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Resistive Drift-Alfvén Waves in Sheared Magnetic Fields*

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ABSTRACT

A new class of resistive drift-Alfvén modes is discovered. The new modes are believed to be the generalization of shear-Alfvén waves to sheared magnetic fields. In addition, the finite β (plasma over magnetic pressure) corrections to the electrostatic dissipative drift wave are evaluated.

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I. INTRODUCTION

In the sheared magnetic field of a tokamak, shear Alfvén waves in the vicinity of a rational surface are unable to satisfy $\omega^2 \simeq k_\parallel^2 v_A^2$ because the parallel wavevector $k_\parallel$ is proportional to x, the distance from the rational surface, while the Alfvén speed v_A is roughly constant. Because of this dependence of $k_\parallel$ on x the behavior of shear Alfvén waves in a sheared magnetic field has remained a mystery. Recently, some insight into the nature of resistive drift-Alfvén modes in a sheared magnetic field has been provided by the investigation of Hsu, et al.[1] Here, a general treatment is presented which permits the recovery of both a new class of resistive drift-Alfvén modes and the finite β corrections to the electrostatic dissipative drift wave. The new modes are believed to be the extension of shear Alfvén waves to sheared magnetic fields and in the absence of a density gradient satisfy $\omega^2 \approx k_\parallel^2(x=\rho_s)v_A^2$, where $\rho_s^2 = T_e\rho_i^2/T_i$ with ρ_i the ion gyroradius and T_e and T_i the electron and ion temperature. One of these shear Alfven modes becomes unstable in an inhomogeneous plasma for large enough β. However, the simple analytic results are limited to extremely small $\nu_e/\omega_\star$ for which the rapid spatial variation of the fields casts doubt upon the validity of a differential formulation for the unstable mode. Here, ν_e and $\omega_\star$ are the electron collision and electron diagmagnetic drift frequencies and $\beta = 4\pi N_e T_i/B^2$ with N_e the electron number density and B the magnitude of the magnetic field.

In the following section the relevant differential equation is solved by a method of matched asymptotic expansions and the appropriate eigenvalue equation obtained. In Section III a discussion of the subtleties of analyzing the eigenvalue equation is given and a proper analysis performed to recover a new set of drift-Alfvén modes. In addition, the retention of ion inertia in the solution of Section II allows the finite β corrections to the electrostatic dissipative drift wave to be recovered. In Section III these corrections are evaluated and result in a further stabilization of the drift mode. In the Appendix the derivation of the appropriate equations is summarized.

II. SOLUTION OF DIFFERENTIAL EQUATION

In order to generalize the work of Hsu et al.[1] to include the inertia of the ions, the techniques of Catto et al.[2] are combined with methods similar to those employed in Reference 1 in order to solve the coupled Poisson equation and parallel Ampere's law derived in the Appendix. Combining Eqs. (A2) and (A4) as indicated at the end of the Appendix, the equation to be solved is

$$\left\{\frac{E'}{\lambda\left[1-i(x_r/x)^2\right]^{-1} - \mu^2 x^2}\right\}' - \left(1 - \frac{\nu}{x^2}\right)E = 0 \ , \qquad (1)$$

where x is normalized to the ion gyroradius, prime denotes d/dx, and $E = -d\Phi/dx$ with Φ the potential. Equation (1) is obtained from the coupled Poisson and Ampere's system by assuming $d^2/dx^2 >> k^2$, where k is the poloidal wave vector, and by demanding that Φ be even in x about the rational surface at $x = 0$. In Eq. (1) the following definitions are employed:

$$\lambda = \frac{\omega - \omega_\star}{\omega\tau + \omega_\star} \ , \qquad \mu = \frac{L_n\omega_\star}{L_s\omega\tau}\left[\frac{\omega(\tau+1)}{\omega\tau+\omega_\star}\right]^{1/2} \ ,$$

$$\nu = \frac{\tau\beta L_s^2\,\omega(\omega\tau+\omega_\star)}{L_n^2\,\omega_\star^2} \ , \qquad x_r^2 = \frac{\tau m L_s^2\,\omega\nu_e}{M L_n^2\,\omega_\star^2} \ ,$$

with $\omega_\star = kcT_e/eBL_n$, $\tau = T_e/T_i$, L_s and L_n the shear and density scale lengths, and m and M the electron and ion masses. Equation (1) is derived from a number conserving Krook model by assuming that $\nu_e > \omega, k_\parallel v_e$ where v_e is the electron thermal speed.

In Ref. 1 the $\mu \to 0$ limit was treated. Here the $\mu^2 x^2$ term is retained so that the equation in the outer region, $x^2 > x_r^2$, becomes

$$\left(\frac{E'}{\lambda - \mu^2 x^2}\right)' - \left(1 - \frac{\nu}{x^2}\right)E = 0. \tag{2}$$

In the inner region, $x^2 < \lambda^{-1}$ and λ/μ^2, the equation is the same as for the inner region of Ref. 1,

$$\left\{\left[1 - i(x_r/x)^2\right] E'\right\}' + (\lambda\nu/x^2)E = 0. \tag{3}$$

However, rather than write the solution of Eq. (3) for dE/dx in terms of associated Legendre functions, Eq. (3) can be solved directly to obtain the series solution odd in x; namely,

$$E = A\,x^3 F(1 - \tfrac{\sigma}{2}, \tfrac{3}{2} + \tfrac{\sigma}{2}; \tfrac{5}{2}; x^2/i\,x_r^2), \tag{4}$$

where A is a constant,

$$\sigma = -\tfrac{1}{2} + \tfrac{1}{2}(1 - 4\lambda\nu)^{1/2}, \tag{5}$$

and F is the hypergeometric function,

$$F(a,b;c;t) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\,\Gamma(n + b)\,t^n}{\Gamma(n + c)\,n!}$$

with Γ a gamma function. For $x^2 > x_r^2$, a linear transformation for the hypergeometric function[3] may be employed to write E as

$$E = A\Gamma(5/2)x^3\left\{\frac{\Gamma(\sigma+\frac{1}{2})}{\left[\Gamma(\frac{\sigma}{2}+\frac{3}{2})\right]^2}\left[(x^2/x_r^2)\exp(i\pi/2)\right]^{-1+\frac{\sigma}{2}} F(1-\frac{\sigma}{2}, -\frac{1}{2}-\frac{\sigma}{2}; -\sigma+\frac{1}{2}; ix_r^2/x^2)\right.$$

$$\left. + \frac{\Gamma(-\sigma-\frac{1}{2})}{\left[\Gamma(1-\frac{\sigma}{2})\right]^2}\left[(x^2/x_r^2)\exp(i\pi/2)\right]^{-\frac{3}{2}-\frac{\sigma}{2}} F(\frac{3}{2}+\frac{\sigma}{2}, \frac{\sigma}{2}; \sigma+\frac{3}{2}; ix_r^2/x^2)\right\}. \quad (6)$$

In writing (6), $-\pi/2 < \arg\omega < 3\pi/2$ has been employed.

To solve Eq. (2) the method employed in Appendix B of Ref. 2 is used. Because both solutions are not given there, the essential details are presented in the next two paragraphs. Taking

$$E = \exp(-i\mu x^2/2)\sum_{n=0}^{\infty} a_n x^{2n+\delta} \quad (7)$$

and

$$\frac{E'}{\lambda - \mu^2 x^2} = \exp(-i\mu x^2/2)\sum_{n=0}^{\infty} b_n x^{2n+\delta-1} \quad (8)$$

and demanding that Eqs. (7) and (8) give the same E' yields

$$(2n+\delta)a_n - i\mu a_{n-1} = \lambda b_n - \mu^2 b_{n-1}, \quad (9)$$

while inserting Eqs. (7) and (8) into (2) gives

$$\nu a_n - a_{n-1} = -(2n+\delta-1)b_n + i\mu b_{n-1}. \quad (10)$$

For $n = 0$, Eqs. (9) and (10) give the indicial equation $\delta(\delta - 1) + \lambda\nu = 0$ having the roots

$$\delta_{\pm} = \frac{1}{2}\left| 1 \pm (1 - 4\lambda\nu)^{1/2} \right| = \begin{cases} \sigma+1 \\ -\sigma \end{cases} . \tag{11}$$

Eliminating a_{n-1} and b_{n-1} from Eqs. (9) and (10) results in

$$b_n = \frac{(2n + \delta) - i\mu\nu}{\lambda + i\mu(2n + \delta - 1)} a_n$$

which can be inserted into (9) or (10) to obtain

$$a_n = \frac{(i\mu)^n \left|2n + \delta - 1 + (\lambda/i\mu)\right| \Gamma(\delta + \frac{1}{2})\Gamma\left|n + \frac{1}{2}\delta - \frac{1}{4} -(i\mu\nu/4) + (\lambda/4i\mu)\right|}{n! \left[\delta - 1 + (\lambda/i\mu)\right] \Gamma(n + \delta + \frac{1}{2})\Gamma\left[\frac{1}{2}\delta - \frac{1}{4} - (i\mu\nu/4) + (\lambda/4i\mu)\right|} , \tag{12}$$

where unlike Ref. 2, Eq. (12) is valid for both $\delta_+ = \sigma + 1$ and $\delta_- = -\sigma$; that is, $a_n^{(\pm)} \equiv a_n(\delta = \delta_{\pm})$.

Writing

$$E = \exp(-i\mu x^2/2) \sum_{n=0}^{\infty} \left[a_n^{(+)} x^{2n+1+\sigma} + C a_n^{(-)} x^{2n-\sigma} \right] , \tag{13}$$

the constant C is determined from the condition that the wave be outgoing as $x \to \infty$. The result is

$$C = - \frac{\left|(\lambda/i\mu) - 1 - \sigma\right| \Gamma(\frac{3}{2} + \sigma)\Gamma\left|(\lambda/4i\mu) - (i\mu\nu/4) - \frac{1}{4} - \frac{1}{2}\sigma\right|}{(i\mu)^{\sigma+\frac{1}{2}} \left[\lambda/i\mu) + \sigma\right| \Gamma(\frac{1}{2} - \sigma)\Gamma\left|(\lambda/4i\mu) - (i\mu\nu/4) + \frac{1}{4} + \frac{1}{2}\sigma\right|} . \tag{14}$$

Matching the $x \to 0$ limit of the outer solution (13) to the $x \to \infty$ of the inner solution (6), results in two equations which can be combined to eliminate A and obtain the desired eigenvalue equation,

$$1 + \frac{\left[(\lambda/i\mu) - 1 - \sigma\right]\Gamma(\sigma + \frac{3}{2})\Gamma(\sigma + \frac{1}{2})\left[\Gamma(1 - \frac{\sigma}{2})\right]^2 \Gamma\left[(\lambda/4i\mu) - (i\mu\nu/4) - \frac{1}{4} - \frac{\sigma}{2}\right]}{(\mu x_r^2)^{\sigma+\frac{1}{2}}\left[(\lambda/i\mu) + \sigma\right]\Gamma(-\sigma + \frac{1}{2})\Gamma(-\sigma - \frac{1}{2})\left[\Gamma(\frac{\sigma}{2} + \frac{3}{2})\right]^2 \Gamma\left[(\lambda/4i\mu) - (i\mu\nu/4) + \frac{1}{4} + \frac{\sigma}{2}\right]} = 0 . \qquad (15)$$

Equation (15) can be shown to reduce to the result of Ref. 1 for $\mu \to 0$ or, more precisely, for $|\lambda/4\mu| > 1$. Equation (15) is appropriate as long as $\lambda x_r^2 < 1$ and $\lambda > \mu^2 x_r^2$ since these are the conditions for the validity of the matched asymptotic analysis. Finally, it is useful to note that the eigenvalue equation is unchanged if $\sigma \to -\sigma - 1$ so that only $\mathrm{Re}\sigma \geq -1/2$ need be considered.

III. ANALYSIS OF EIGENVALUE EQUATION

When $\mu \to 0$, Eq. (15) is valid provided $\lambda x_r^2 < 1$. For $\lambda x_r^2 \to 0$, the $\mu \to 0$ version of Eq. (15) requires that σ approach a half integer. Unfortunately, the matching fails for $\sigma = 1/2, 3/2, 5/2, \ldots$ since for such values of σ a seemingly higher order term in $(x_r/x)^2$ from $F(1 - \frac{\sigma}{2}, \frac{1}{2} - \frac{\sigma}{2};\ -\sigma + \frac{1}{2};\ ix_r^2/x^2)$ becomes the same order as the leading term from $F(\frac{3}{2} + \frac{\sigma}{2}, \frac{\sigma}{2};\ \sigma + \frac{3}{2};\ ix_r^2/x^2)$ in Eq. (6). If the $\sigma \to 1/2, 3/2, 5/2, \ldots$ limits of the $x \to \infty$ inner solution and the $x \to 0$ outer solution are written down taking account of this subtlety then it is found that the inner and outer solutions cannot be matched. Indeed, because the eigenvalue equation is unchanged if $\sigma \to -\sigma - 1$, the only remaining half integer value of σ to be considered is $\sigma = -1/2$. Fortunately, the case $\sigma \to -1/2$ does not suffer from the preceding difficulty because it is the case for which the higher order term and the leading term of $F(1 - \frac{\sigma}{2}, \frac{1}{2} - \frac{\sigma}{2}; -\sigma + \frac{1}{2}; ix_r^2/x^2)$ are the same. As a result, $\sigma \to -1/2$ will be shown to be the source of the new class of drift-Alfvén modes.

A simpler, but less general, way of seeing that $\sigma = 1/2, 3/2, 5/2,...$ are not permitted is to note that $\sigma = (2\ell - 1)/2$ with $\ell = 1, 2, 3,...$ results in $\lambda\nu = -(\ell^2 - 1/4) < 0$ where $\lambda\nu \propto \omega^2$ for $L_n \to \infty$. As a result, instability <u>appears</u> to be possible. However, if Eq. (1) is multiplied by E^*, integrated from $-\infty$ to $+\infty$, and integrated by parts assuming that $E'E^*|_{-\infty}^{\infty} = 0$, then for $\mu \to 0$ there results for $L_n \to \infty$ the expression

$$\int_{-\infty}^{\infty} dx\left\{\left[1 - i(x_r/x)^2\right]|E'|^2 + \tau^{-1}\left[1 - (\nu/x^2)\right]|E|^2\right\} = 0 .$$

Taking the real and imaginary parts shows that there can be no unstable or marginally stable modes for $L_n \to \infty$. Consequently, this simple, independent check also leads to the conclusion that $\sigma = 1/2, 3/2, 5/2...$ cannot be allowed.

Returning to the eigenvalue equation, letting $\sigma = -\frac{1}{2} + \varepsilon$ with $|\varepsilon| << 1$, and employing $(\mu x_r^2)^\varepsilon \to \exp(i2\pi n)$ as $\varepsilon \to 0$ with $n = \pm 1, \pm 2, \pm 3,...,$ reduces Eq. (15) to

$$\varepsilon = \sigma + \frac{1}{2} \simeq i2\pi n\left\{\ell n(i/\mu x_r^2) - \psi\left[(\lambda/4i\mu) - (i\mu\nu/4)\right] - 2\left[(\lambda/i\mu) - \frac{1}{2}\right]^{-1} - 4\gamma - 2\psi(5/4)\right\}^{-1}, \tag{16}$$

where $\psi(t) = (d/dt)\ \ell n\Gamma(t)$ and γ is Euler's constant. Equation (16) is not valid for $n = 0$ because for $\sigma \equiv -1/2$ the inner and outer solutions do not match, as can be verified by substitution into the $\sigma \to -1/2$ limits of Eqs. (6), (13), and (14).

To lowest order Eq. (16) gives $\sigma = -1/2$, resulting in $\lambda v = 1/4$ or

$$\omega_0/\omega_* = \frac{1}{2}\left\{1 \pm \left[1+(L_n^2/\tau\beta L_s^2)\right]^{1/2}\right\} . \tag{17}$$

Equation (17) is believed to be the lowest order dispersion relation for shear Alfvén waves in a sheared magnetic field. In order to obtain the higher order corrections to ω_0, the limit $|\lambda/4\mu| > 1$ is considered in order to simplify Eq. (16). Using $\lambda\nu \simeq 1/4$, $\psi(t) \sim \ell n\ t$ for $t > 1$, and $\psi(5/4) = 4 - \gamma - (\pi/2) - 2\ell n4$, Eq. (16) gives $\varepsilon \simeq i2\pi n\left[\ell n(i10\nu/x_r^2)\right]^{-1}$, so that writing $\omega = \omega_0 + \omega_1$ yields

$$\frac{\omega_1}{\omega_*} = \pm \frac{4\pi^2 n^2 L_n^2/\tau\beta L_s^2}{\left[1 + (L_n^2/\tau\beta L_s^2)\right]^{1/2}\left\{\ell n\left[(i10\beta M/m\nu_e)(\omega_0\tau + \omega_*)\right]\right\}^2} , \tag{18}$$

where ω_0 is given by (17) and the upper and lower signs correspond to those of Eq. (17). Consequently, if $\omega_0\tau + \omega_* > 0$ the lower sign in Eq. (17) can result in an instability with a growth rate

$$\mathrm{Im}\omega_1/\omega_* = \frac{2\pi^3 n^2 L_n^2/\tau\beta L_s^2}{\left[1 + (L_n^2/\tau\beta L_s^2)\right]^{1/2}\left\{\ell n^2\left[(10\beta M/m\nu_e)(\omega_0\tau + \omega_*)\right] + \pi^2/4\right\}} , \tag{19}$$

where $\omega_0/\omega_* = \frac{1}{2}\left\{1 - \left[1 + (L_n^2/\tau\beta L_s^2)\right]^{1/2}\right\}$. The instability occurs whenever β exceeds a critical value given by the condition that $\omega_0\tau + \omega_* > 0$; namely,

$$(\tau + 1)\beta > \tau L_n^2/4L_s^2 \equiv \beta_{crit} . \tag{20}$$

Equation (19) is expected to be appropriate whenever $|\lambda x_r^2| < 1$ and $|\lambda/4\mu| > 1$ so that using $\lambda\nu \simeq 1/4$ gives the inequalities

$$\frac{m\nu_e}{4M(\omega_0\tau + \omega_*)} < \beta < \frac{L_n\omega_*}{16L_s\left|\omega_0(\tau + 1)(\omega_0\tau + \omega_*)\right|^{1/2}} \tag{21}$$

For $\tau \sim 1$ the right portion of inequality (21) leads to the conclusion that the assumption $|\lambda/4\mu| > 1$ is appropriate as long as $\beta << 1$. Because the derivation of Eq. (1) is valid only when compressional Alfvén waves are neglected ($\beta << 1$), there is no need to consider the $\mu\nu >> 1$ limit of the $\psi\left[(\lambda/4i\mu) - (i\mu\nu/4)\right]$ function in Eq. (16). Consequently, the $\mu \to 0$ limit employed by Hsu et al.[1] is capable of recovering this shear Alfvén mode. However, the $\mu \to 0$ limit is not appropriate for $\beta \to 0$ as will be shown at the end of this section.

It is important to realize that even though the growth rates of the radial eigenmodes appear to increase as n^2, the assumption $|\omega_1| < \omega_0$ severely restricts the range of validity of Eq. (19). For $n = 1$, Eq. (19) is valid only if $(10\beta M/m)(\omega_*/\nu_e) \gtrsim \exp(4\pi) \simeq 3 \times 10^5$, that is, for extremely small values of ν_e/ω_*. Of course, Eq (15) is valid more generally, and so can be employed to follow this mode for arbitrary ν_e/ω_*. Whether instability persists for a larger range of parameters is presently being investigated by numerical solutions of Eq. (1) and Eq. (15), and will be considered in a later paper. The important point for the moment is that a new class of drift-Alfvén modes has been found. These new drift-Alfvén modes are believed to be the form a shear Alfvén wave takes in a sheared magnetic field. Note that for $L_n \to \infty$, Eq.(17) reduces to two identical modes of opposite phase velocities, $\omega^2 \simeq k_{\parallel}^2(x = \rho_s) v_A^2$. The shear Alfvén wave given by the upper sign of Eq. (17) appears to be related to the mode observed numerically by Tsang et al.[4]

Finally, it must be noted that in this $\mu \to 0$ limit the asymptotic form of E goes like $\exp(-x\lambda^{1/2})$. Because the original differential equation, Eq. (1), is derived under the assumption that $E(x)$ varies on a spatial scale larger than an ion gyroradius, the description presented here for $\mu \to 0$ is strictly valid only if $|\lambda^{1/2}| \lesssim 1$. For the upper sign in Eq. (17), $|\lambda| < 1$ can be satisfied for most parameters, while for the lower sign $\tau > 1$ and $|\omega| > \omega_*$ is required. However, both the condition for instability, Eq. (20), and $\tau > 1, |\omega| \gtrsim \omega_*$ cannot be satisfied for the lower sign of (17). As a result, for an unstable $\sigma = -1/2$ mode $|\lambda| \gtrsim 1$, with $|\lambda|$ approaching unity only for $|\omega| < \omega_*, \omega_*/\tau$. Consequently, the unstable modes may require an integral formulation in order to decide upon their validity. But because the analysis for the unstable modes is restricted to extremely small ν_e/ω_*, Eq. (15) is presently being solved numerically to determine whether the modes remain unstable for larger ν_e/ω_* and if so, $|\lambda|$ can be evaluated to see whether it is less than unity.

In order to obtain the mode that reduces to the dissipative drift wave[5,6] for $\beta \to 0$, Eq. (15) can be analyzed by considering $|\sigma| < 1$. Letting

$$(\lambda/4i\mu) - (i\mu\nu/4) + (1/4) + (\sigma/2) = \varepsilon \qquad (22)$$

with $|\varepsilon| << 1$, and neglecting terms of $O(\varepsilon\sigma)$, ε is found to be

$$\varepsilon = -\frac{1}{2}(\pi\mu x_r^2)^{1/2} \frac{\lambda}{\lambda - i\mu} . \tag{23}$$

For $\beta \to 0$, these two equations can be solved for $\lambda/i\mu$. Rejecting the extraneous which will not satisfy $|\varepsilon| << 1$, and recalling that Eq. (15) is only valid for $\mu x_r^2 < 1$ gives $\lambda/i\mu \simeq -1 - (\pi\mu x_r^2)^{1/2}$ so that the result of Guzdar et al.[5] is recovered. For $(\mu x_r^2)^{1/2} \sim \mu\nu < 1$, Eqs. (22) and (23) can be solved by taking $\lambda = -i\mu$ to lowest order. The resulting expression is $\lambda/i\mu = -1 + i\mu\nu + 2\mu^2\nu^2 + (\pi\mu x_r^2)^{1/2}$ or using $\omega = \omega_*$ except in λ ,

$$\frac{\omega - \omega_*}{\omega_*} = \frac{-i(\tau + 1)L_n}{\tau L_s}\left[1 + i\beta(\tau + 1)\frac{L_s}{L_n} + \frac{2\beta^2(\tau + 1)L_s^2}{L_n^2} + \left(\frac{\pi m L_s \nu_e}{M L_n \omega_*}\right)^{\frac{1}{2}}\right] \tag{24}$$

Consequently, the finite β corrections stabilize the mode still further. In addition, for $x_r \to 0$, Eq. (22) with $\varepsilon = 0$ and the full σ retained is a legitimate solution of Eq. (15). Solving $\lambda/i\mu - i\mu\nu + (1 - 4\lambda\nu)^{1/2} = 0$ gives $\lambda/i\mu = -1 - i\mu\nu$ so that shear stabilization appears to remain effective for all β . The other radial eigenmodes can be recovered by letting $\varepsilon \to \varepsilon - n$ in Eq. (22), with $n = 1,2,...,$ and redoing the analysis of Eq. (15).

Finally, it should be noted that there are no other obvious values of σ to expand about in analyzing Eq. (15) since seemingly higher order terms in x_r^2/x^2 in Eq. (6) become important whenever $\mathrm{Re}\sigma \geq \frac{1}{2}$. Consequently, only the range $-\frac{1}{2} < \mathrm{Re}\sigma < \frac{1}{2}$ need be considered.

IV. DISCUSSION

The eigenvalue equation for resistive drift-Alfvén modes is derived. A careful analysis of the eigenvalue equation reveals the existence of a new set of resistive drift-Alfvén modes. These new modes are believed to be the extension of shear Alfvén waves to a sheared magnetic field. One of these shear Alfvén modes can become unstable in the presence of a density gradient for large enough β. However, the simple analytic results are limited to extremely shall ν_e/ω_* for which the rapid spatial variation of the fields calls into question the validity of the differential formulation for the unstable mode. Larger values of ν_e/ω_* are being investigated numerically. Finally, the retention of ion inertia in the analysis allows the finite β corrections to the electrostatic dissipative drift wave to be evaluated. They are found to result in further stabilization.

APPENDIX

In order to derive Eq. (1) the unperturbed distribution function F is taken to be of the form

$$F = N(\Psi)(M/2\pi T)^{3/2}\exp(-Mv^2/2T),$$

with $\Psi = \psi + \Omega^{-1}\underset{\sim}{v}\times\hat{\underset{\sim}{n}}\cdot\nabla\psi$, $\hat{\underset{\sim}{n}} = \underset{\sim}{B}/B$, $\underset{\sim}{v} = v_{\parallel}\hat{\underset{\sim}{n}} + v_{\perp}(\hat{\underset{\sim}{\psi}}\cos\phi + \hat{\underset{\sim}{n}}\times\hat{\underset{\sim}{\psi}}\cos\phi)$, and $2\pi\psi$ the poloidal flux so that the poloidal magnetic field $\underset{\sim}{B}_p = R^{-1}\underset{\sim}{\zeta}\times\nabla\psi$ with ζ the toroidal angle and R the distance from the axis of symmetry to the point of interest ($|\nabla\zeta| = 1/R$). This form for F is appropriate to lowest order when the collision frequency is greater than the bounce frequency and magnetic drifts are neglected. For simplicity the concentric magnetic coordinates r, θ, ζ will be employed, where r is the distance from the magnetic axis to the point of interest and θ is the poloidal angle. In this system $R = R_o + r\cos\theta$ $\underset{\sim}{B} = (B_oR_o/R)[\hat{\underset{\sim}{\zeta}} + (\varepsilon/q)\hat{\underset{\sim}{\theta}}]$ with R_o the distance from the axis of symmetry to the magnetic axis, $\varepsilon = r/R_o$, and q the safety factor. The distinction between $\hat{\underset{\sim}{n}}$ and $\hat{\underset{\sim}{\zeta}}$ will not be retained in the following.

Assuming that the perturbed vector potential may be written as $\underset{\sim}{A} = A_{\parallel}\hat{\underset{\sim}{n}}$ because only $\beta \ll 1$ are of interest, seeking solutions of the form $\exp(-i\omega t + i\underset{\sim}{k}\cdot\underset{\sim}{r})$, defining the perturbed distribution function f via

$$f = g - (ZeF/cT)(c\Phi + v_{\star}A_{\parallel})$$

with $v_{\star} = (cRT/ZeN)\partial N/\partial\psi$, and employing the gyrokinetic[7] variables $\underset{\sim}{R} = \underset{\sim}{r} + \Omega^{-1}\underset{\sim}{v}\times\hat{\underset{\sim}{n}}$ and $\underset{\sim}{v}$, then the gyro-averaged equation for g becomes

$$(\omega - k_\parallel v_\parallel)g = (ZeF/cT)(\omega - \omega^*)(c\Phi - v_\parallel A_\parallel)J_0(k_\perp v_\perp/\Omega)$$
$$+ \left\langle C_1 + (Ze/cT)(c\Phi + v_\star A_\parallel)C_0 \right\rangle \quad \text{(A1)}$$

where C_0 and C_1 are the unperturbed and perturbed Fokker-Planck collision operators, $\langle \ldots \rangle = (2\pi)^{-1}\oint d\phi(\ldots)$, and $\omega^* = (m/qR_0)v_\star$ with m the poloidal mode number.

The perturbed current $J_\parallel$ can be formed from Eq. (A1) in the usual fashion by integrating over all $\underset{\sim}{v}$, multiplying by Ze, and summing over all species. Combining the result with the parallel Ampere's law $k_\perp^2 A_\parallel = (4\pi/c)J_\parallel$ gives

$$x(A_\parallel'' - bA_\parallel) = \alpha(\Phi'' - b\Phi) \quad \text{(A2)}$$

where x is normalized to the ion gyroradius ρ_i, $b = (k\rho_i)^2$, $k = m/r$, and $\alpha = \nu k_\parallel' c/\omega$ with $k_\parallel' = k/L_s$ and ν as defined in the text. In obtaining (A2), $k_r^2 \to -\partial^2/\partial x^2$ is employed.

In order to form the Poisson equation a number conserving Krook model is employed in Eq. (A1), namely, $C_1 + (Ze/cT)(c\Phi + v_\star A_\parallel)C_0 = -\nu_e[g - (F/N)\int d^3vg]$. Inserting the Krook model, returning to the $\underset{\sim}{r}$ variable, and carrying out the ϕ integral in $d^3v = d\phi d\, v_\parallel\, dv_\perp\, v_\perp$ gives

$$[1 -(i\nu_e/N)\int d^3vF(\omega + i\nu_e - k_\parallel v_\parallel)^{-1}]\int d^3vg =$$
$$(2\pi Ze/cT)(\omega - \omega^*)\int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} dv_\perp \frac{v_\perp F J_0^2(c\Phi - v_\parallel A_\parallel)}{\omega + i\nu_e - k_\parallel v_\parallel} \quad , \quad \text{(A3)}$$

where $\nu_e \equiv 0$ for the ions. Carrying out the integrations for the electrons and ions, taking $\nu_e >> |k_\parallel|v_e$, $\nu_e >> |\omega| >> |k_\parallel|v_i$, $k\rho_i < 1$, and $k\rho_e \to 0$

where v_e and v_i are the electron and ion thermal speeds and ρ_e is the electron gyroradius; and inserting the resulting expressions into the quasi-neutrality conditions gives

$$\Phi'' - b\Phi = \{\lambda[1 - i(x_r/x)^2]^{-1} - \mu^2 x^2\}[\Phi - (\omega A_\parallel/k_\parallel c)] \ , \tag{A4}$$

where $k_\parallel = k_\parallel' x$ and the definitions in the text are employed; again $k_r^2 \to -\partial^2/\partial x^2$ is used.

For $\partial^2/\partial x^2 >> b$, Eq. (A2) becomes $xA_\parallel'' = \alpha\Phi''$ which can be integrated (from x = 0 to x) once to obtain $(A_\parallel/x)' = (\alpha/x^2)\Phi'$ for Φ even and $A_\parallel$ odd. Dividing Eq. (A4) by $\lambda[1 - i(x_r/x)^2]^{-1} - \mu^2 x^2$ and differentiating, $(A_\parallel/x)'$ may be eliminated to obtain Eq. (1).

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