.

7.2 Concluding Remarks

Throughout this work the general structure of the noise model is fixed. The autoregressive model requires that the input noise spectrum be flat in the bandwidth of interest. In general this assumption often is not satisfied. This causes variation in the information contained in the output signal. The results have shown that the time constants obtained from the noise analysis are consistently on the higher side, or in some cases, the analysis simply fails and provides no results at all. In some instances the results are close to the LCSR test and in other cases they vary widely. The noise analysis time constant for the fast sensors (Turkey Point) shows less variation between the hotleg and coldleg sensors.

It is noticed that the noise power spectrum at low frequencies is often not flat. This apparent behavior in the spectrum is better predicted by fitting a generalized ARMA model which makes no restricted assumptions about the noise statistic. This procedure has been applied to sensors for which larger time constants were predicted by the AR modeling as shown in table 6.2. It is reported [U3] that this modification improves the time constant estimates by 25 to 40 percent over the AR modeling.

Discussion of this procedure is beyond the scope of this report.

Since the noise signal measurement is influenced by many factors, such as - location of sensor, extraneous noise interference, property of randomness in temperature fluctuation - the quantitative estimate of the time constant using autoregressive noise modeling is subject to error. The approach may be used as a means for sensor monitoring. During the life of the sensor if variation in time constant is noticed, this may be caused either due to degradation in sensor performance or because of changes in noise characteristics. At this point an LCSR test may be performed to check for the time constant itself. Based on the above observations we have to conclude that the AR modeling can be used with confidence only as a means of monitoring changes in sensor characteristics.

REFERENCES

[A1] T. W. Anderson, An Introduction to Multivariate Statistical Analysis, John Wiley & Sons, Inc., New York, 1958

[A2] H. Akaike, "A New Look at the Statistical Model Identification", IEEE Tran. Aut. Cont., Vol. AC-19, pp 716-723, 1974.

[A3] Special Issue: System Identification and Time Series Analysis, IEEE Tran. Aut. Cont., Dec. 1974.

[A4] H. Akaike, "A Method of Statistical Identification of Discrete Time Parameter Linear Systems," Ann. Inst. Stat. Math (Tokyo), Vol. 21, pp 225-242, 1969

[A5] H. Akaike, "Fitting Autoregressive Model for Prediction," Ann. Inst. Stat. Math (Tokyo), Vol. 21, pp 243-247, 1969.

[A6] H. Akaike, "Power Spectrum Estimation Through Autoregressive Model Fitting," Ann. Inst. Stat. Math (Tokyo), Vol. 21, pp 407-419, 1969.

[A7] H. Akaike, "Statistical Predictor Identification," Ann. Inst. Stat. Math (Tokyo), Vol. 22, pp 203-217, 1970

[A8] H. Akaike, "A Fundamental Relationship Between Predictor Identification and Power Spectrum Estimation, "Ann. Inst. Stat. Math (Tokyo), Vol. 22, pp 219-223, 1970

[A9] H. Akaike, "Autoregressive Model Fitting for Control," Ann. Inst. Stat. Math (Tokyo), Vol. 23, pp 163-180, 1971.

[A10] T. W. Anderson, "On Asymptotic Distributions of Estimates of Parameters of Stochastic Difference Equations," Ann. Math. Stat., Vol. 30, pp 676-687, 1959.

[A11] R. L. Anderson, "Distribution of the Serial Correlation Coefficient," Ann. Math. Stat., Vol. 13, p1, 1942.

[B1] J. S. Bendat and A. G. Piersol, Random Data: Analysis and Measurement Procedures, John Wiley & Sons, Inc., New York, 1971.

[B2] G. E. P. Box and G. M. Jenkins, Time Series Analysis: Forecasting and Control, Holden-Day, San Francisco, 1970.

[B3] G. E. P. Box and D. A. Pierce, "Distribution of Residual Auto-correlations in Autoregressive Integrated Moving-Average Time Series Models," Jour. Amer. Stat. Assoc., Vol. 65, 1970.

[G1] U. Grenander and G. Szego, Toeplitz Forms and Their Applications, University of California Press, Berkeley, 1958.

[G2] J. D. Gibbons, Nonparametric Statistical Inference, McGraw-Hill Book Co., N. Y., 1971

[H1] E. J. Hannan, Time Series Analysis, Methuen, London, 1969

[H2] E. J. Hannan, Multiple Time Series, John Wiley & Sons, Inc., New York, 1970

[J1] G. M. Jenkins and D. G. Watts, Spectral Analysis and Its Applications, Holden-Day, San Francisco, 1968

[K1] R. L. Kashyap and A. R. Rao, Dynamic Stochastic Models From Empirical Data, Academic Press, New York, 1976.

[K2] T. W. Kerlin, Frequency Response Testing in Nuclear Reactors, Academic Press, New York 1974

[M1] H. B. Mann and A. Wald, "On the Statistical Treatment of Linear Stochastic Difference Equations", Econometrica, Vol. 11, p 173, 1943.

[P1] A. Papoulis, Probability, Random Variables and Stochastic Processes, McGraw-Hill Book Co., New York, 1965

[P2] E. Parzen, "An Approach to Time Series Analysis", Ann. Math. Stat., Vol. 32, pp 951-989, 1961.

[P3] R. E. A. C. Paley and N. Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc., Providence, 1943

[Q1] M. H. Quenouille, Analysis of Multiple Time Series, Hafner, New York, 1957.

[R1] M. Rosenblatt (Editor), Time Series Analysis, Proc. of the Symp. at Brown University, June 1962, John Wiley and Sons, Inc., New York, 1963.

[R2] C. R. Rao, Linear Statistical Inference and Its Applications, John Wiley & Sons, Inc. New York, 1965.

[R3] In-Situ Response Time Testing of Platinum Resistance Thermometers, Annual Progress Report, Jan. 1977, Dept. of Nuclear Engineering, The Univ. of Tenn; EPRI Report NP-459.

[R4] "Periodic Testing of Electric Power and Protection Systems," U.S. Nuclear Regulatory Commission, Regulatory Guide 1.118, June 1976.

[S1] H. W. Sorenson, "Least-Squares Estimation: from Gauss to Kalman", IEEE Spectrum, Vol. 7, pp 63-68

[S2] N. V. Smirnov, "Table for Estimating the Goodness of Fit of Empirical Distributions, "Ann. Math. Stat. Vol. 19, pp 279-281, 1948.

[T1] J. A. Thie, Reactor Noise, Rowman and Littlefield, Inc., New York, 1963.

[T2] M. G. Natrella, Experimental Statistics, Handbook 91, National Bureau of Standards, 1966

[T3] Z. W. Birnbaum, "Numerical Tabulation of the Distribution of Kolmogorov Statistic for Finite Sample Size," J. Am. Stat. Assoc., Vol. 47, p. 431, Table 2, 1952.

[U1] R. E. Uhrig, Random Noise Techniques in Nuclear Reactor Systems, The Ronald Press Co., New York, 1970.

[W1] G. Walker, "On Periodicity in Series of Related Terms," Proc. Royal Soc., Vol. A131, p 518, 1931.

[W2] S. S. Wilks, Mathematical Statistics, John Wiley & Sons, Inc., New York, 1963, Chapter 13.

[Y1] G. U. Yule, "On a method of investigating periodicities in disturbed series, with special reference to Wolfer's Sunspot numbers," Phil. Trans. Vol. A226, P267, 1927.

[K3] M. Kaveh and G. R. Cooper, "An Empirical Investigation of the properties of the Autoregressive Spectral Estimator," IEEE Tran. Info. th., vol. IT-22, pp 313-323, 1976.

[U2] B. R. Upadhyaya and T. W. Kerlin, "Response Time Testing of Temperature Sensors Using a Noise Analysis Method," Tran. Am. Nucl. Soc., Vol. 26, p 445, 1977.

[K4] R. L. Kashyap, "A Bayesian Comparison of Different Classes of Dynamic Models Using Empirical Data," IEEE Tran. Aut. Cont., Vol. AC-22, pp 715-727, 1977.

[U3] B. R. Upadhyaya and T. W. Kerlin, "Modified Noise Analysis Method for the Estimation of Temperature Sensor Response Time Characteristics," Tran. Am. Nucl. Soc., June 1978.

APPENDIX A

Recursive Estimation of Autoregressive Models

In chapter 4 we described the estimation of AR models using Yule-Walker equations. The parameter estimate is given by

$$\underline{a} = P_n^{-1}\underline{C} \tag{A-1}$$

Where P_n is the n x n matrix of correlation functions of the observation $\{y_k\}$. If P_{n+1} is the correlation matrix of order $(n + 1)$ then we can write this in terms of P_n as

$$P_{n+1} = \left[\begin{array}{ccc|c} & & & C_n \\ & P_n & & C_{n-1} \\ & & & \vdots \\ & & & C_1 \\ \hline C_n & C_{n-1} \cdots & C_1 & C_0 \end{array}\right] \tag{A-2}$$

Instead of determining the inverse of P_{n+1} directly we can express P_{n+1}^{-1} in terms of P_n^{-1}.

Let us rewrite P_{n+1} as

$$P_{n+1} = \begin{bmatrix} P_n & \underline{C}_{n1} \\ \underline{C}_{n1}^T & C_o \end{bmatrix} \tag{A-3}$$

where $\underline{C}_{n1}^T = (C_n\ C_{n-1} \ldots C_1)$. Then the inverse of P_{n+1} is given by

$$P_{n+1}^{-1} = \begin{bmatrix} \dfrac{(P_n^{-1} + P_n^{-1}\ \underline{C}_{n1}\ \underline{C}_{n1}^T\ P_n^{-1})}{\lambda} & \dfrac{-P_n^{-1}\ \underline{C}_{n1}}{\lambda} \\ \dfrac{-\underline{C}_{n1}^T\ P_n^{-1}}{\lambda} & \dfrac{1}{\lambda} \end{bmatrix} \tag{A-4}$$

$$\lambda = C_o - \underline{C}_{n1}^T P_n^{-1}\ \underline{C}_{n1} \tag{A-5}$$

APPENDIX B

Kolmogorov-Smirnov Goodness-of-fit Test of the Normality of the given Time Series

In chapter 4 we presented the statistical test to ascertain the goodness of a given assumed distribution of a time series. In this appendix we will describe this procedure as applied to the Millstone 2 temperature data.

B.1 Computation of Distribution Function

Consider N data points $\{y_k, k=1,2,..,n\}$ with mean zero. The probability density function

$$p(y) = \frac{N_y}{NW} \tag{B.1}$$

where W is a narrow interval centered at y and N_y is the number of data values which fall in the range $y \pm \frac{W}{2}$.

Select the number of intervals of width W equal to M such that

$$M = \frac{y_{max} - y_{min}}{W} \tag{B.2}$$

Define the end point of the ith interval as

$$d_i = y_{min} + iW, \quad i=0,1,2,..,M. \tag{B.3}$$

Define $\{N_i, i=o,1,...,M\}$ as

N_o = [number of y such that $y \leq d_o$]

N_i = [number of y such that $d_o < y \leq d_1$]

.

.

N_M = [number of y such that $d_{M-1} < y < d_M$]

The procedure will sort out N data values such that

$$N = \sum_{i=0}^{M} N_i \tag{B.4}$$

B.2 Distribution and Density Functions

Now we define the probability

$$\hat{P}_i = \text{Prob}\,[d_{i-1} \leq y \leq d_i] = \frac{N_i}{N} \tag{B.5}$$

$$i = 0,1,2, \ldots,M.$$

The density function is estimated as

$$\hat{p}_i = \frac{\hat{P}_i}{W}, \quad i=1,2,\ldots,M \tag{B.6}$$

The distribution function is given by

$$F_i = \sum_{j=0}^{i} \hat{P}_j, \quad i=0,1,2,\ldots,M \tag{B.7}$$

For a given Normal density with mean zero and variance σ^2, $N(0,\sigma^2)$

$$p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} \tag{B.8}$$

The distribution function is

$$F(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{y} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy \tag{B.9}$$

F(y) is computed numerically using a trapezoidal integration scheme at the same points as the F_i are calculated.

B.3 Application to the Millstone Hotleg Temperature noise data

The first step in computing the Kolmogorov-Smirnov test is to estimate p_i and F_i as discussed in section B.2. The best fitting normal density is determined such that the squared error between F_i and F(y) is a minimum.

This is done by incrementing the variance σ^2 in steps so that a minimum value of

$$\text{Error} = \sum_{i=0}^{M} (F(y_i) - F_i)^2 \tag{B.10}$$

is obtained.

Figures (B.1) and (B.2) show the estimated values of the amplitude probability density function and the corresponding distribution function. These are compared with a normal density and distribution function whose standard deivation is

$$\sigma = 1.082 \tag{B.11}$$

Determine the quantity

$$\max_{y} |F(y_N) - F_N| = D_N \tag{B.12}$$

The value of D_N is determined as

$$D_N = 0.00833, \; N = 31787$$

To apply K-S test, for a given significant level α

$P\{D_N > D_{N,\alpha}\} = \alpha$ is selected and the corresponding value of $D_{N,\alpha}$ is obtained from table (4A.5b) of K-S statistics.

For $N = 31787$, $\alpha = 0.05$, $D_{N,\alpha} = 0.00763$

For $N = 31787$, $\alpha = 0.01$, $D_{N,\alpha} = 0.00914$

Thus, we have $D_{N,0.05} < D_N < D_{N,0.01}$. The normality of the given time series is satisfied for a significance level between 95% and 99%.

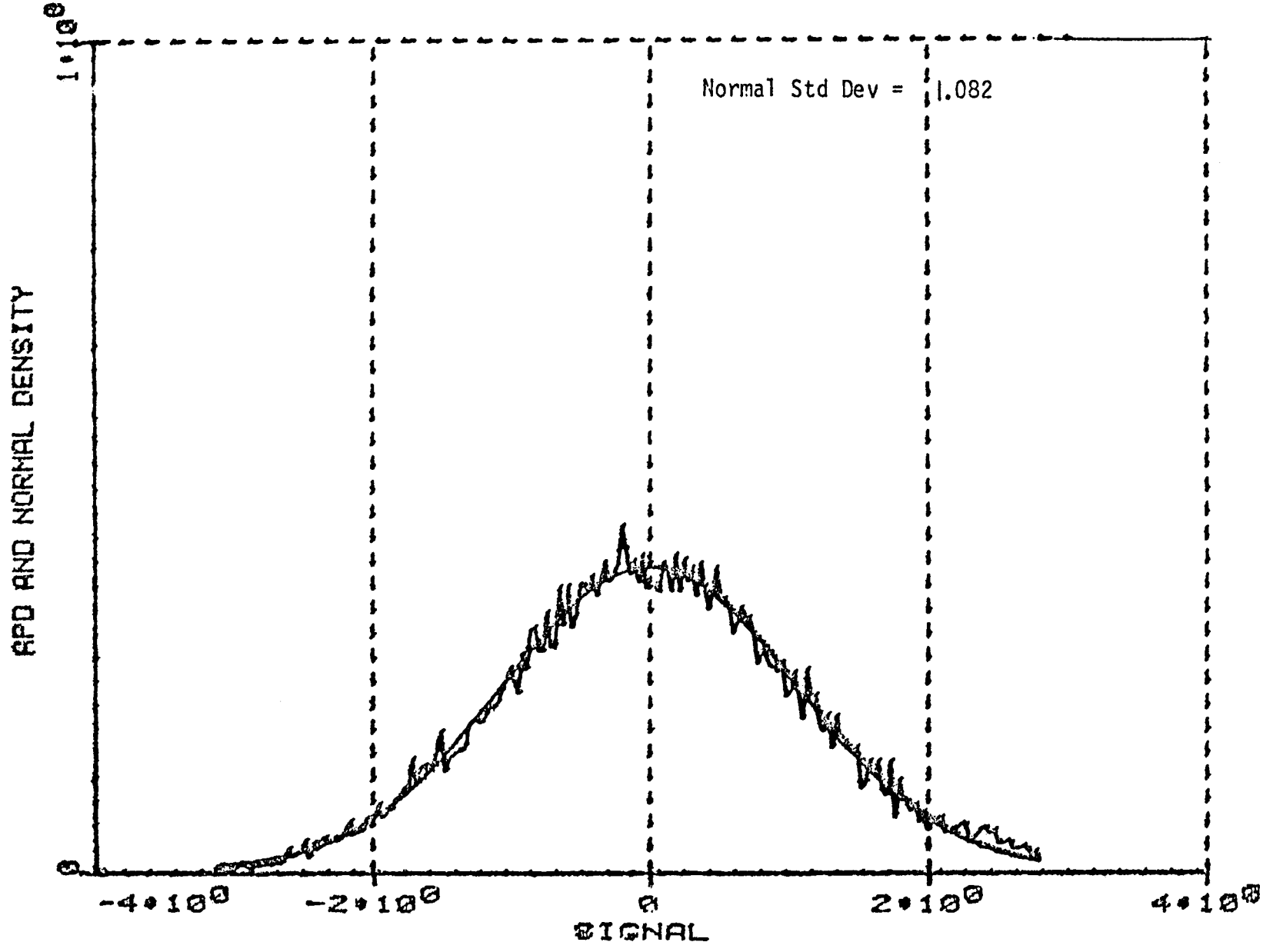


Figure B.1 Comparison of Probability Density Functions for Millstone 2 Hotleg Temperature.

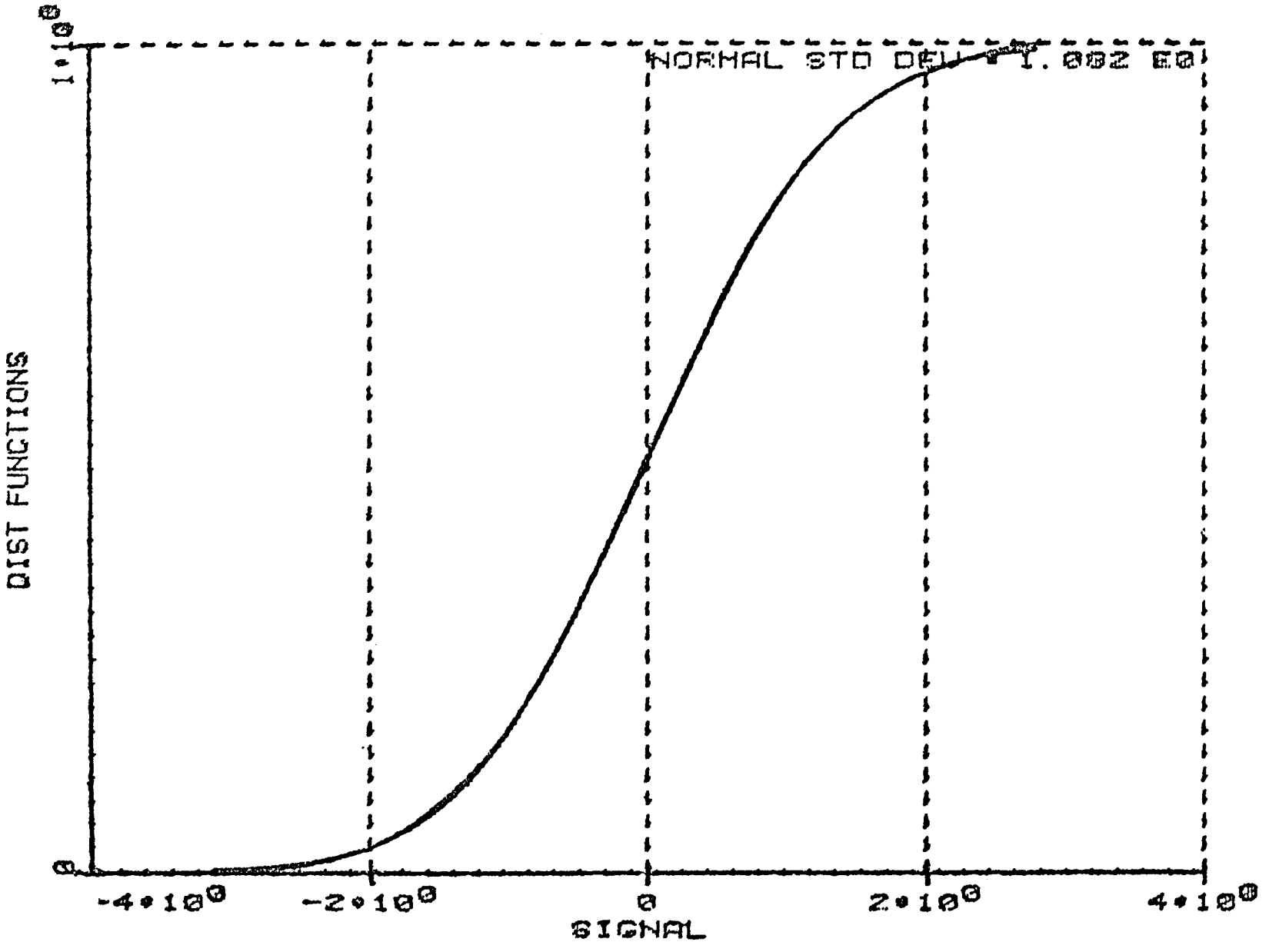


Figure B.2 Comparison of Distribution Functions for Millstone 2 Hotleg Temperature.

APPENDIX C

RESULTS OF RTD NOISE ANALYSIS FOR FOUR POWER REACTORS

In this appendix results from RTD tests at Millstone, Oconee, St. Lucie and Turkey Point power plants are summarized. There are nine RTD's and for each plots of autocorrelation function, AR power spectrum, step response and residual power spectrum are shown. The results are tabulated in section 6.3, table 6.2.

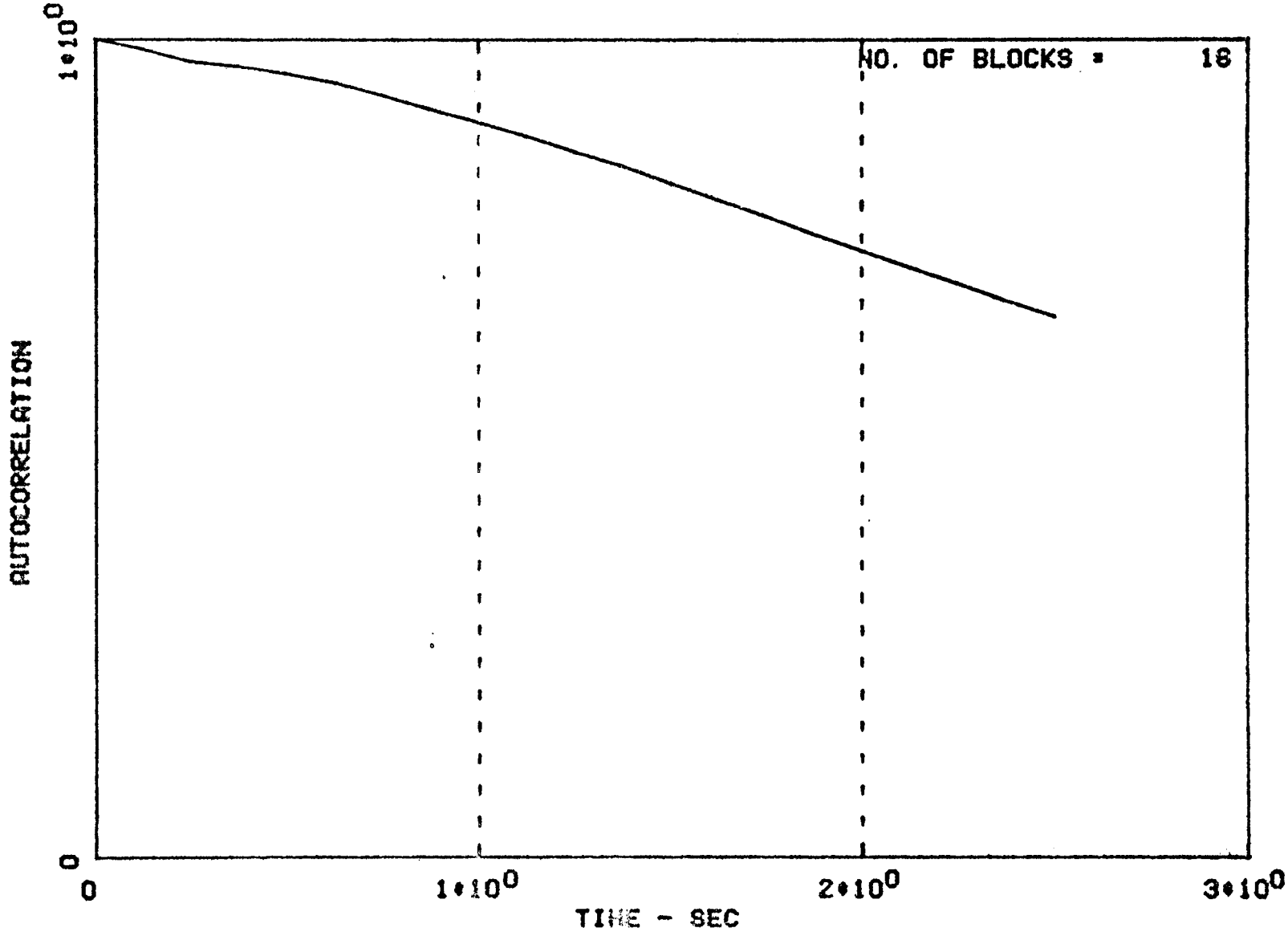


Figure C.1 Millstone - Signal Autocorrelation Function (DT8).

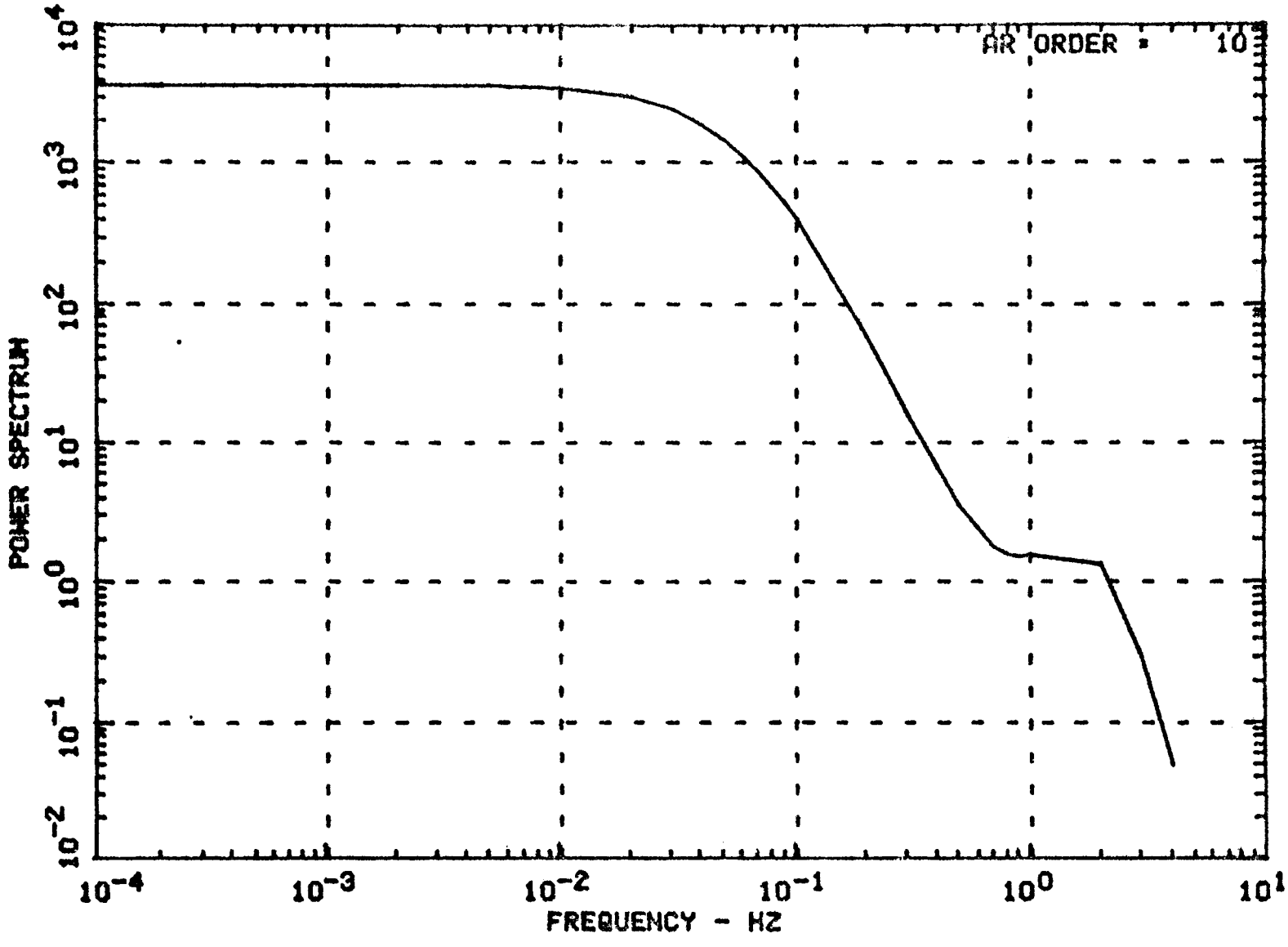


Figure C.2 Millstone - AR Power Spectrum (DT8).

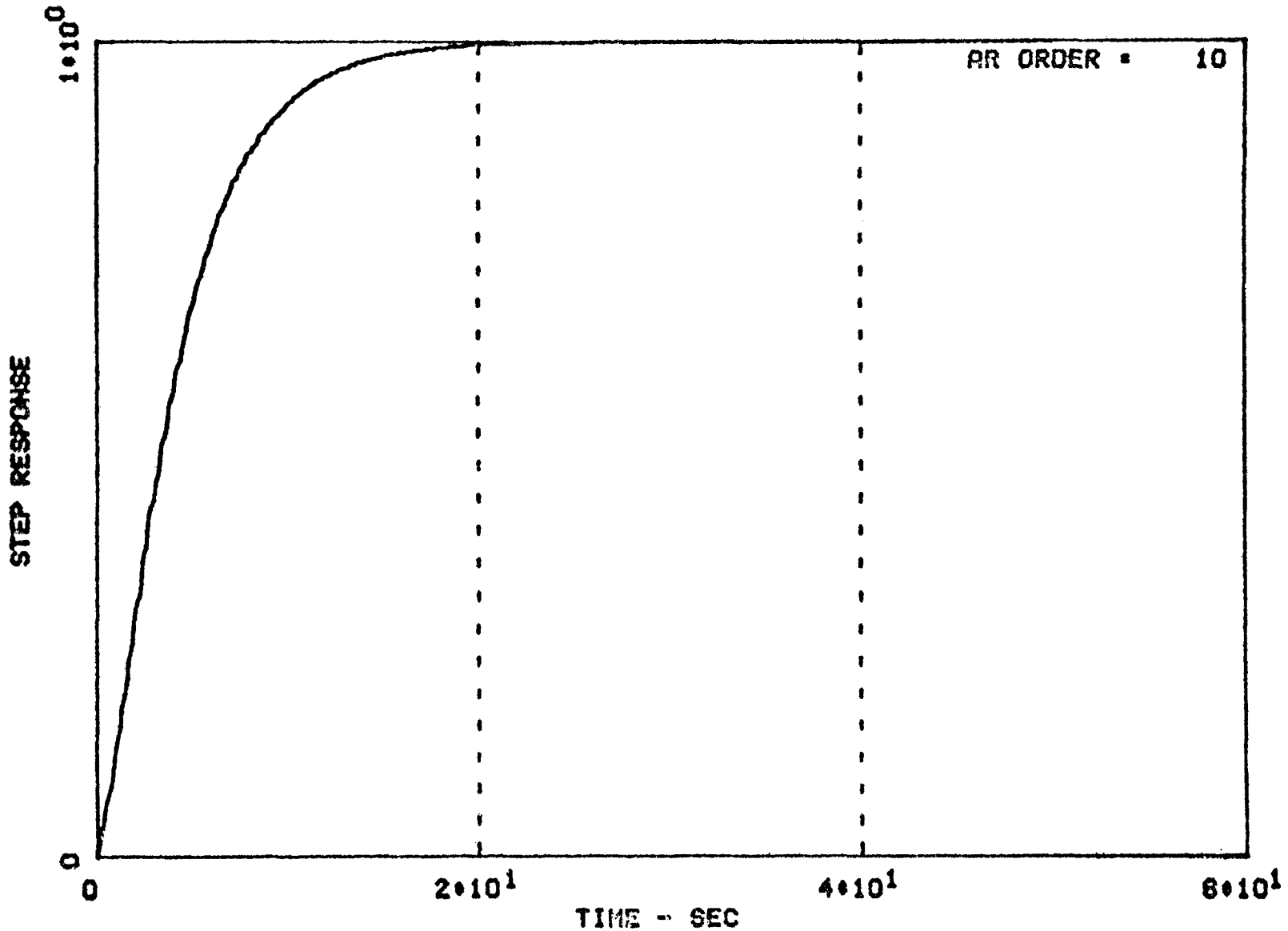


Figure C.3 Millstone - Step Response from AR Model (DT8).

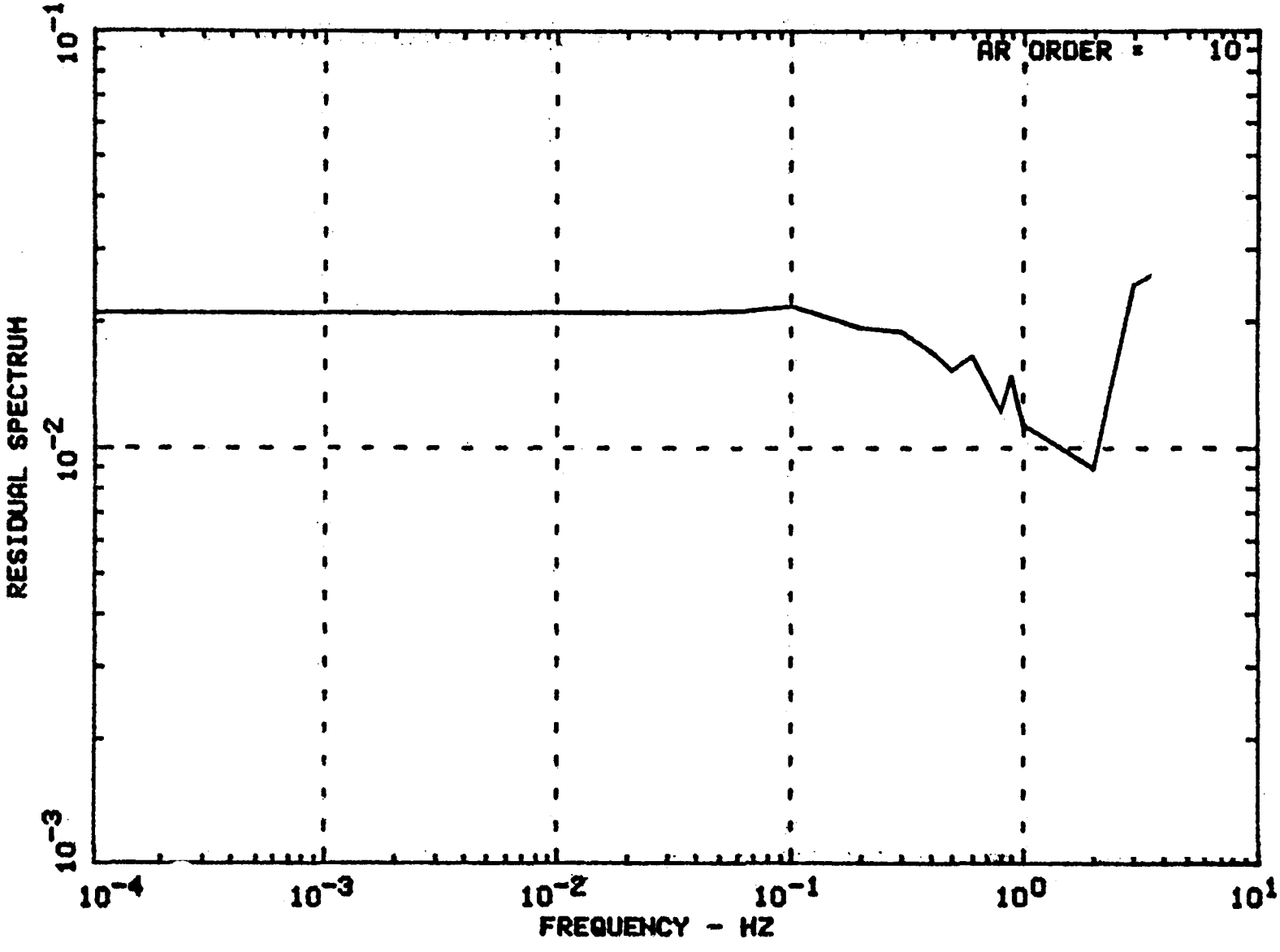


Figure C.4 Millstone 2 -- Residual Power Spectrum (DT8).

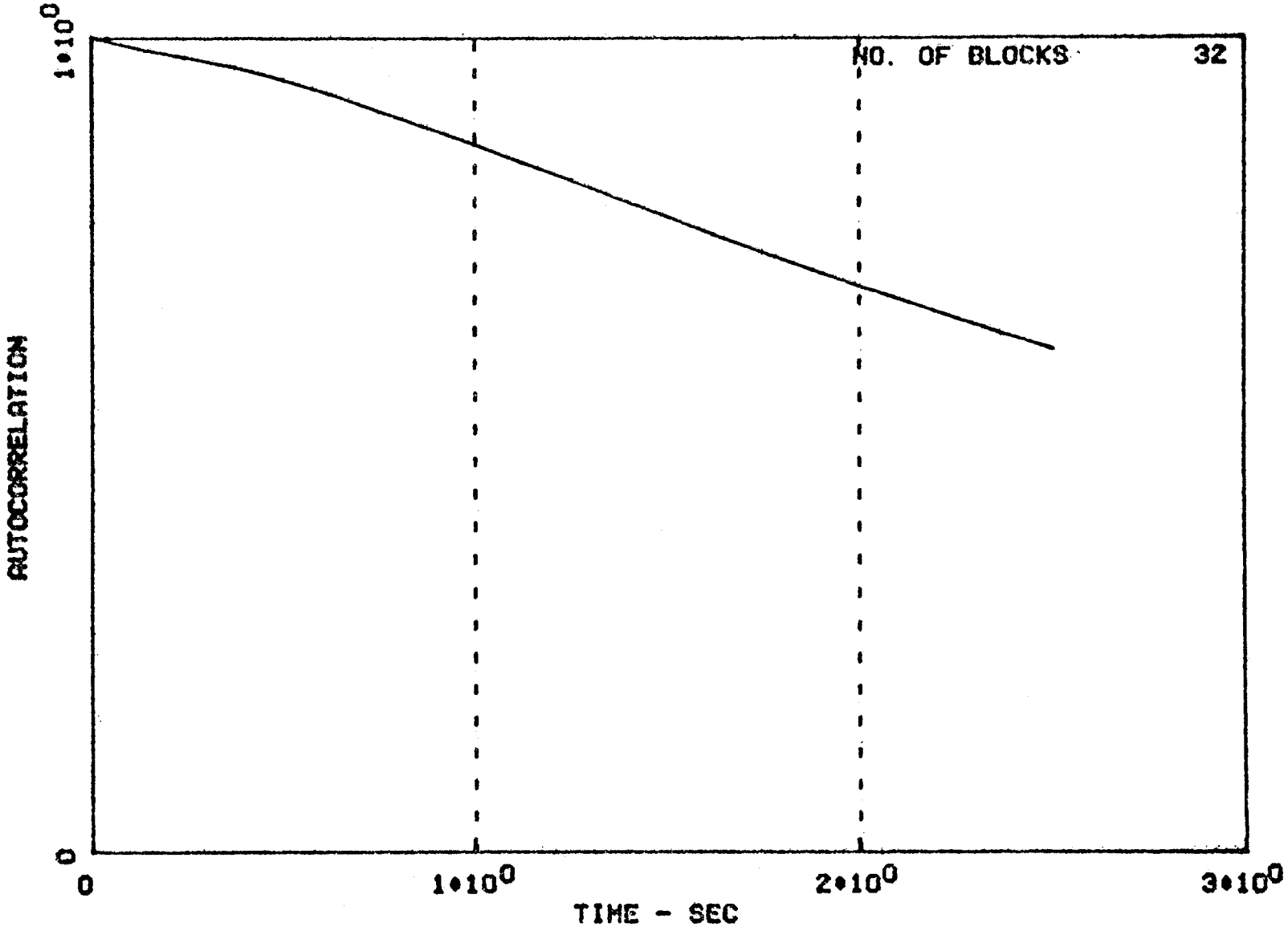


Figure C.5 St. Lucie - Signal Autocorrelation Function (T63). (Sensor 1)

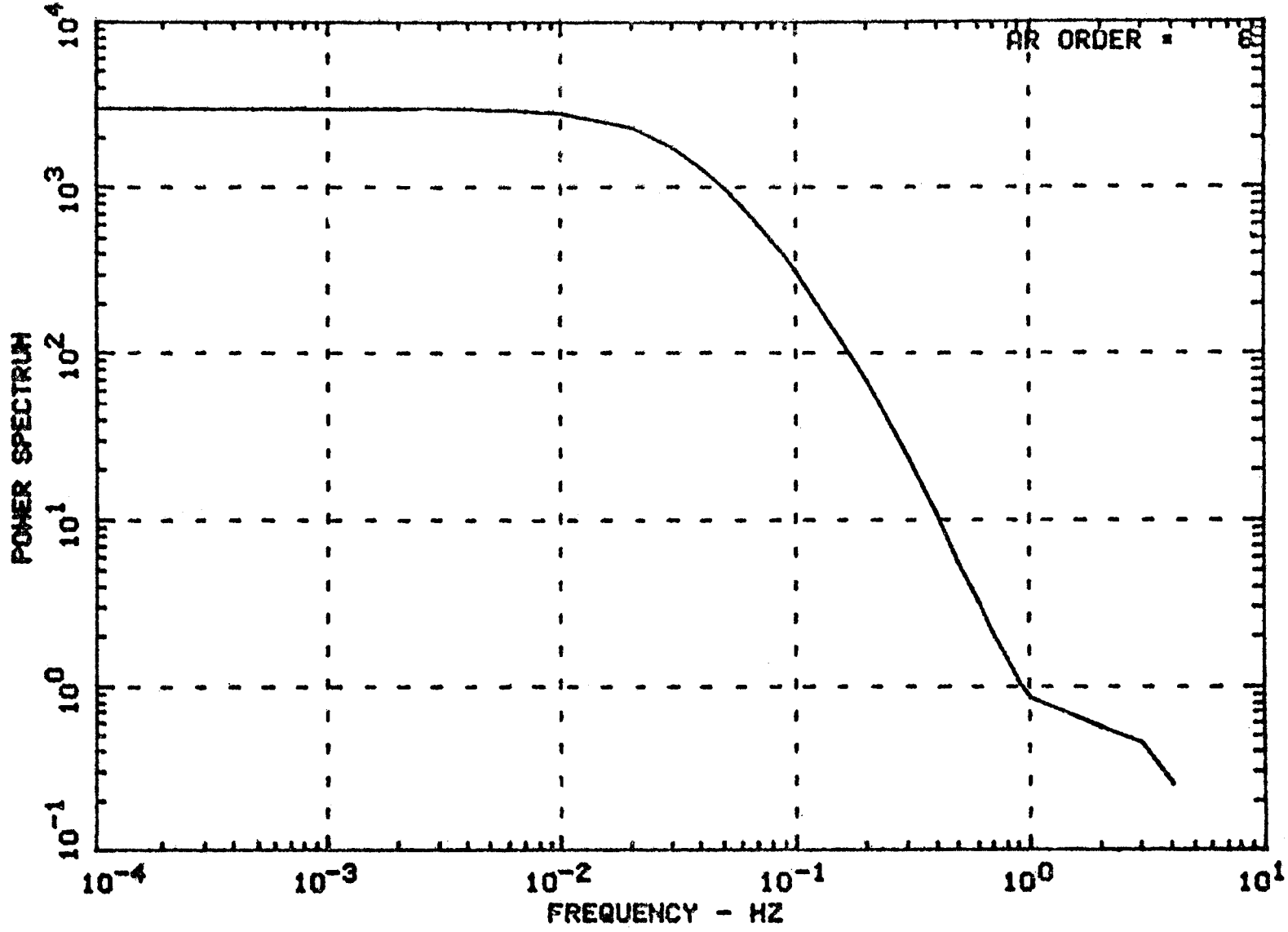


Figure C.6 St. Lucie - AR Power Spectrum (T63). (Sensor 1)

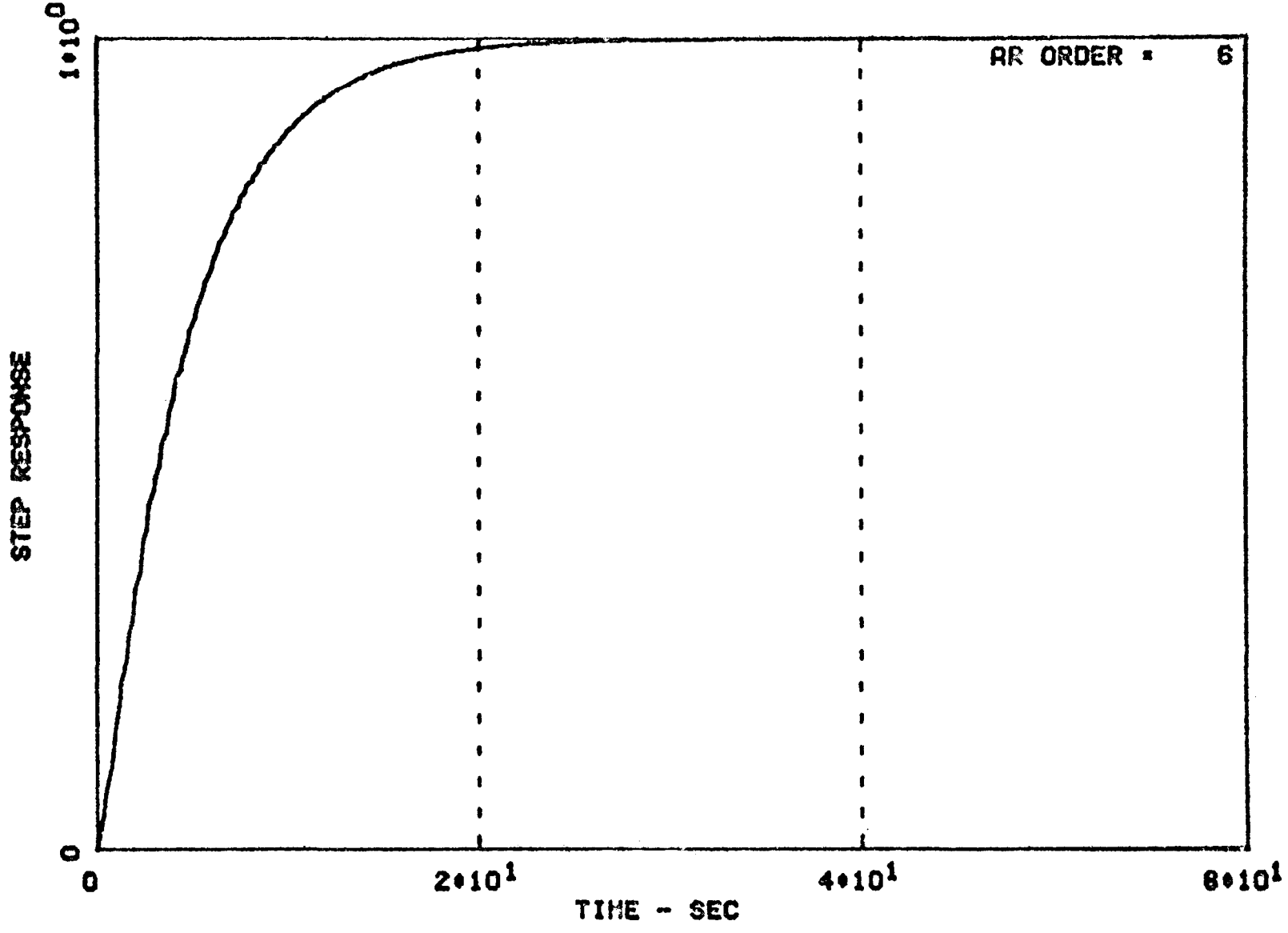


Figure C.7 St. Lucie - Step Response from AR Model (T63). (Sensor 1)

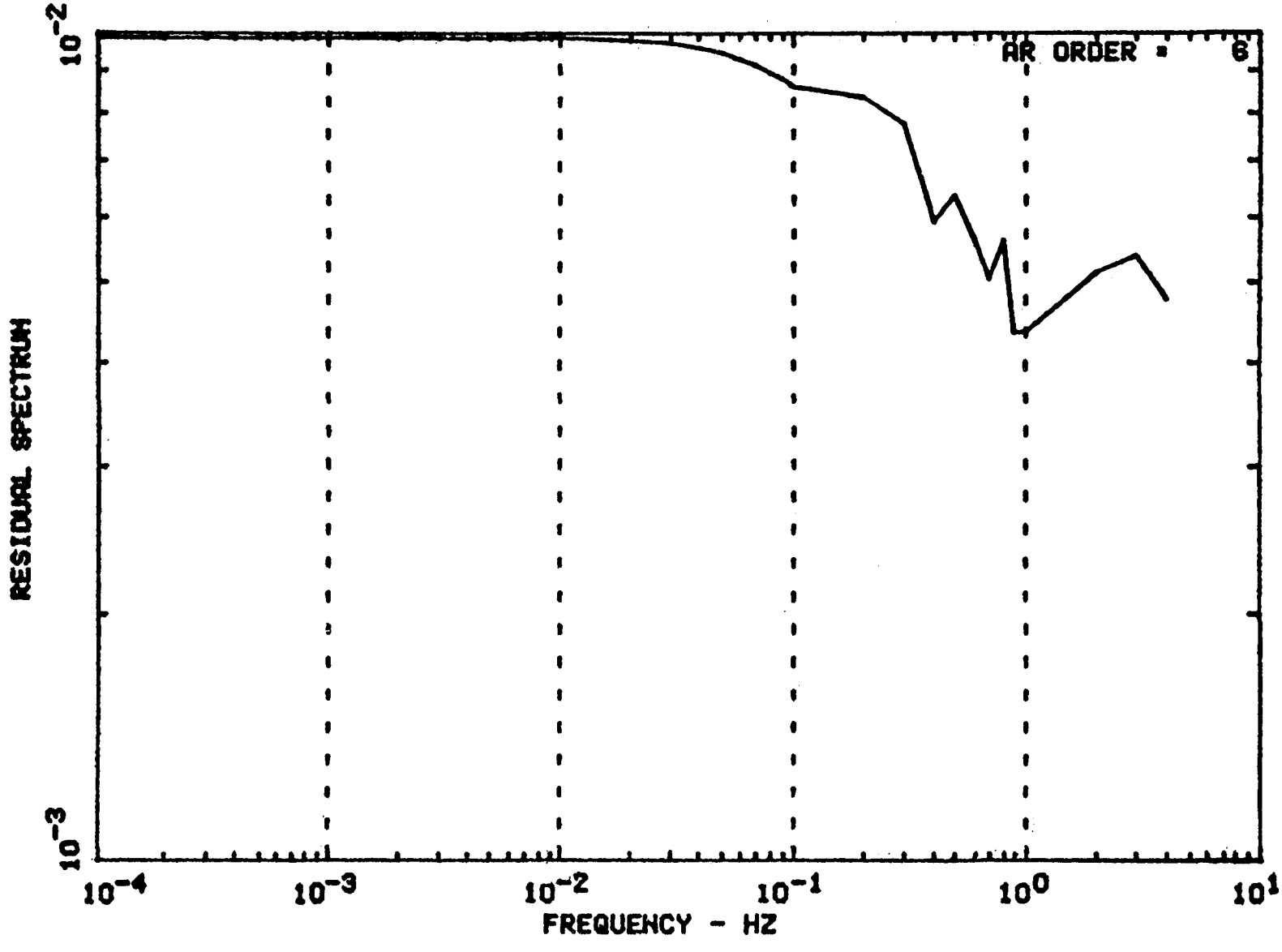


Figure C.8 St. Lucie -- Residual Power Spectrum (T63, Sensor 1).

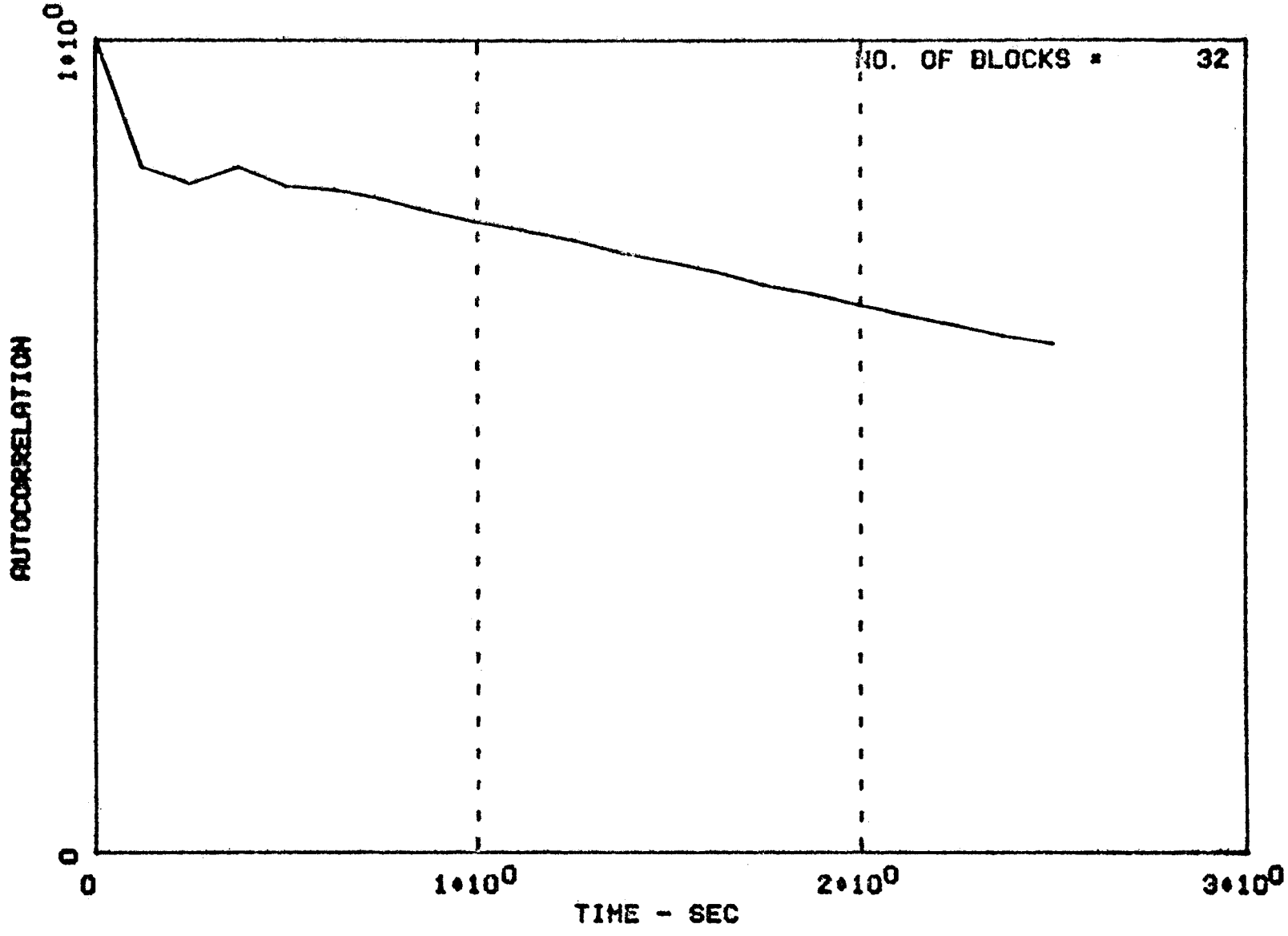


Figure C.9 St. Lucie - Signal Autocorrelation Function (T63). (Sensor 2)

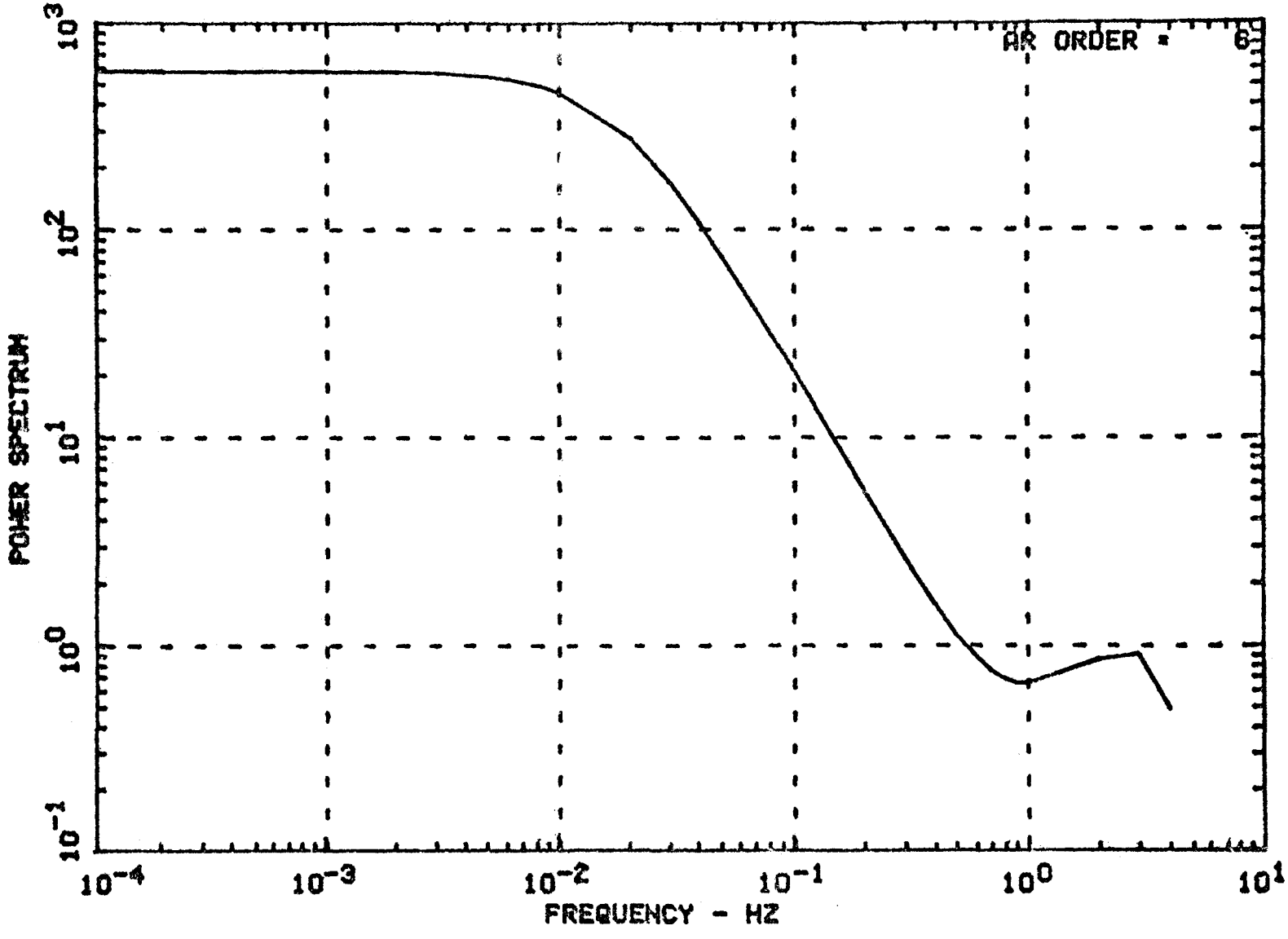


Figure C.10 St. Lucie - AR Power Spectrum (T63). (Sensor 2)

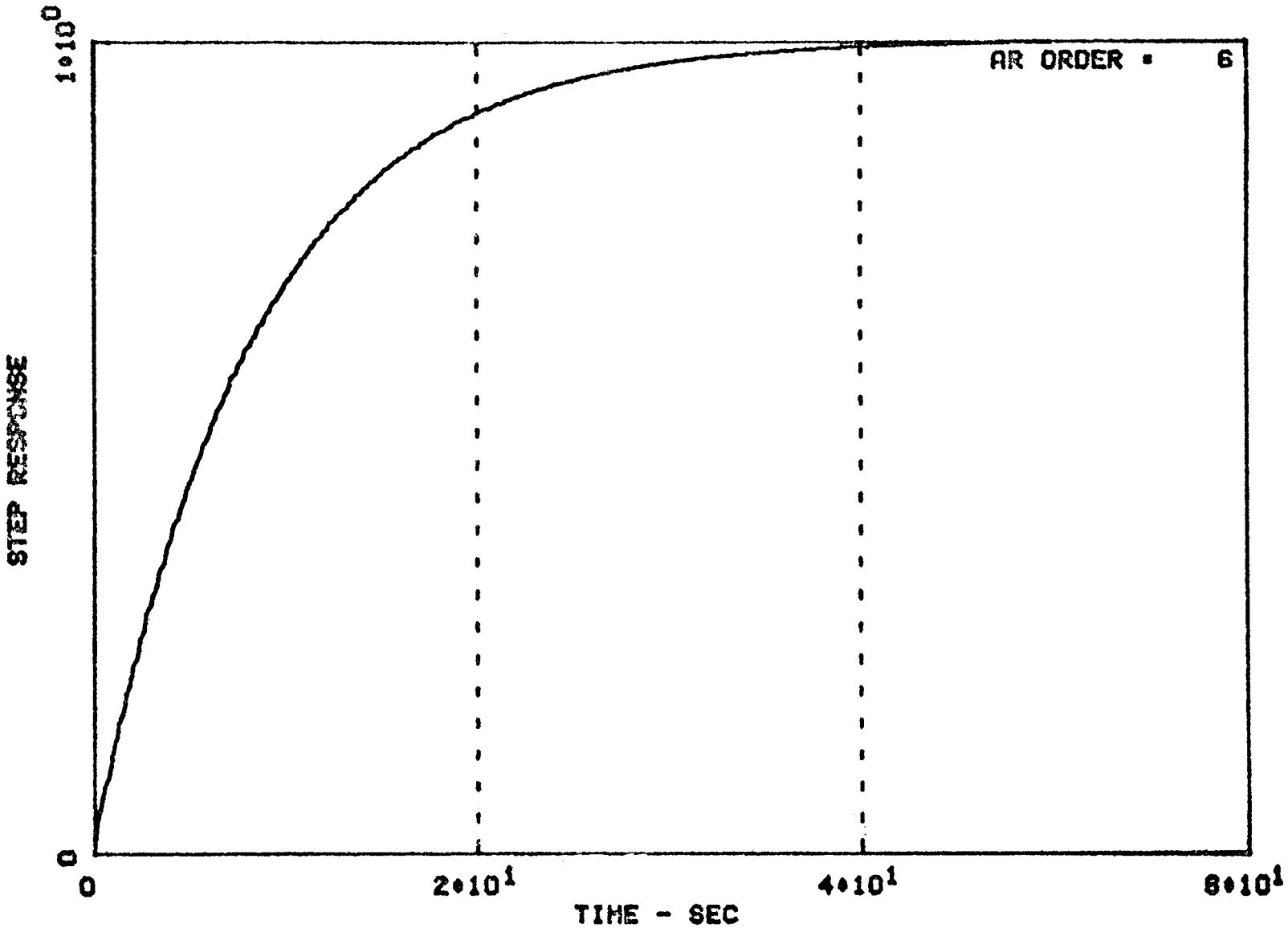


Figure C.11 St. Lucie - Step Response from AR Model (T63). (Sensor 2)

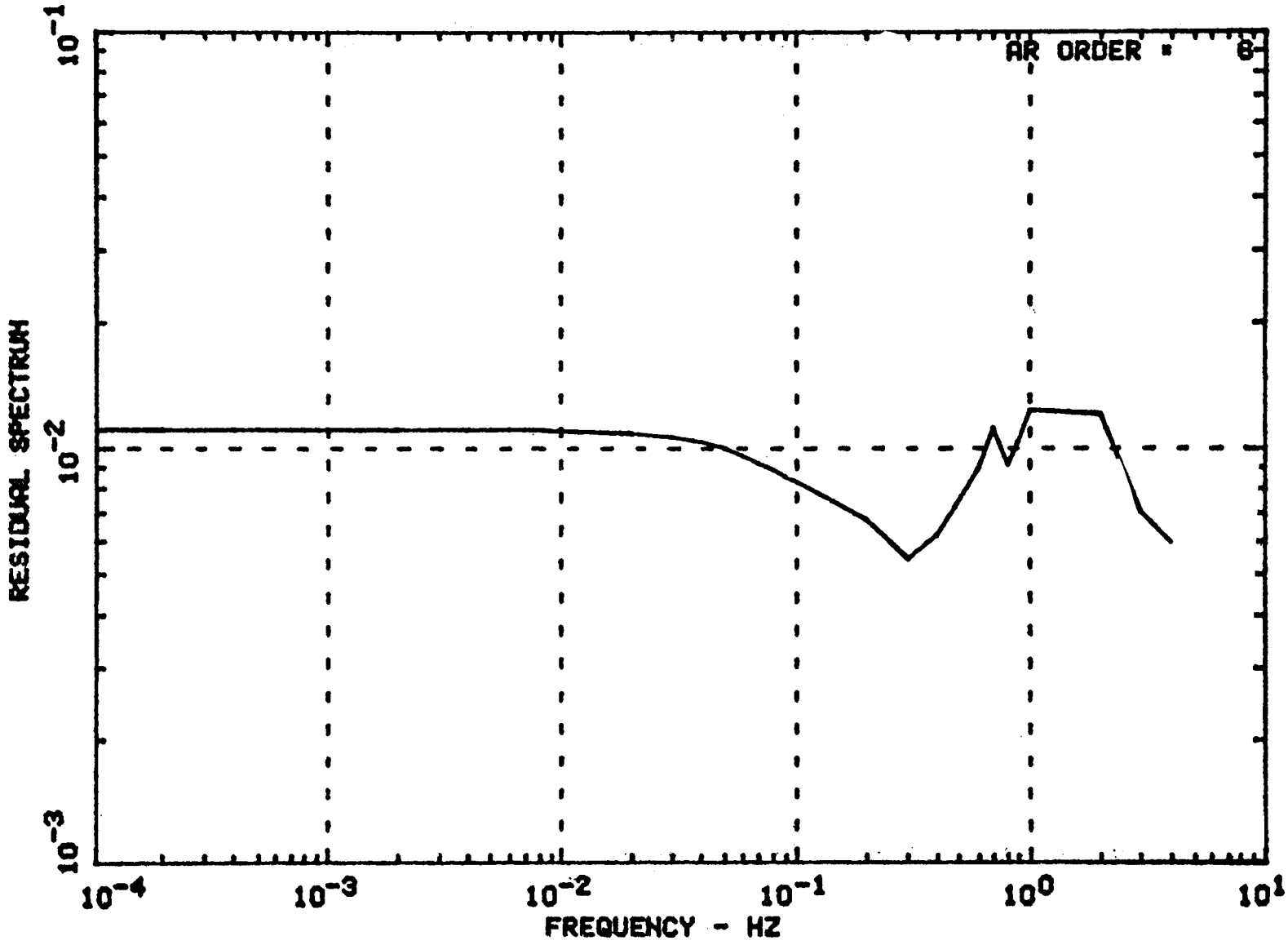


Figure C.12 St. Lucie -- Residual Power Spectrum (T63, Sensor 2).

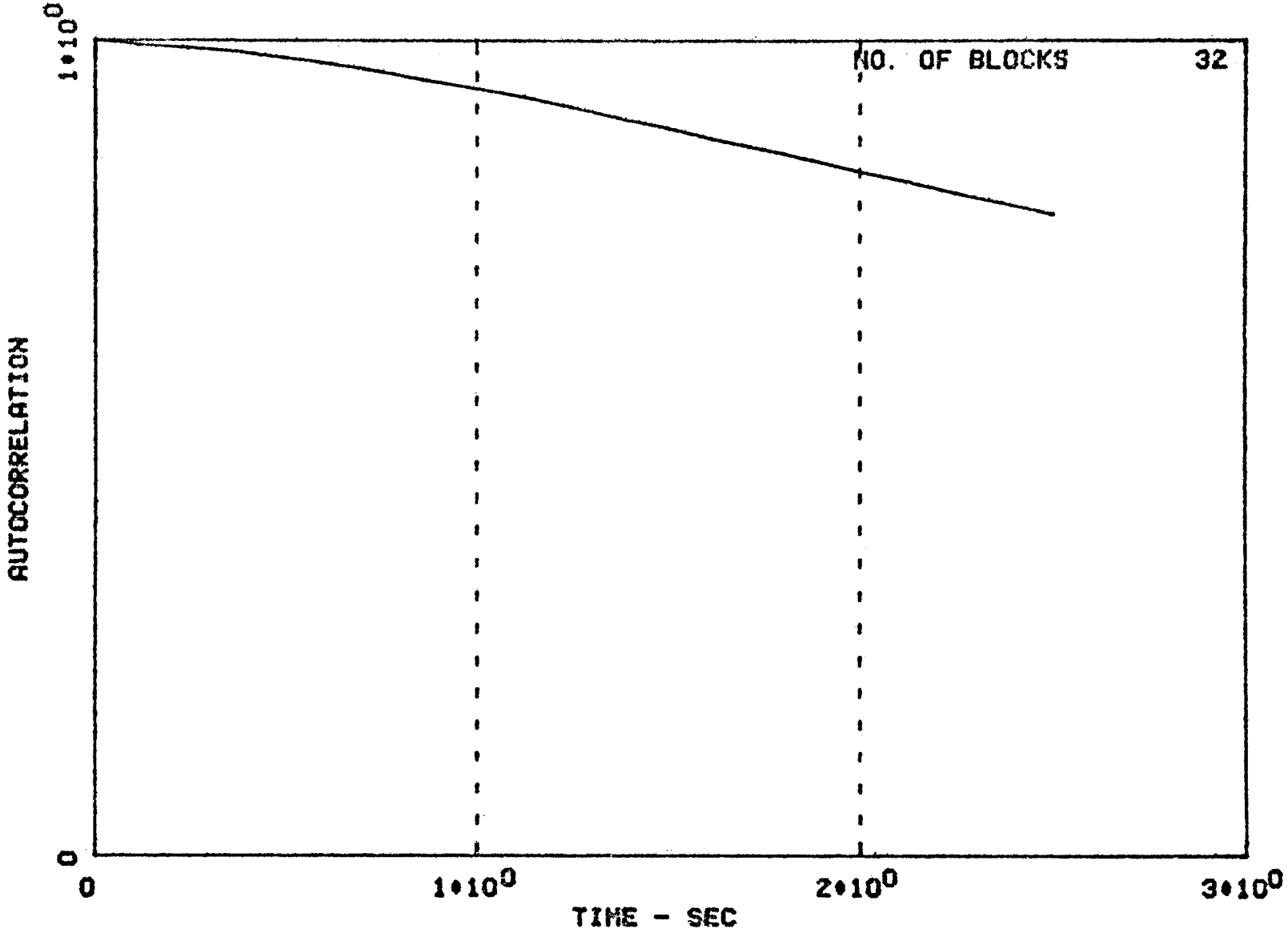


Figure C.13 St. Lucie - Signal Autocorrelation Function (T07). (Sensor 1)

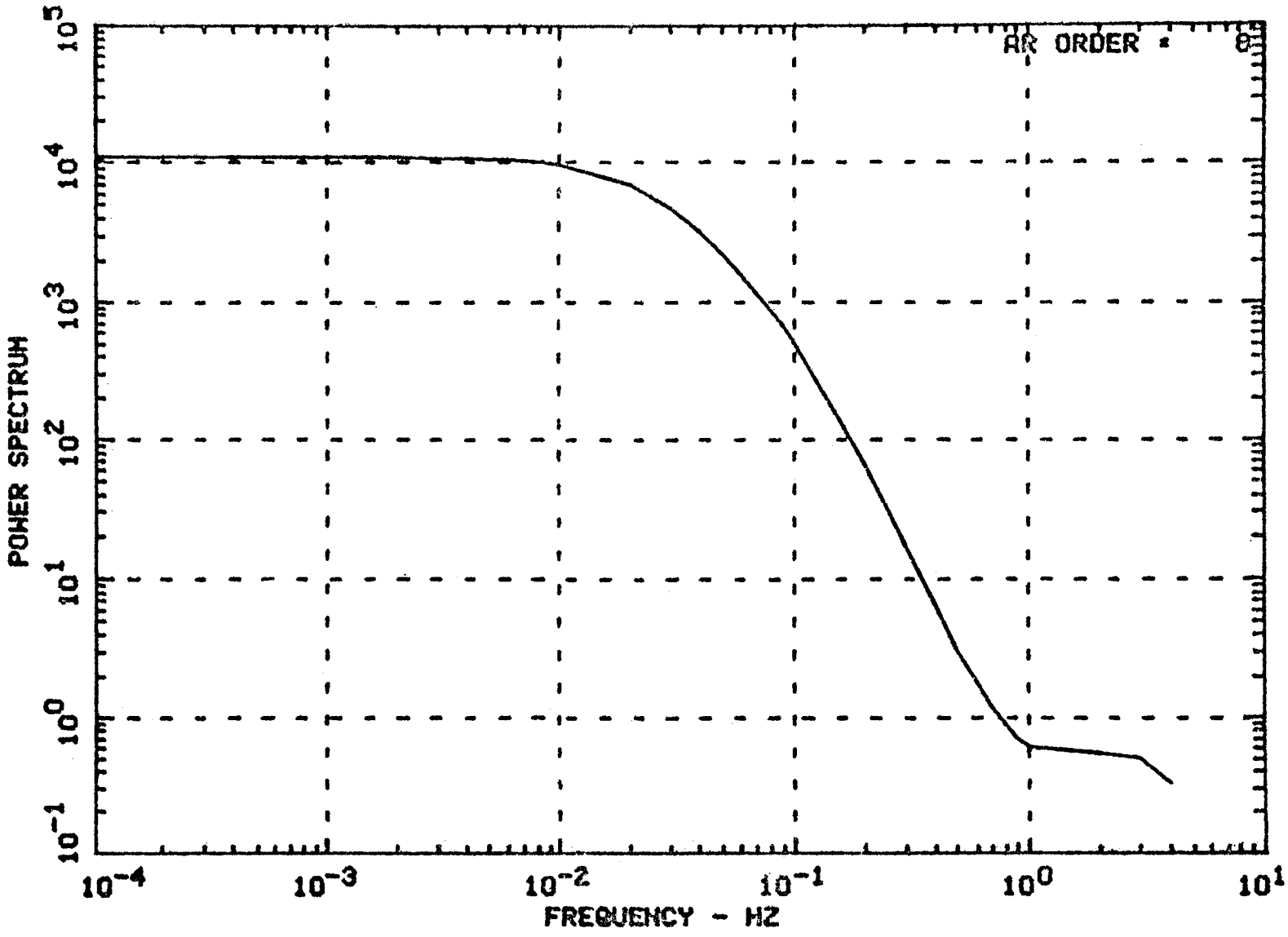


Figure C.14 St. Lucie - AR Power Spectrum (T07). (Sensor 1)

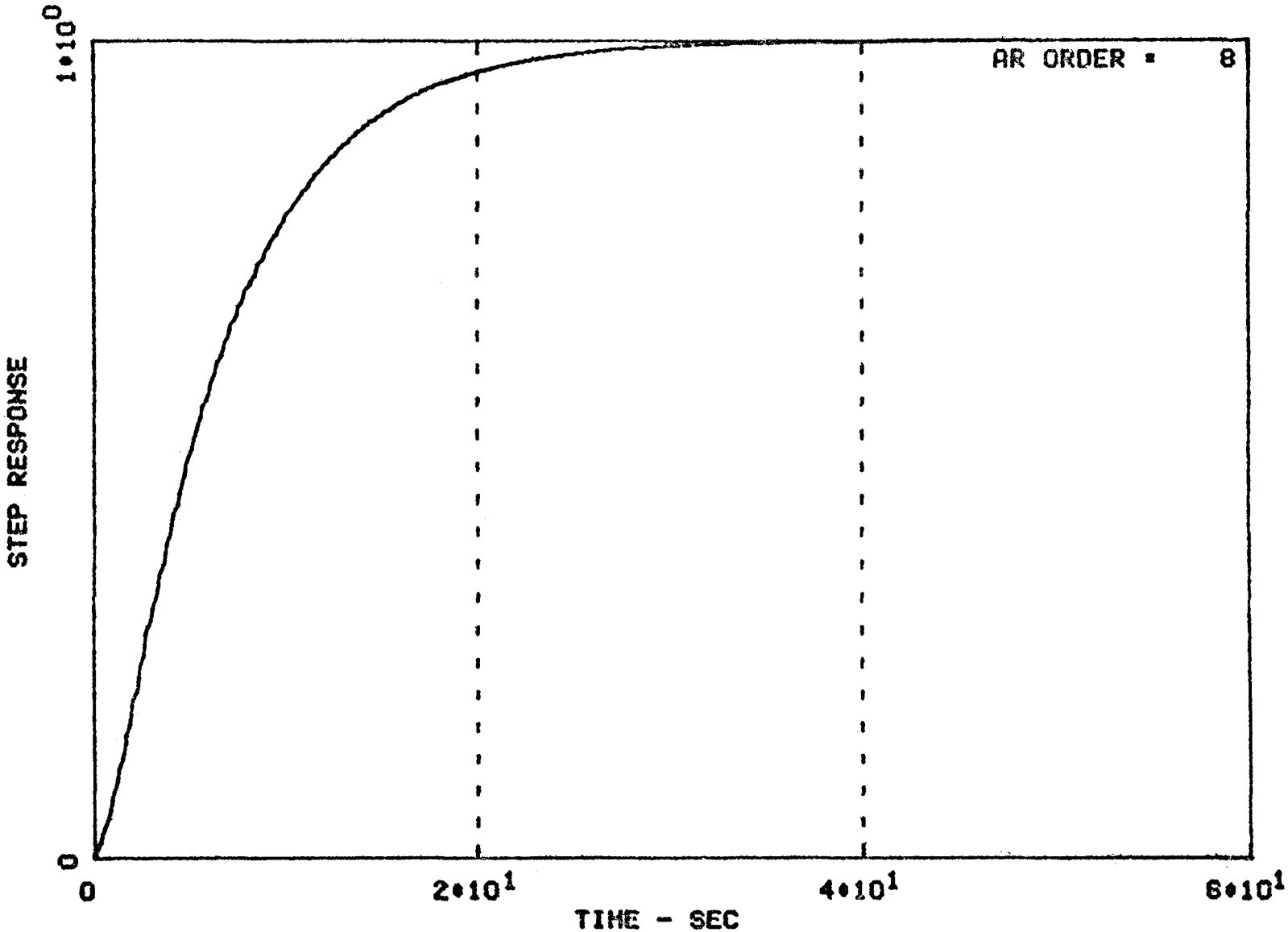


Figure C.15 St. Lucie - Step Response from AR Model (T07). (Sensor 1)

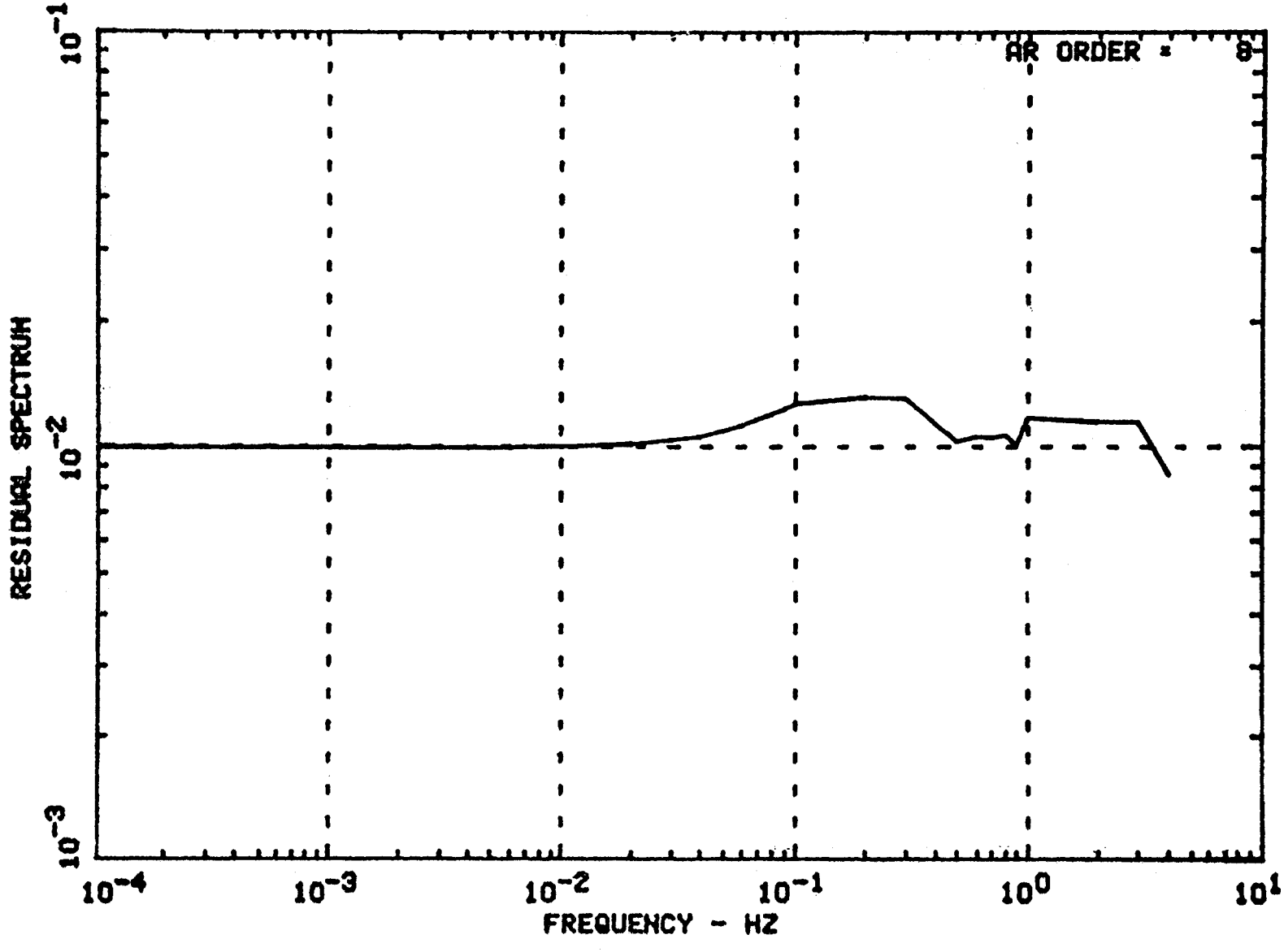


Figure C.16 St. Lucie -- Residual Power Spectrum (T07, Sensor 1).

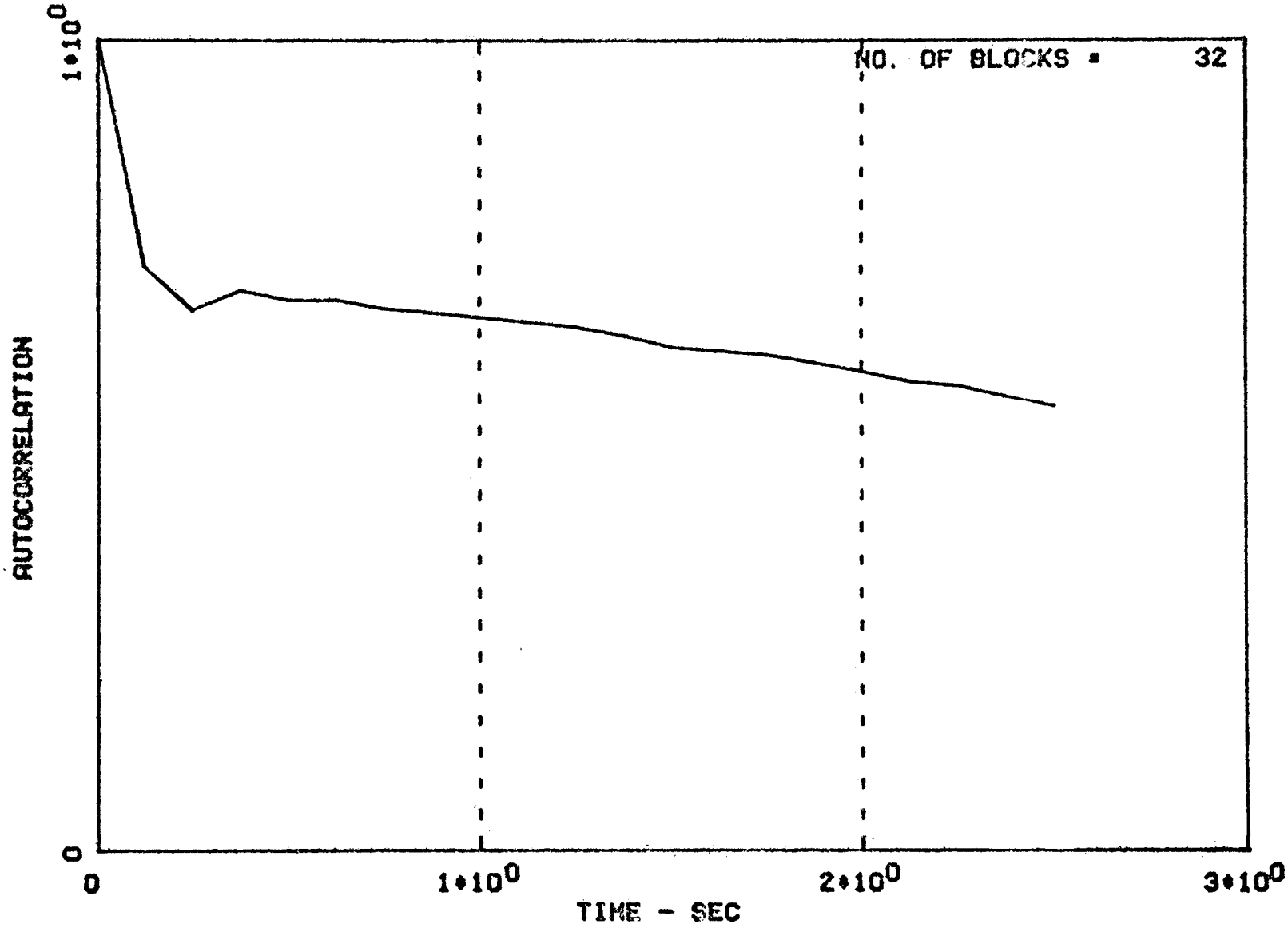


Figure C.17 St. Lucie - Signal Autocorrelation Function (T07). (Sensor 2)

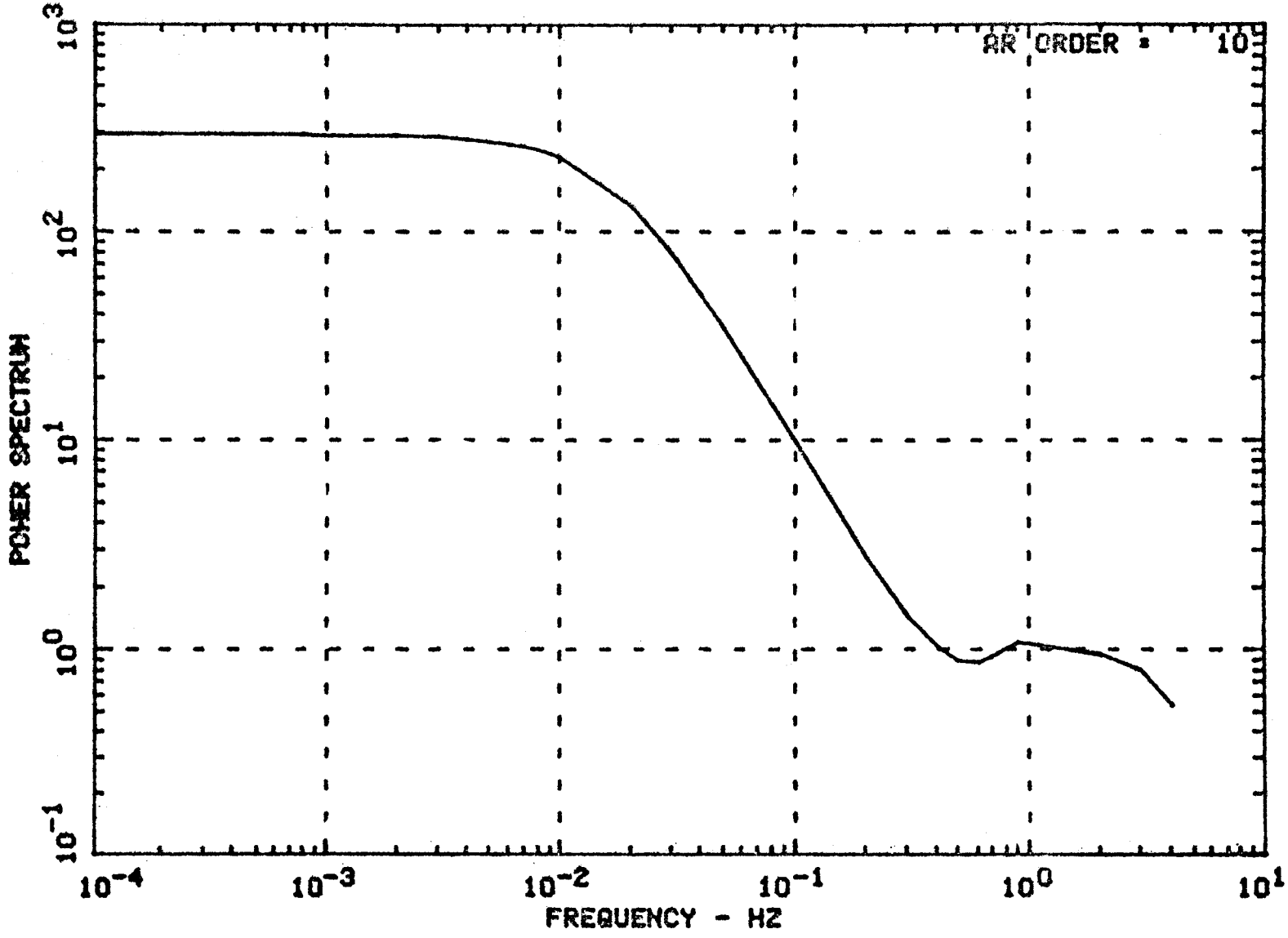


Figure C.18 St. Lucie - AR Power Spectrum (T07). (Sensor 2)

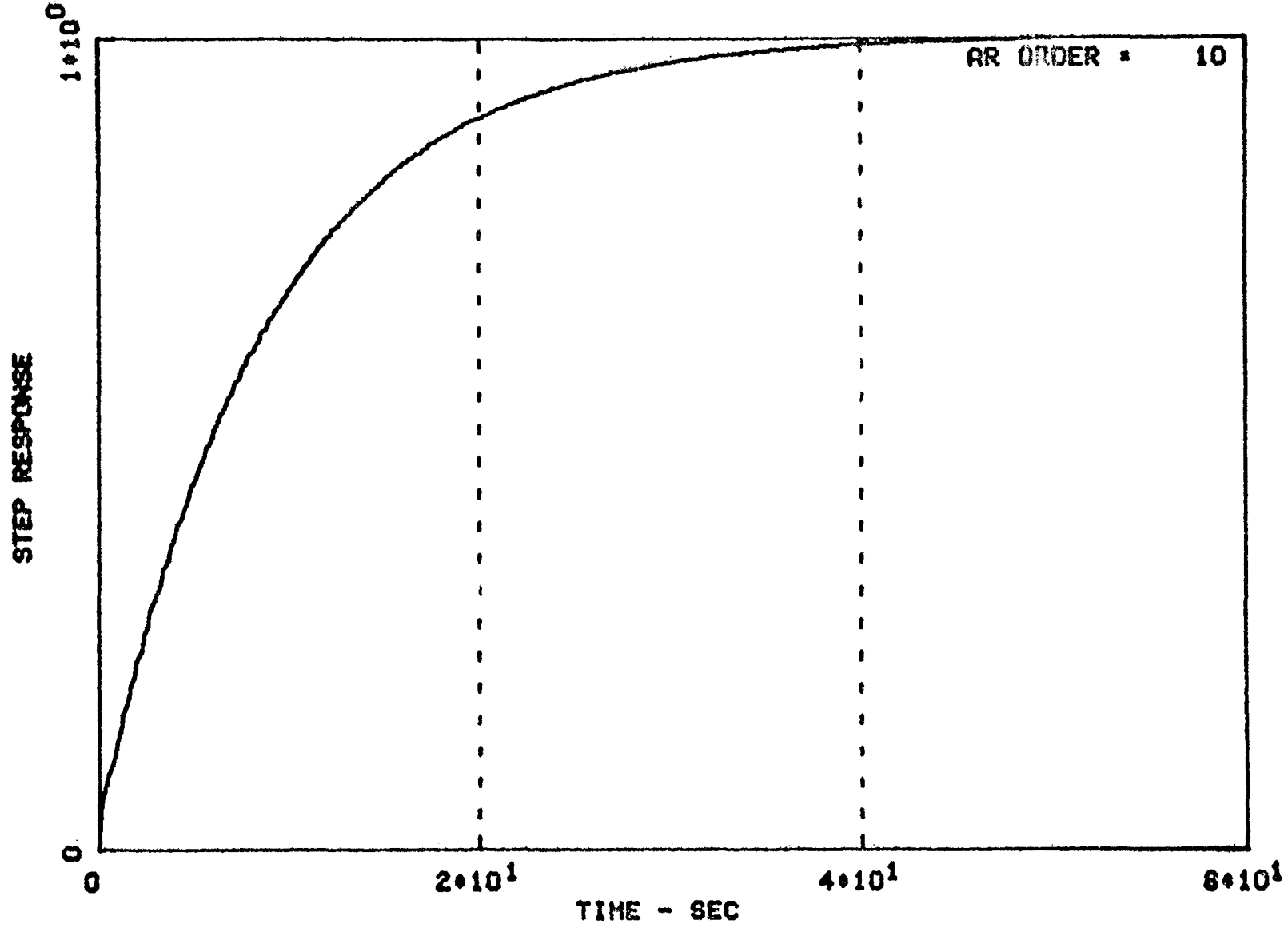


Figure C.19 St. Lucie - Step Response from AR Model (T07).
(Sensor 2(

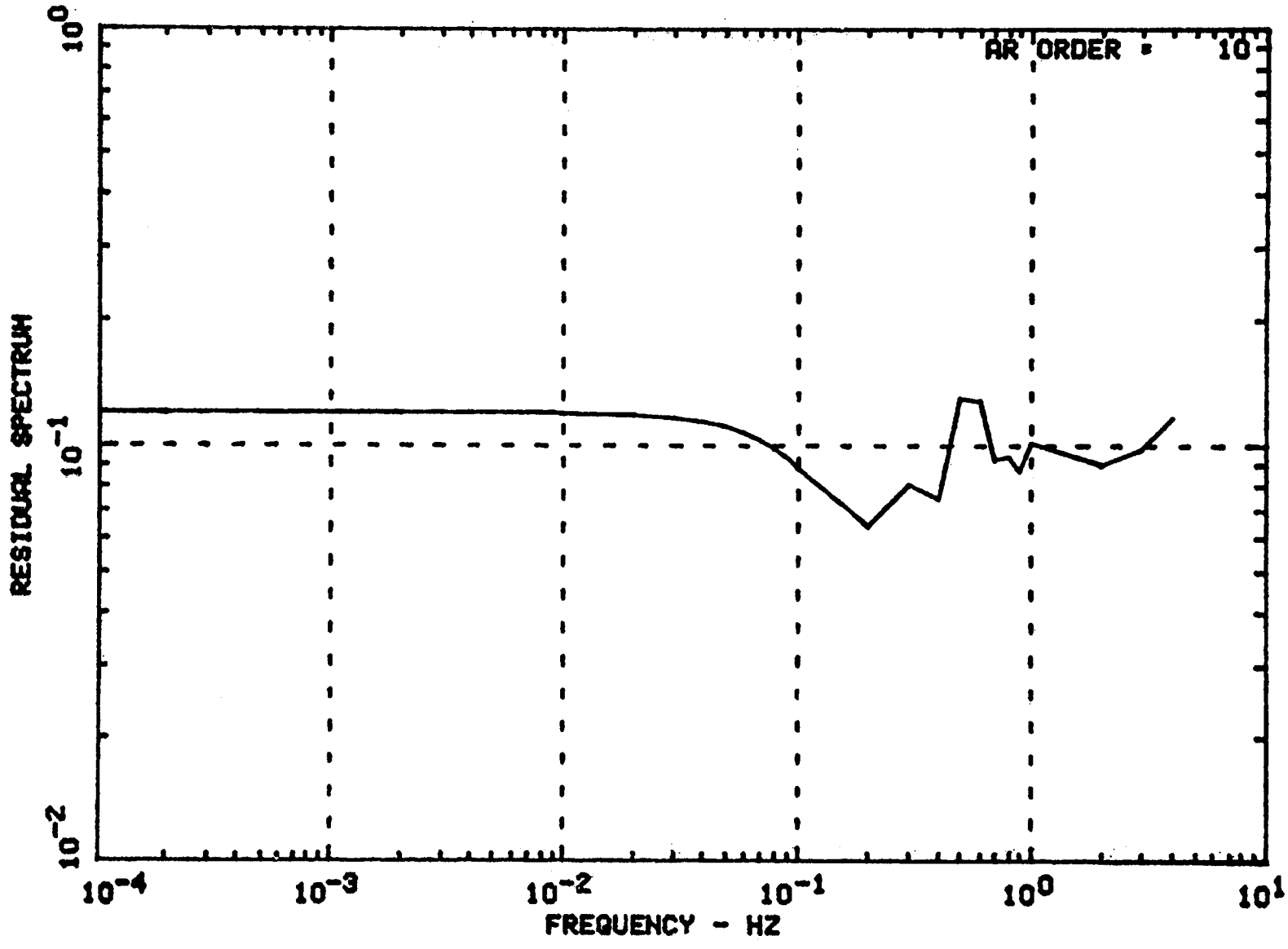


Figure C.20 St. Lucie -- Residual Power Spectrum (T07, Sensor 2).

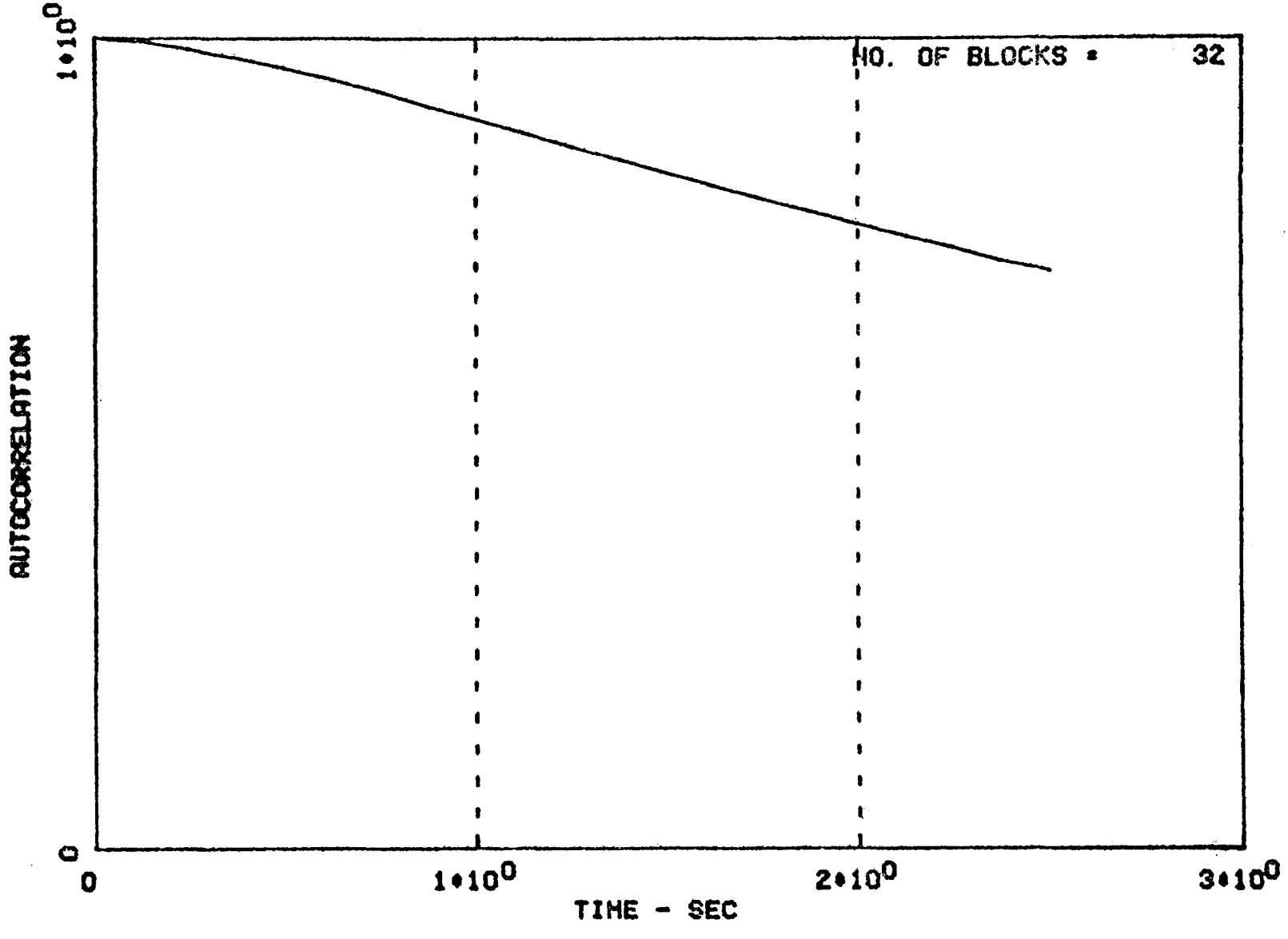


Figure C.21 Oconee - Signal Autocorrelation Function (T55). (Sensor 1)

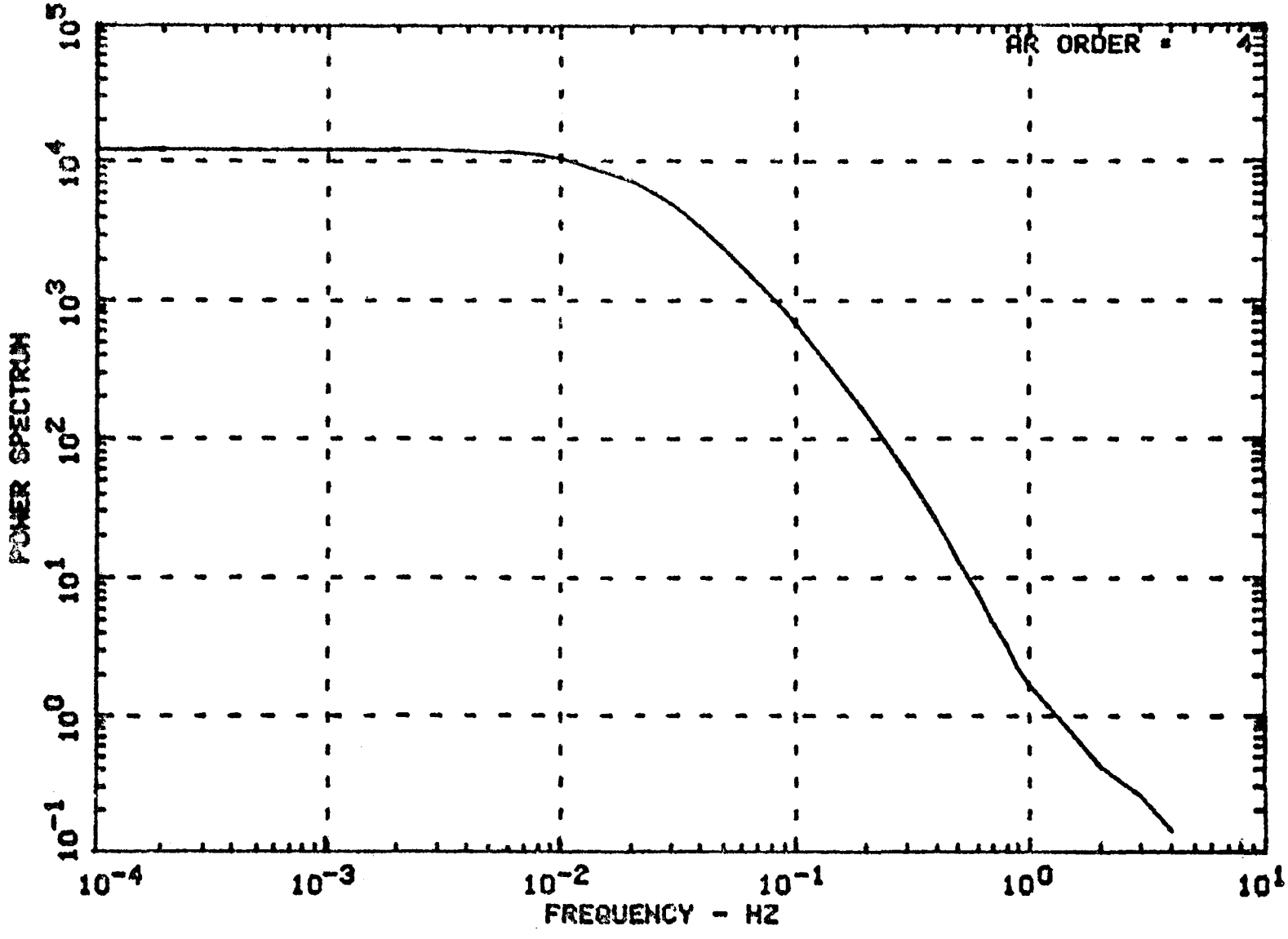


Figure C.22 Oconee - AR Power Spectrum (T55). (Sensor 1)

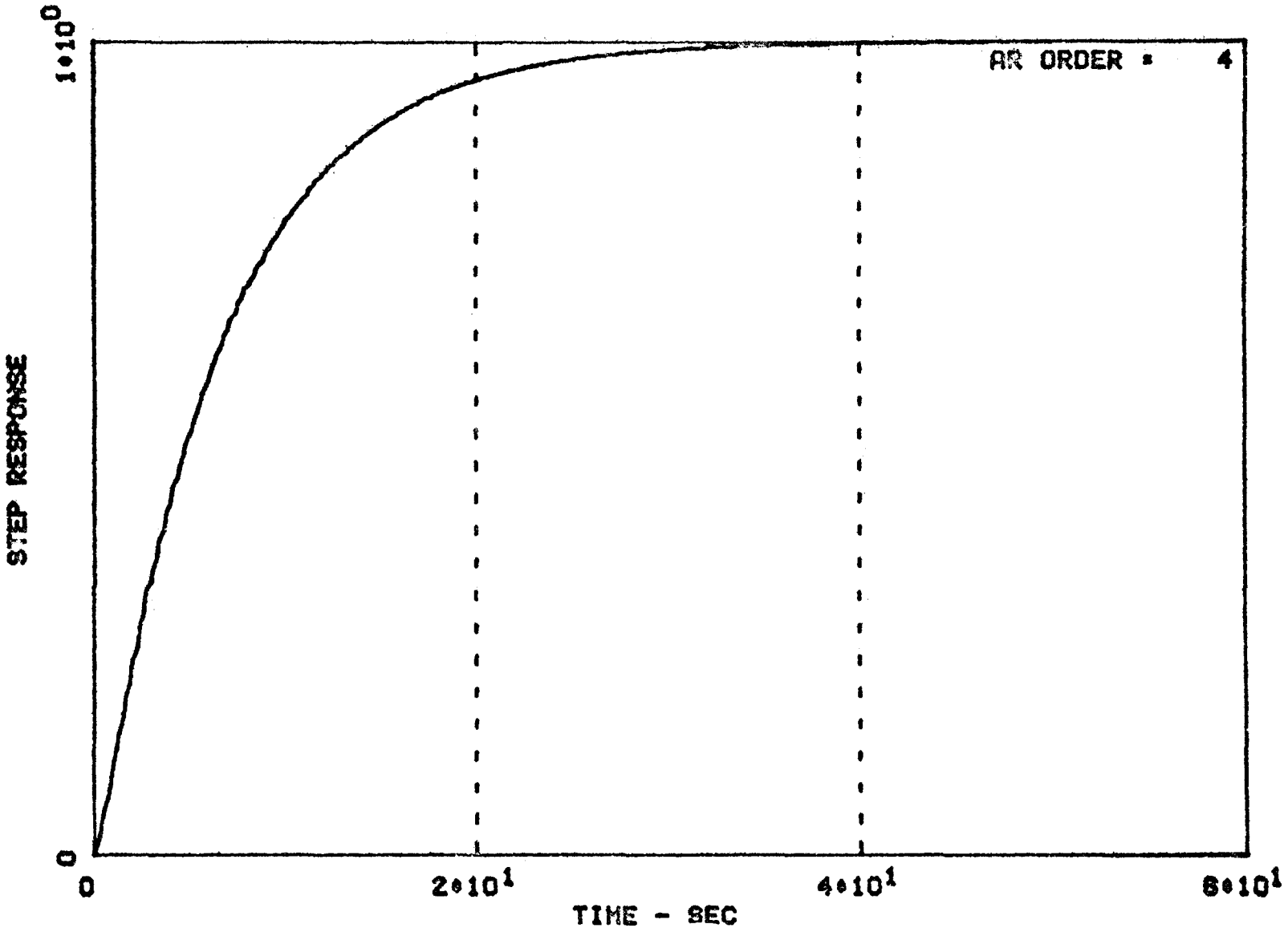


Figure C.23 Oconee - Step Response from AR Model (T55). (Sensor 1)

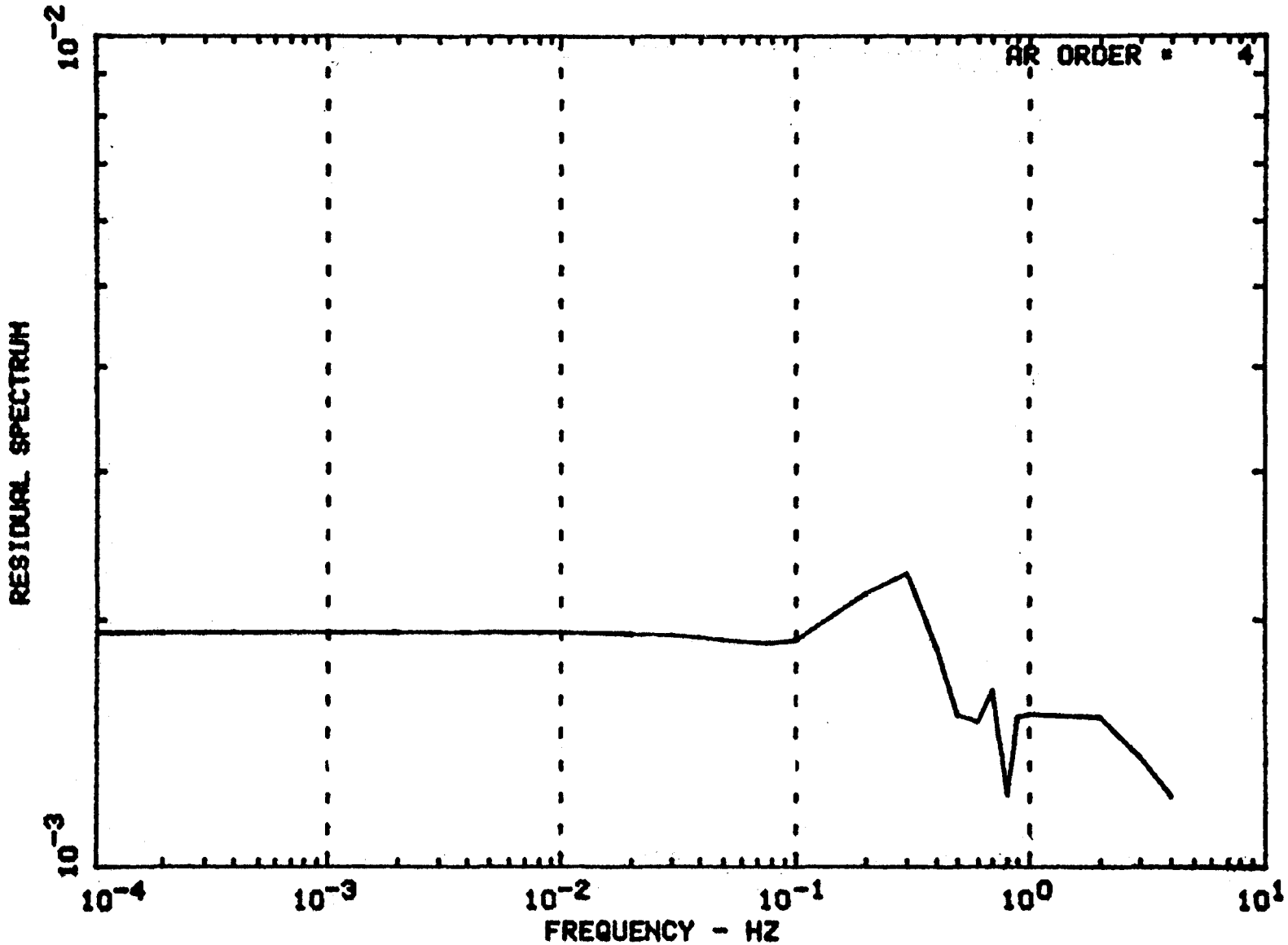


Figure C.24 Oconee 1 -- Residual Power Spectrum (T55, Sensor 1).

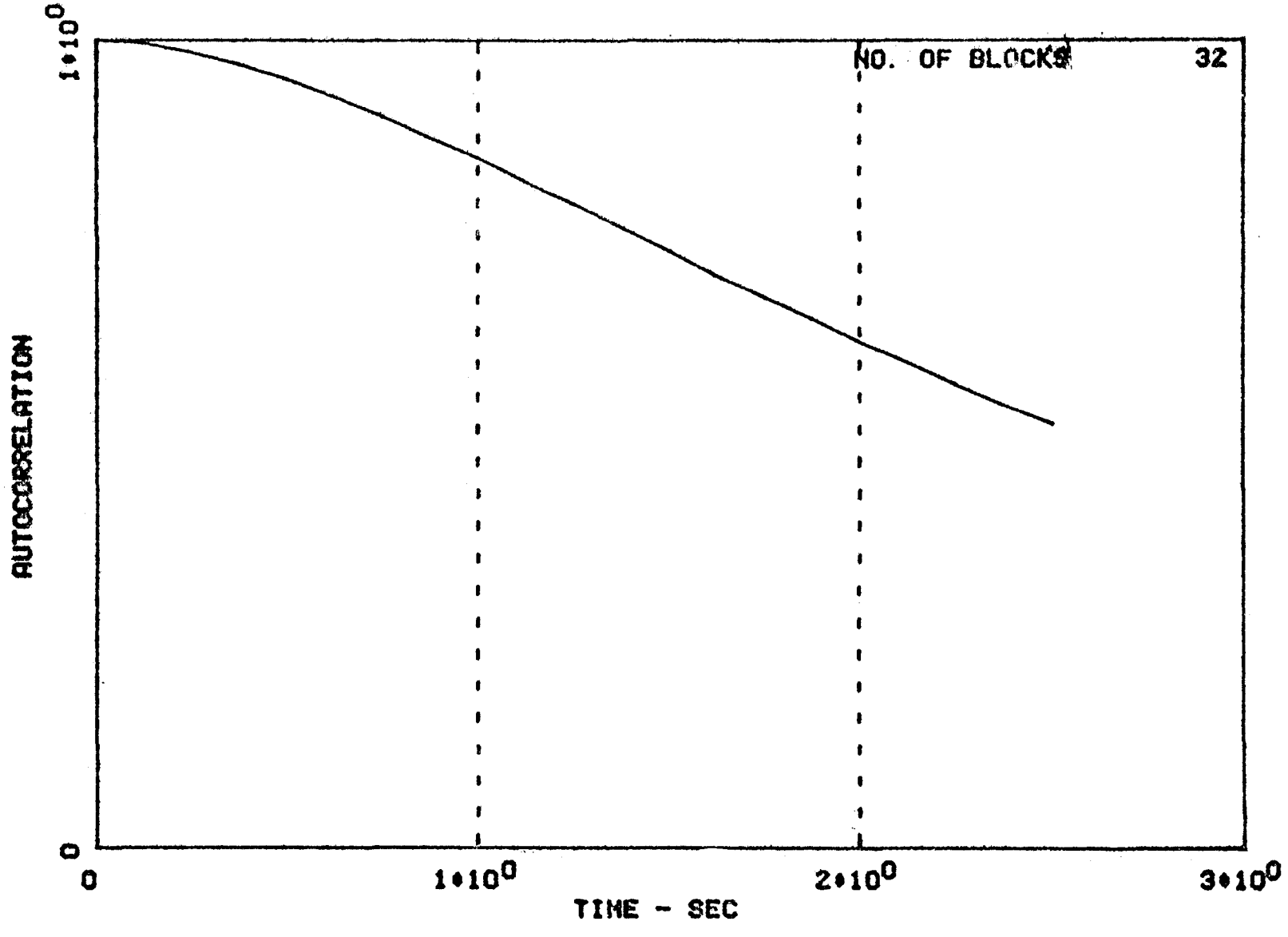


Figure C.25 Oconee - Signal Autocorrelation Function (T55). (Sensor 2)

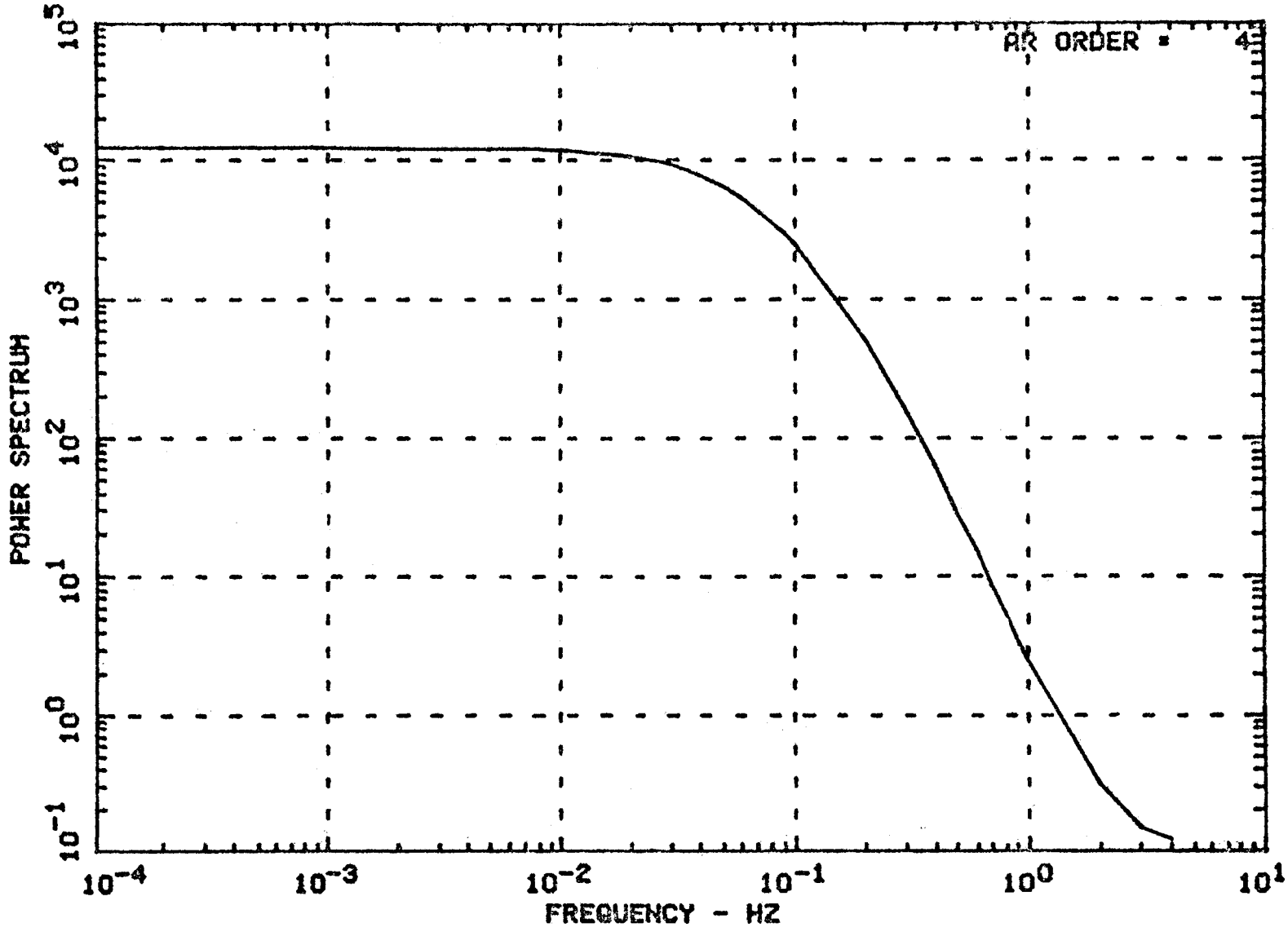


Figure C.26 Oconee - AR Power Spectrum (T55). (Sensor2)

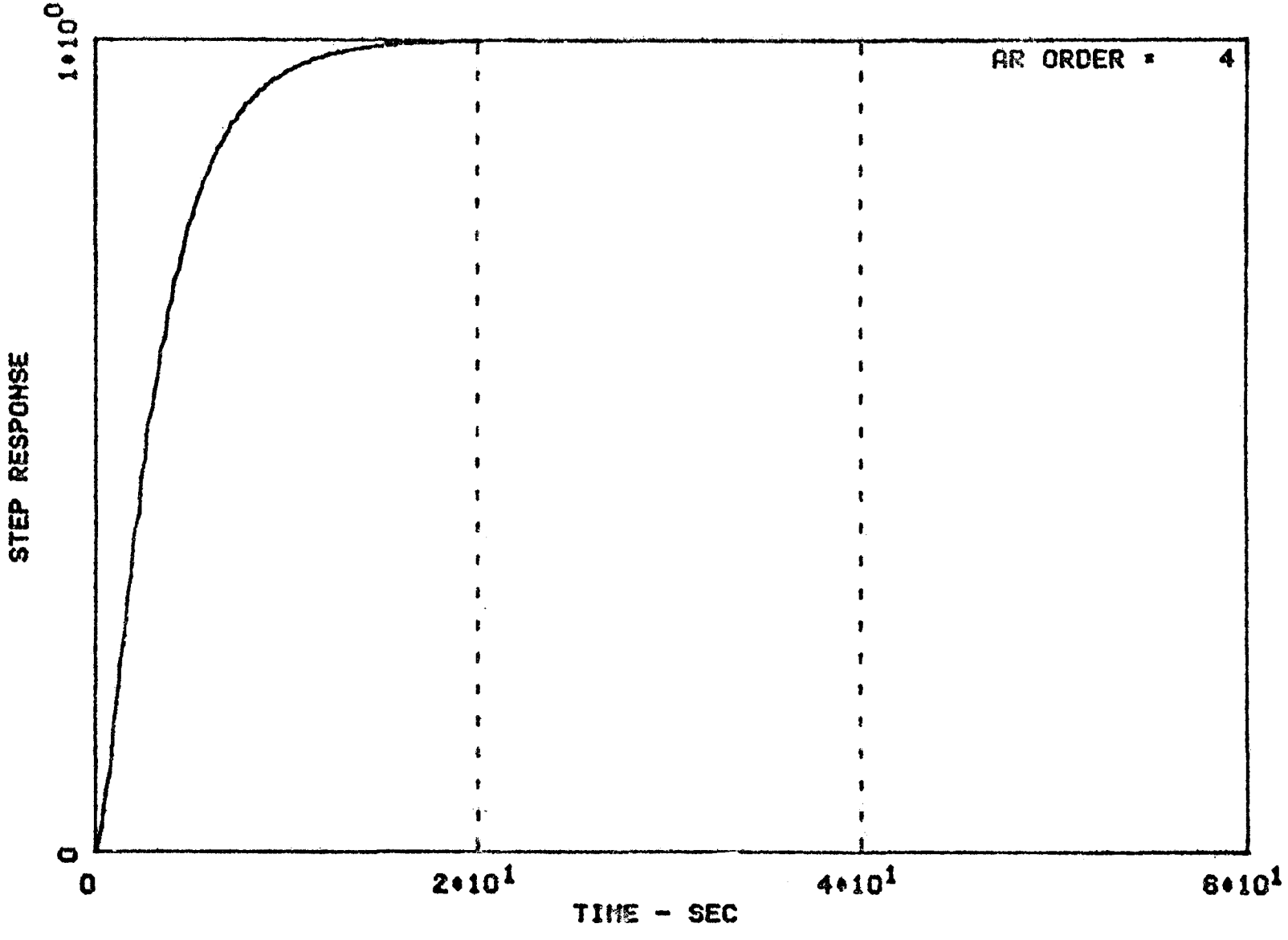


Figure C. 27 Oconee - Step Response from AR Model (T55). (Sensor 2)

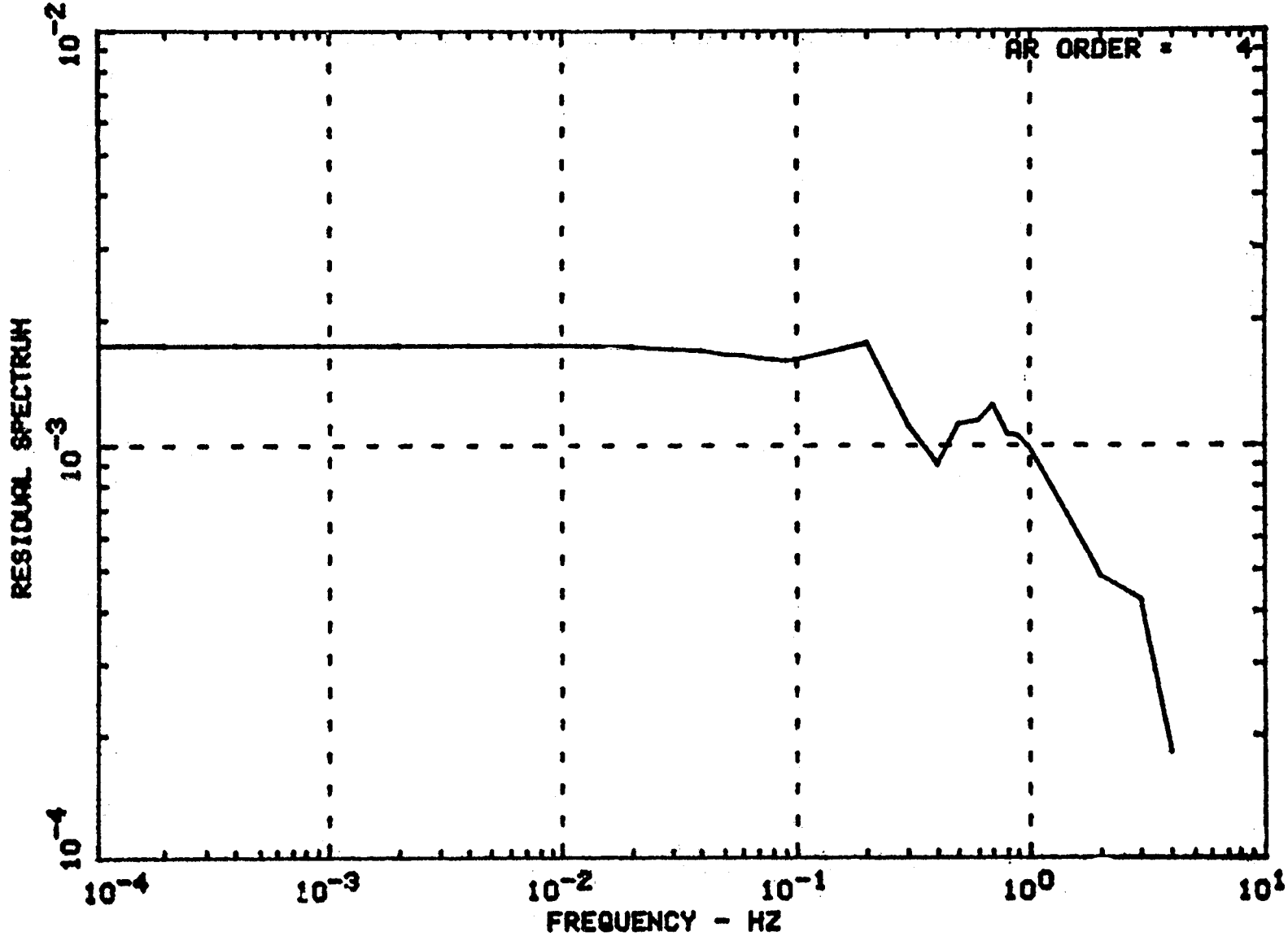


Figure C.28 Oconee 1 -- Residual Power Spectrum (T55, Sensor 2).

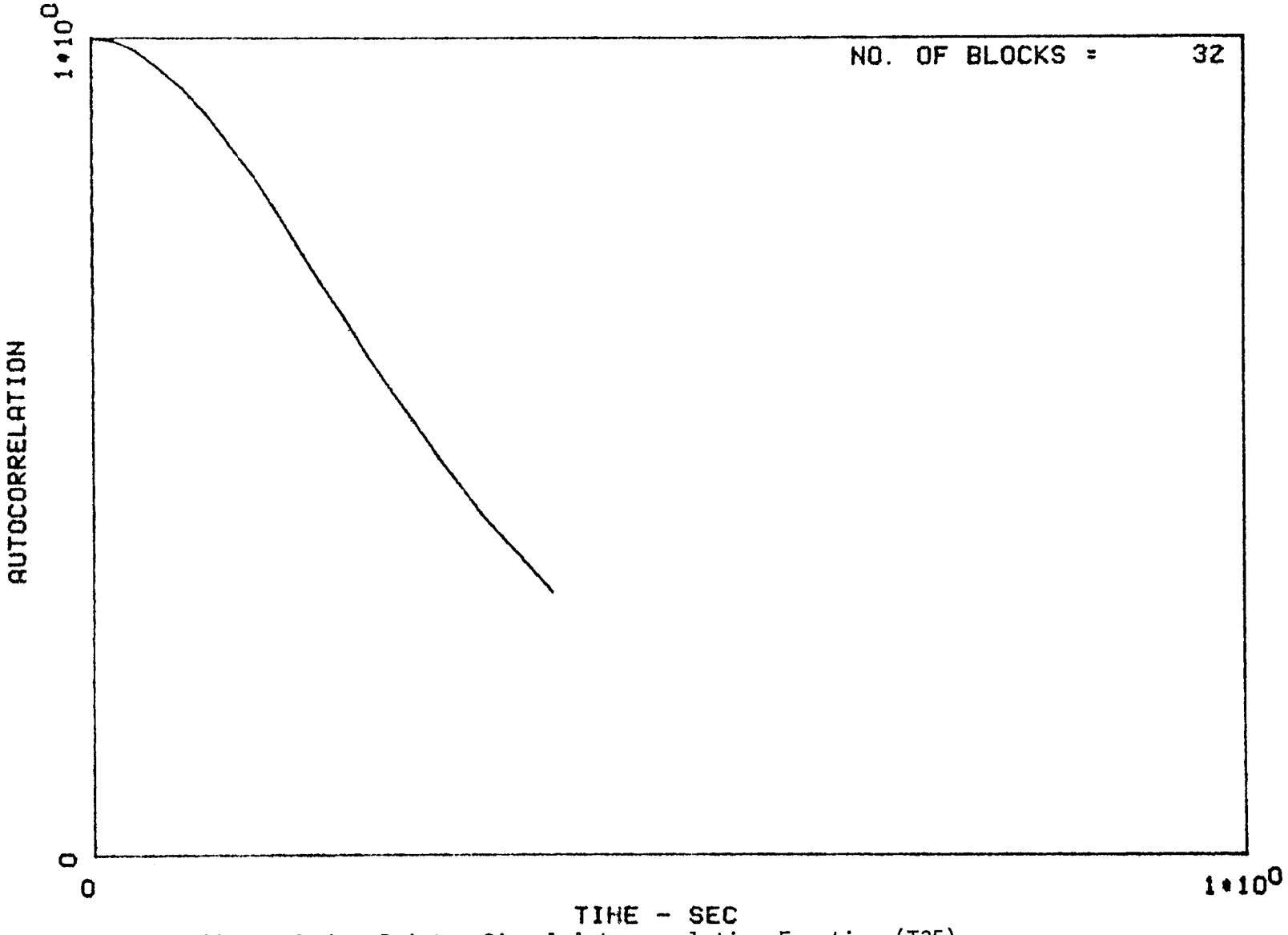


Figure C.29 Turkey Point - Signal Autocorrelation Function (T35). (Sensor 1)

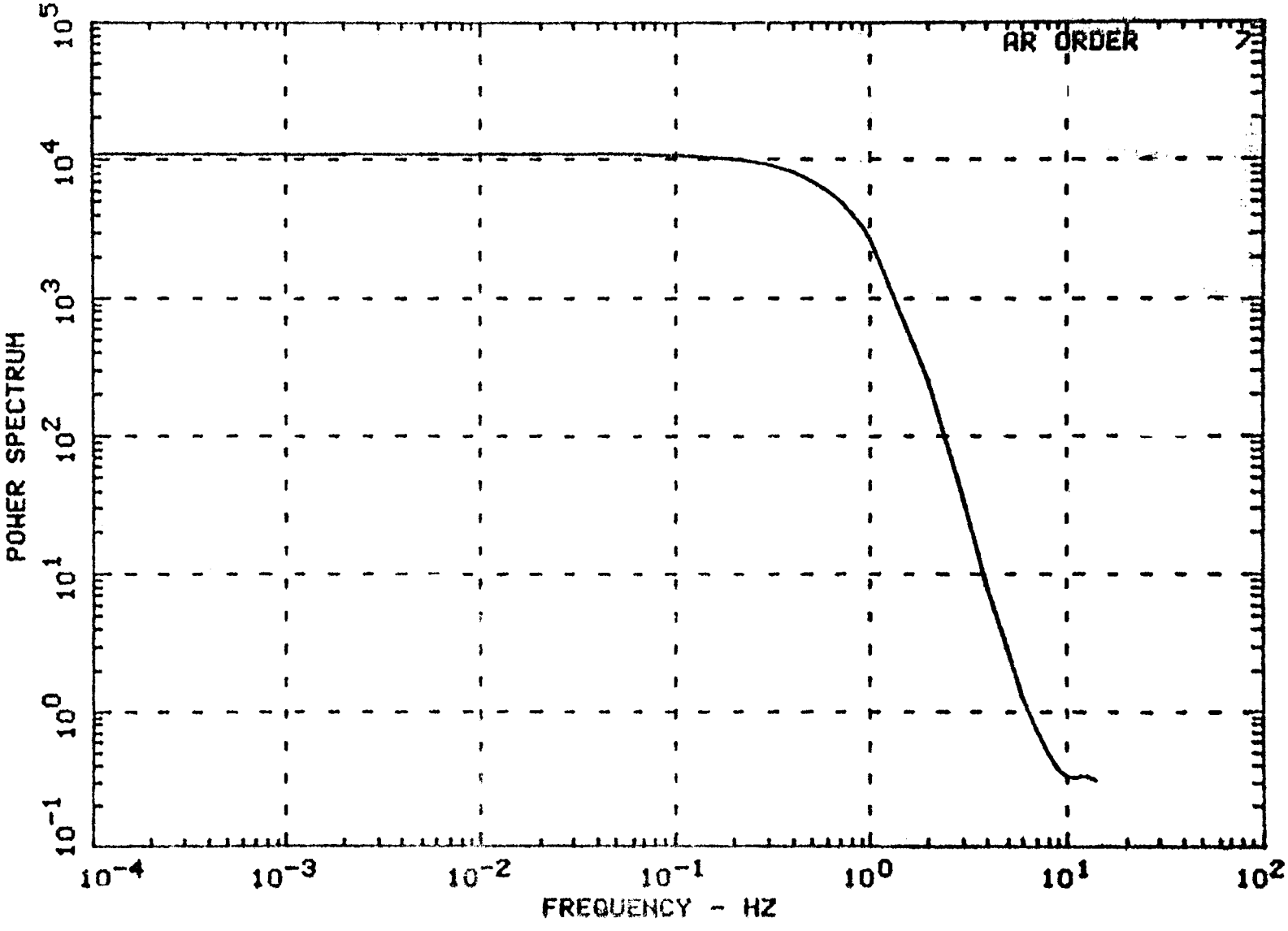


Figure C. 30 Turkey Point - AR Power Spectrum (T35). (Sensor 1)

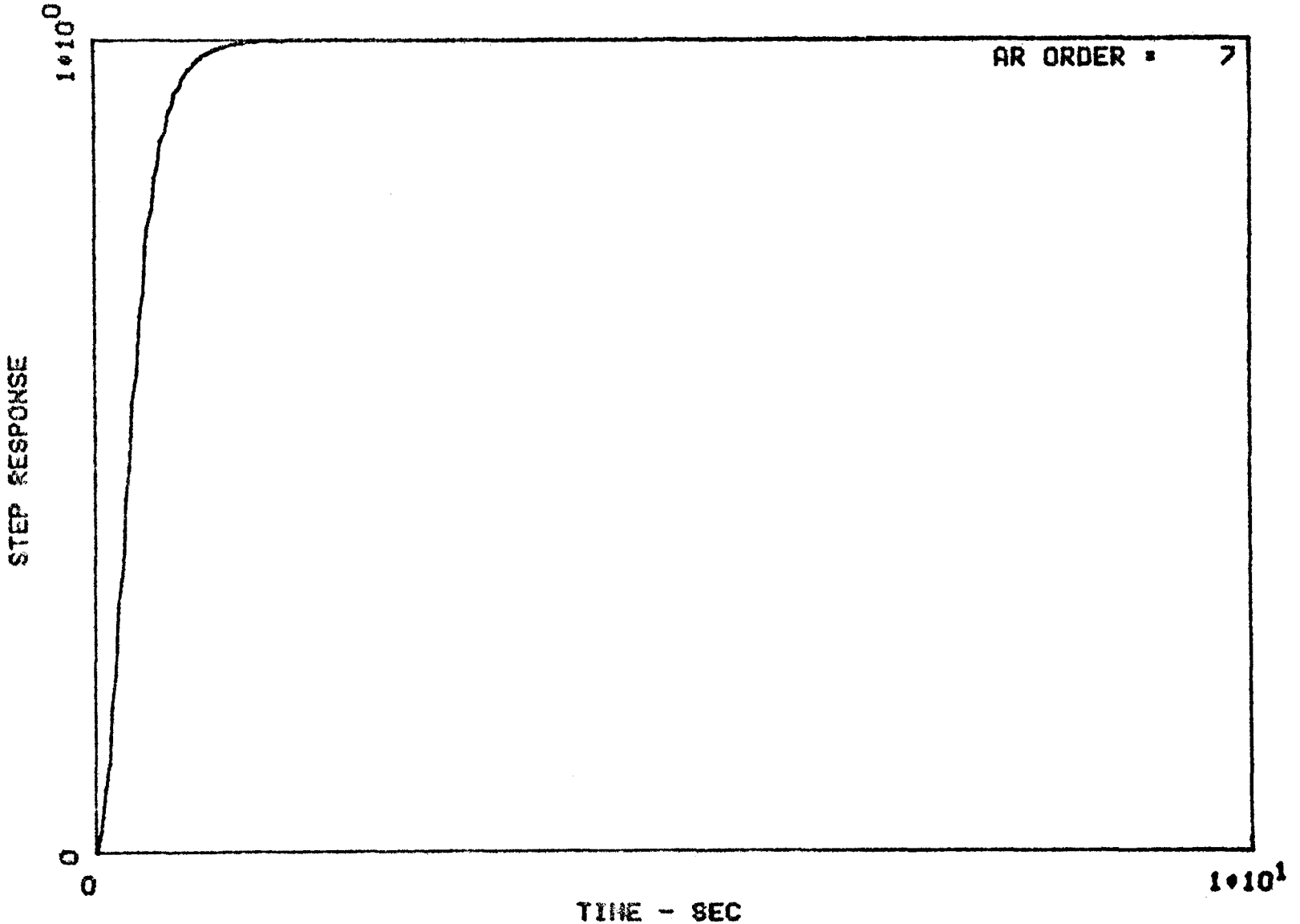


Figure C.31 Turkey Point - Step Response from AR Model (T35). (Sensor 1)

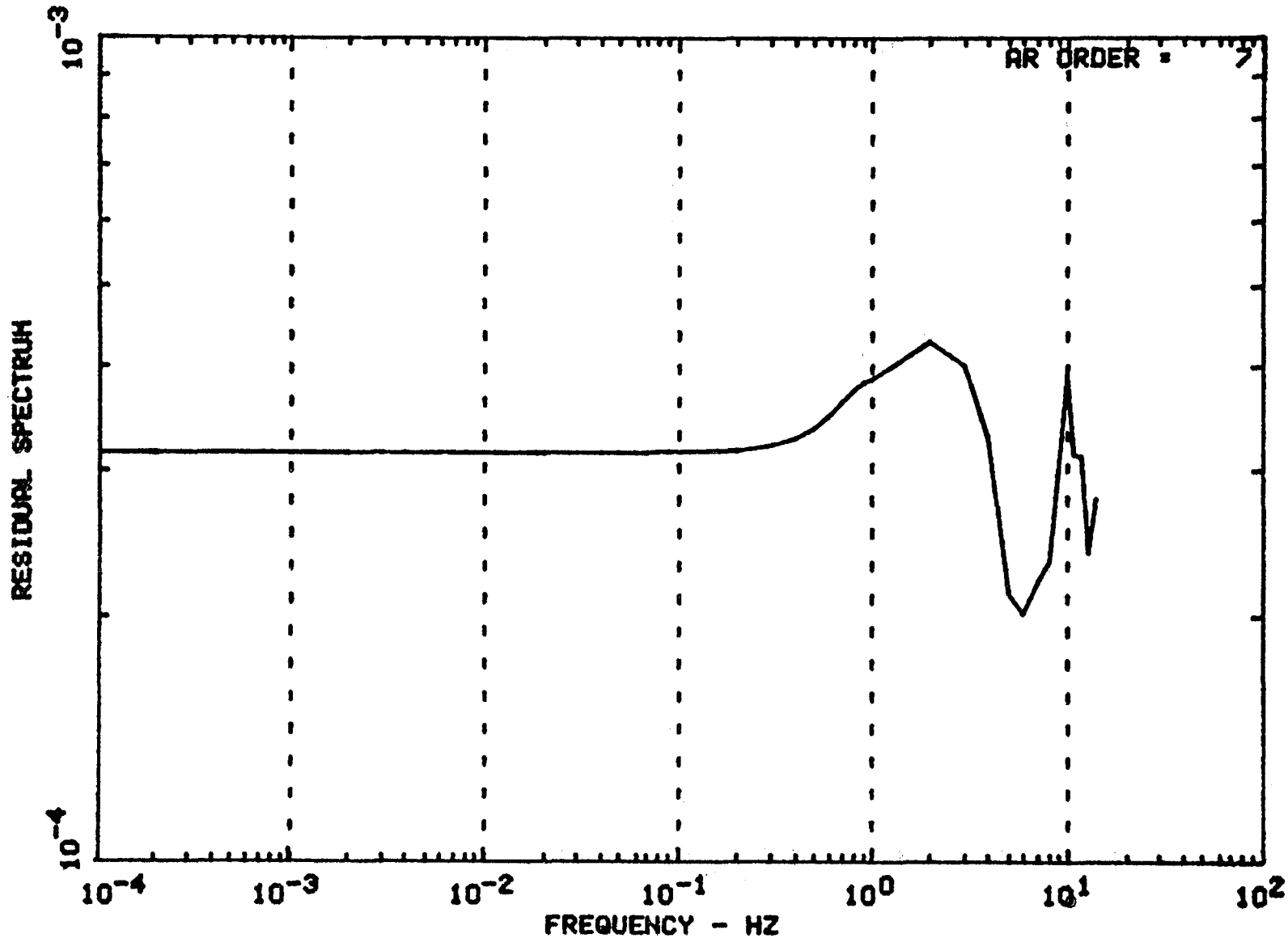


Figure C.32 Turkey Point -- Residual Power Spectrum (T35, Sensor 1).

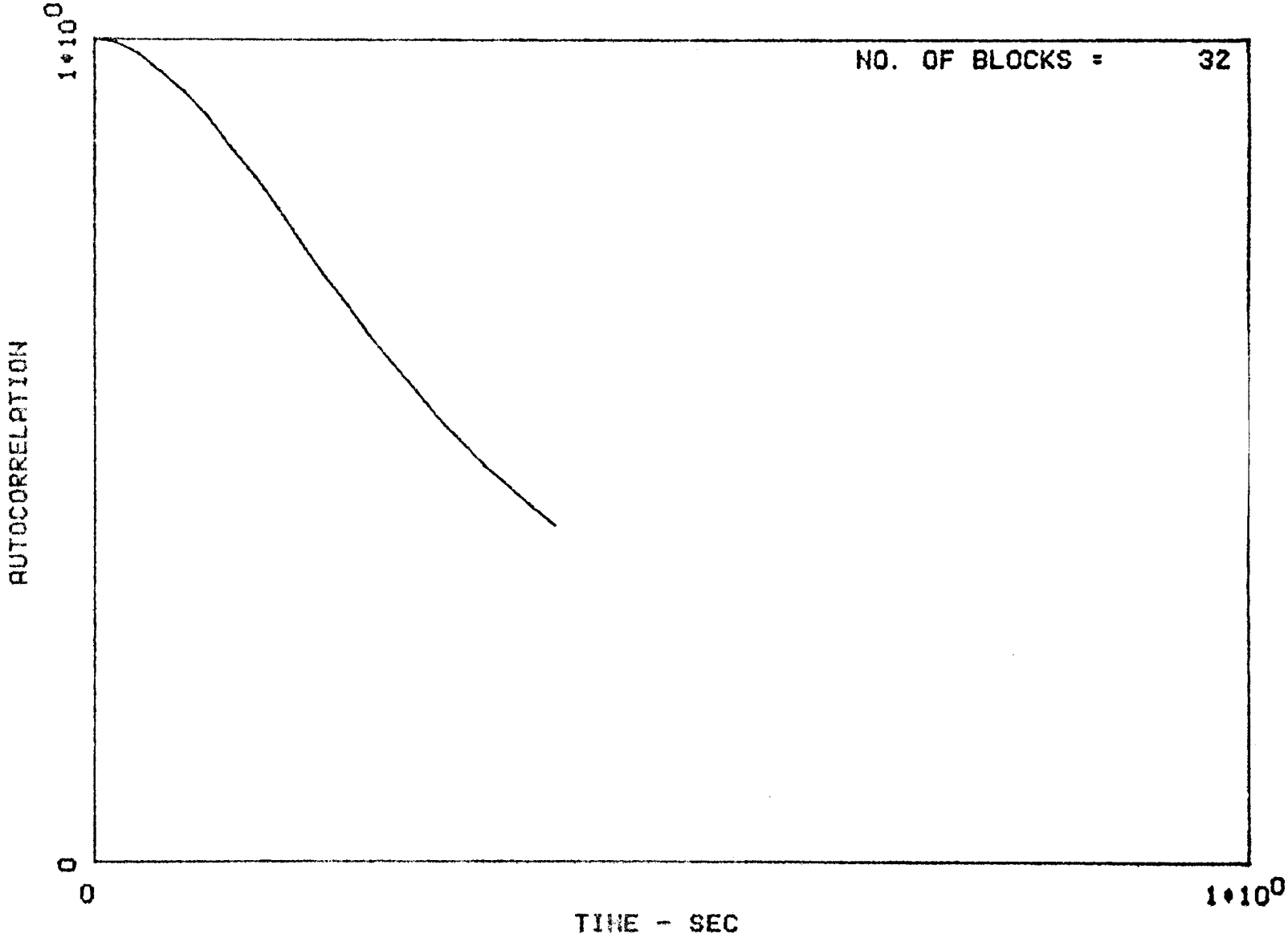


Figure C.33 Turkey Point - Signal Autocorrelation Function (T35). (Sensor 2)

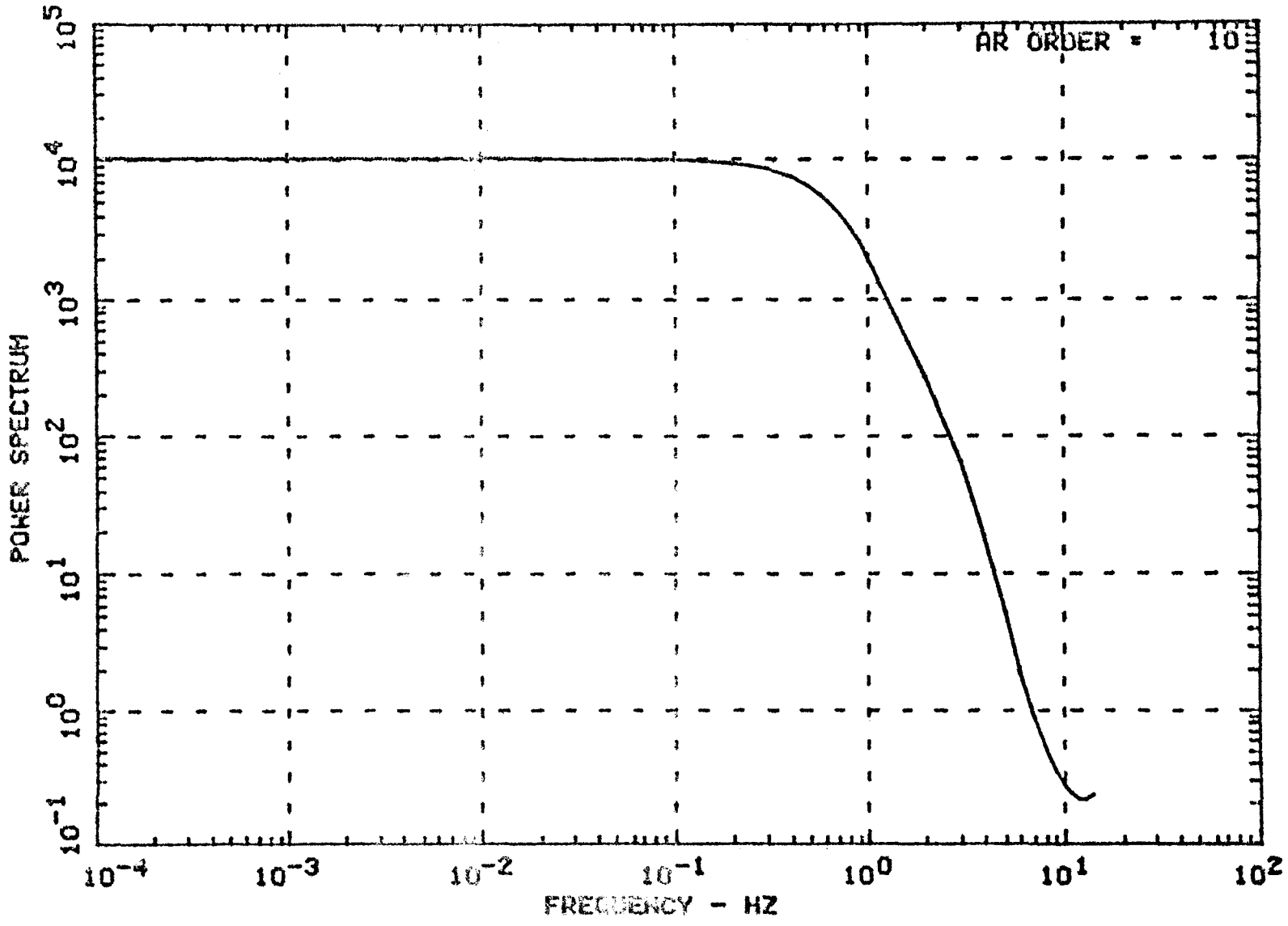


Figure C. 34 Turkey Point - AR Power Spectrum (T35). (Sensor 2)

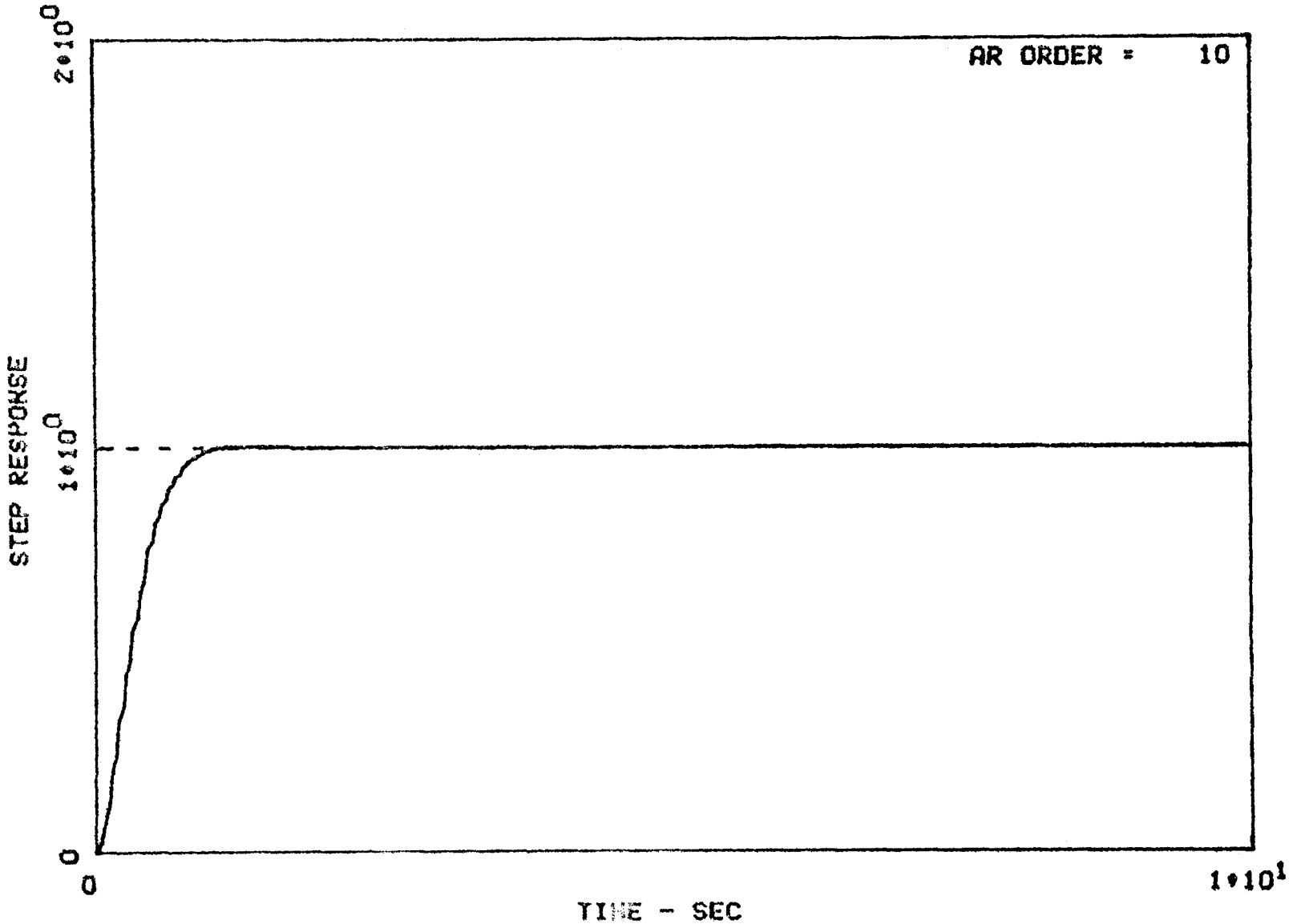


Figure C. 35 Turkey Point - Step Response from AR Model (T35). (Sensor 2)

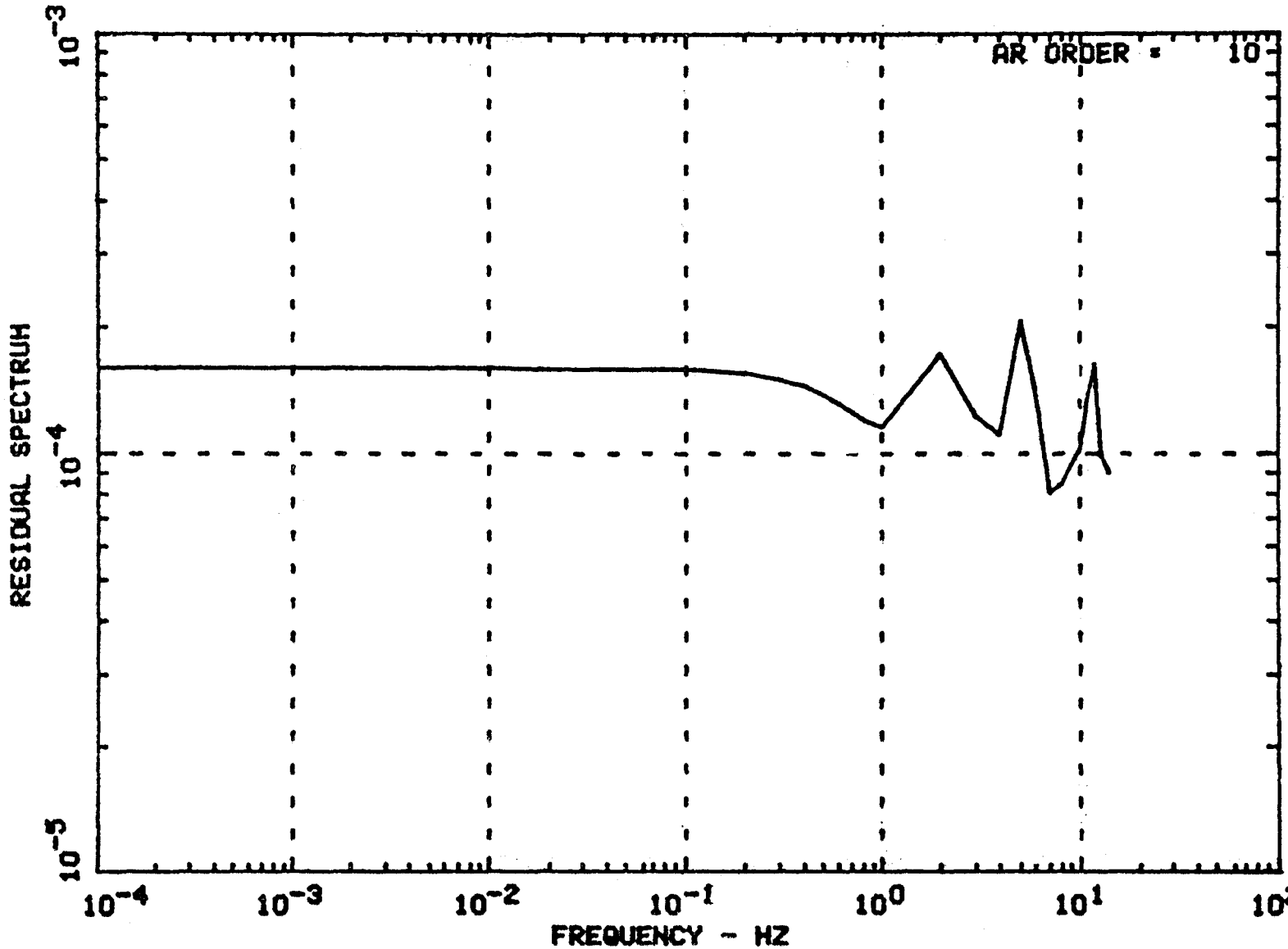


Figure C.36 Turkey Point -- Residual Power Spectrum (T35, Sensor 2).