SLAC-PEP-NOTE--321

DE89 006350

TRANSVERSE INSTABILITY EXCITED BY R.F. DEFLECTING MODES FOR PEP

A. W. Chao and C. Y. Yao*

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

We have looked at the possible transverse instability effects which are caused by the deflecting modes of the R.F. cavities in PEP. The results are obtained by applying the expression of the instability damping (or anti-damping if $\tau_{\mu m}^{-1} < 0$) rate:

$$\tau_{\mu m}^{-1} = \frac{3 N e^2 c}{4\pi T_0 E_0 \nu_\beta} \sum_{k=-\infty}^{\infty} R_\perp \left(3k + \mu + \nu_\beta + m\nu_s\right) J_m^2 \left[(3k + \mu + \nu_\beta + m\nu_s - \frac{\xi}{\alpha}) a\right] \quad (1)$$

where we have assumed that there are three equal bunches equally spaced in PEP. Symbols are defined in Table 1; μ and m are mode numbers. Derivation of Eq. (1) will be given in Appendix IV.

In Appendix I, we have worked out the equivalent of Eq. (1) for a single bunch beam. The analysis follows that of Sacherer's[1]. The effect of chromaticity ξ is included as a frequency shift in the bunch mode spectra. In Appendix II, we will rewrite this result in terms of the transverse wake field instead of the impedance.

We include in Appendix III an application of the Sacherer formalism to the case of resistive wall. The resulting expression of the damping rate contains two terms. The first term corresponds to the effect of the short wake fields; it agrees with the result of the head-tail instability as derived by Sands.[2] A numerical estimate of this resistive-wall head

---

* Visiting scientist from the University of Science and Technology, Hofei, China

tail case for PEP is given at the end of Appendix III. It re-confirms that the resistive wall instability is not a serious problem for PEP. The second term gives the effect of long wake fields and it agrees with the result of Courant & Sessler[3]. In particular, if $\xi = 0$ and $m = 0$ (rigid dipole mode) the stability criterion is $n < \nu_\beta < n + \frac{1}{2}$ for some integer n. In case $\xi \neq 0$ and $m \neq 0$, we find that the stability criterion is $n < \nu_\beta + m\nu_s < n + \frac{1}{2}$.

In Appendix IV we give a derivation of Eq. (1) for a three-bunch beam. The head-tail type of instability is pronounced for a broad-band impedance for which the wake field decays before the next bunch arrives; the instability growth rate is sensitive to the chromaticity ξ. For the narrow-band impedances such as the ones we will consider, the instability is more sensitive to ν_β than to ξ and in the following we will ignore the head-tail effect by setting $\xi = 0$.

For a high-Q deflecting mode of an rf cavity, we have the impedance[10]

$$R_\perp \left(\frac{\omega}{\omega_o}\right) = \frac{R_s \quad \omega_R / \omega}{1 + Q^2 \left(\frac{\omega}{\omega_R} - \frac{\omega_R}{\omega}\right)^2} \tag{2}$$

where R_s, ω_R and Q are the peak value in unit of ohms per meter, the resonant frequency and the Q-value of the impedance, respectively. This impedance has two peaks located around $+\omega_R$ and $-\omega_R$. The half-width around each of the peaks is about $\Delta\omega = \omega_R/2Q$. If this width is narrower than the revolution frequency ω_o i.e., $\Delta\omega < \omega_o$, the only significant contribution in the summation of Eq. (1) occurs if $3k + \mu + \nu_\beta + m\nu_s$ is equal to $\pm\,\omega_R/\omega_o$ within a range of $\Delta\omega/\omega_o$.

If $3k + \mu + \nu_\beta + m\nu_s$ is equal to $+\,\omega_R/\omega_o$ with a relative error $\delta(|\delta| \leq 1/2Q)$, the impedance (2) contributes to a damping of the (μ, m)

mode with a damping rate

$$\tau_{\mu m}^{-1} \approx \frac{3N e^2 c}{4\pi T_o E_o \nu_\beta} R_s \frac{F_m}{1 + 4Q^2\delta^2} \qquad (3)$$

$$F_m = \frac{1}{2^{2m}(m!)^2} \left(\frac{\omega_R}{\omega_o} a\right)^{2m}$$

where we have assumed $\frac{\omega_R}{\omega_o} a \ll 1$, i.e. the wavelength of the transverse wake field is much longer than the bunch length. We have used the small argument approximation of the Bessel fur[illegible]n: $J_m(x) \approx x^m/2^m m!$ if x [illegible]. The form factor F_m decreases rapidly wi[illegible] increasing m and suppresses the effect of the impedance on modes with higher values of m. In the later discussions, we will consider therefore only the cases m=0 and m=1.

If $3k + \mu + \nu_\beta + m\nu_s$ is equal to $-\omega_R/\omega_o$ with a relative error of δ ($|\delta| < 1/2Q$), we find that the (μ,m) mode is anti-damped with a growth rate whose magnitude is the same as the damping rate predicted from Eq. (3).

If we do not exactly know the frequency spectrum of all the rf deflecting modes, we may have to obtain the stability criterion by statistical considerations.[4] If the real part of impedance integrated over all PEP rf cavities contains $\bar{N}$ impedance peaks typically of height $\bar{R}_s$, Q-value $\bar{Q}$ and resonant frequency $\bar{\omega}_R$, the damping/anti-damping rate of a given mode (μ,m) is estimated to be roughly

$$\left|\tau_{\mu m}^{-1}\right| \sim \frac{3Ne^2 c}{4\pi T_o E_o \nu_\beta} \frac{\bar{R}_s}{\bar{Q}} F_m \sqrt{\frac{2\bar{N}}{3}} \frac{\bar{\omega}_R}{\omega_o} \qquad (4)$$

One may try to avoid the instabilities by tuning the cavity or changing the betatron tune ν_β. But since stability of the beam requires all modes with different values of μ and m be stable simultaneously, this may not be easy to achieve in practice. To get some idea, we show this in the following by a semi-quantitative example.

The impedance of one of the rf cavity cell designs to be used in PEP has been measured by Perry Wilson[5]. It consists of $\sim$ 23 narrow impedance peaks somewhat evenly distributed in the frequency ranging from $\sim$ 600 MHz (corresponding to the lowest transverse rf mode) to $\sim$ 2 GHz (corresponding to the vacuum chamber cut-off). We assume that the impedance of the other cavity cells have at least roughly the same number of impedance peaks; each peak has about the same peak value and resonance width but the location of each of those resonance peaks is more or less randomly shifted by as much as ± 3%. We have ignored the impedance beyond cut-off.

To obtain the total rf cavity impedance for PEP, we thus take the impedance of the rf cavity cell measured by Wilson and randomly shift the location of each of the 23 resonance peaks by up to ± 3% (keeping the peak value and width unchanged) and call that the impedance for another cavity cell. The justification of this procedure lies in: (i) the 5 cells composing a PEP cavity station are coupled; therefore, the impedance of a cavity is not five times the impedance of a single cell, but is more like the impedance of the single cell with each of the impedance peak splitted into five peaks; and (ii) there are 8 different rf cell designs to be used in PEP, each having slightly different geometry and thus slightly different locations for the impedance peaks from the others.

The above mentioned procedure of generating the rf cell impedance is first repeated 5 x 8 = 40 times. Then since there are 3 cavity stations for each cell design, the height of each of the 23 x 40 impedance peaks is increased by a factor of 3. In order to take into account of the possible construction errors of the cavity cells, we have taken a slightly reduced Q-value of 5000 for all the impedance peaks. The total rf cavity impedance is then the sum of all the impedance thus obtained.

RESULTS

When this total impedance is substituted into Eq. (1) we obtain the results shown in Figs. 1 (for m=0) and 2 (for m=1) for 4 GeV. The damping (anti-damping if negative) rate is plotted against the betatron tune ν_β for various modes (μ,m). In these calculations, we have used a beam current of 5 mA/bunch and an rms bunch length of 2.7 cm. Using these parameters and $\overline{N} = 800$, $\overline{Q} = 5000$, $\overline{R}_s = 5$ MΩ/m and $\overline{f}_R = 1$ GH$_z$, Eq. (4) gives $|\tau^{-1}| \approx 450$ sec^{-1} for m=0 and 70 sec^{-1} for m=1, in rough agreement with (but slightly more pessimistic than) Figs. 1 and 2. The rigid dipole modes (m=0) can be comfortably damped by a bunch-to-bunch feedback system; the required feedback damping rate would be about 1/4 of the design capability of the PEP feedback system[6] at 4 GeV. The m=1 modes, which can not be easily handled by feedback system and have typically anti-damping rates greater than the radiation damping rate (which is 5 sec^{-1} at 4 GeV), on the other hand, will most likely cause beam instabilities. The current threshold for a stable beam is about 0.4 mA/bunch. These results are only meant to be rough estimates due to the uncertainties in the impedance used.

At 15 GeV and a beam current of 20 mA/bunch, the instability growth rate is essentially the same as shown in Figs. 1 and 2, but the radiation damping rate has increased to 120 sec^{-1}. A feedback system will be needed to damp the m=0 modes but the m=1 modes are taken care of by the radiation damping.

ACKNOWLEDGEMENTS

We are grateful to Chen Shen Yu 陈森玉; a conversation with him has led to the material studied in Appendix III. We thank Perry Wilson for several illuminating discussions concerning the PEP rf cavity impedance.

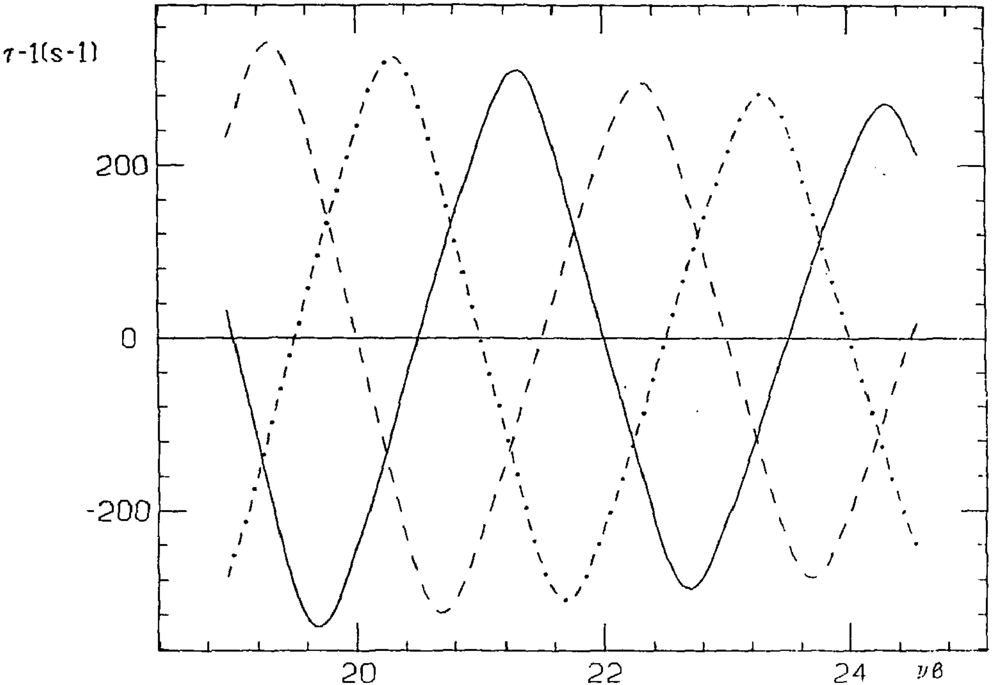

FIG.1 E=4 (GEV) I=5 (MA/BUNCH) M=0

μ=-1: SØLID μ=0: DASH-DØT μ=1: DASH

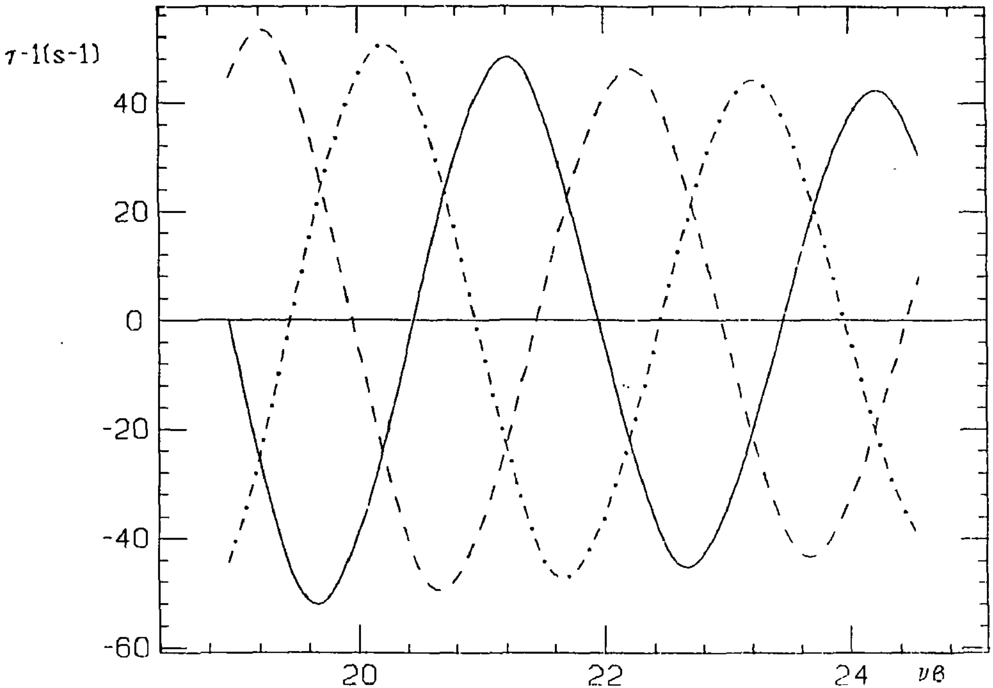

FIG.2 E=4 (GEV) I=5 (MA/BUNCH) M=1

μ=-1: SØLID μ=0: DASH-DØT μ=1: DASH

Appendix I

In this appendix we will derive an expression that is equivalent to Eq. (1), but is valid for a beam with only one bunch. Consider a single bunch executing a coherent dipole oscillation with time dependence given by D exp $(-i\Omega_m t)$, where the mode frequency Ω_m is yet to be determined. Let this oscillation mode also have a longitudinal structure given by the "snap-shot" (i.e., taken at a fixed instant of time) distribution $\rho_m(\theta)$ in unit of charges per radian, where $\theta = z/R$ is the angular coordinate measured relative to the center of the unperturbed bunch. The (dipole moment) x (beam current) seen by an observer is proportional to (*)

$$D\ e^{-i\Omega_m t} \cdot \rho_m(\theta) \cdot e^{i\left(\frac{\xi}{\alpha} - \nu_m\right)\theta} \qquad (I.1)$$

where the extra factor $\exp\left[i\ (\frac{\xi}{\alpha} - \nu_m)\ \theta\right]$ describes the "snap-shot" head-tail betatron phase factor in which $\xi = \Delta\nu_\beta/(\Delta p/p)$ is the chromaticity parameter and α is the momentum compaction factor. Symbols are defined in Table 1. The term involving ξ in this head-tail phase factor has been explained in Refs. 1 and 2; it comes from an accumulation of the single-particle betatron phase during its synchrotron motion. The additional term involving ν_m comes from a time-of-flight effect.

(*) We note that a "longitudinal impedance" samples the signal (monopole moment) x (beam current), where the monopole moment is nothing but the total electric charge. The definition of "transverse impedance" given here samples (dipole moment) x (beam current). One can go on and define impedances which sample (quadrupole moment) x (beam current), (sextupole moment) x (beam current), etc. A similar analysis as that given in the appendix should be applicable to the other cases as well.

The signal sampled by the impedance and summed over all revolutions can be written as

$$\omega_o \sum_k De^{-i\Omega_m t} \rho_m(-\omega_o t + 2\pi k) \; e^{i(\frac{\xi}{\alpha} - \nu_m)(-\omega_o t + 2\pi k)} \tag{I.2}$$

If we introduce a Fourier transform of ρ_m by

$$\tilde{\rho}_m(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \; e^{-ip\theta} \; \rho_m(\theta) \tag{I.3}$$

the signal (I.2) can be re-written as

$$\omega_o D \sum_p \tilde{\rho}_m(p + \nu_m -)\tilde{\rho}_m(p + \nu_m - \frac{\xi}{\alpha})e^{-i(p + \nu_m)\omega_o t} \tag{I.4}$$

The signal (I.4) produces a wake field and the transverse kick received by a particle at location θ from the wake field is then obtained, by definition of the transverse impedance $Z_\perp$:

$$e(\vec{E} + \frac{\vec{\upsilon}}{c} \times \vec{B})_\perp = \frac{ie}{2\pi R} \omega_o D \sum_p Z_\perp (p + \nu_m)\tilde{\rho}_m(p + \nu_m - \frac{\xi}{\alpha}) e^{-i(p + \nu_m)\omega_o t}$$

$$= \frac{ie}{2\pi R} \omega_o D e^{-i\Omega_m t} \sum_p Z_\perp (p + \nu_m) \tilde{\rho}_m(p + \nu_m - \frac{\xi}{\alpha}) e^{ip\theta} \tag{I.5}$$

knowing the expression for the transverse kick, one can write down the single particle equations of motion

$$\ddot{x} + \omega_\beta^2 x = \frac{e}{m_o\gamma} (\vec{E} + \frac{\vec{\upsilon}}{c} \times \vec{B})_\perp \equiv F$$

$$\ddot{\theta} + \omega_s^2\theta = 0 \tag{I.6}$$

where x represents the betatron coordinate; whether it is horizontal or vertical does not concern us here. The right hand side of the θ-equation has been ignored, assuming that the wake field does not change significantly in a distance $\frac{\omega_s}{\omega_x}$ (bunch length).

The Vlasov equation that describes the coherent motion of the particle bunch is

$$\frac{\partial\psi}{\partial t} + \dot{x}\frac{\partial\psi}{\partial x} + (-\omega_\beta^2 x + F)\frac{\partial\psi}{\partial \dot{x}} + \dot{\theta}\frac{\partial\psi}{\partial\theta} - \omega_s^2\theta\frac{\partial\psi}{\partial\dot{\theta}} = 0 \qquad (I.7)$$

If we make a transformation of coordinates from $(x,\dot{x})$ and $(\theta,\dot{\theta})$ to (r_x, φ_s) amd (r_s, φ_s) according to

$$x = r_x \cos\varphi_x \qquad \theta = r_s\cos\varphi_s$$

$$\dot{x} = -\omega_\beta r_x \sin\varphi_x \qquad \dot{\theta} = -\omega_s r_s \sin\varphi_s \qquad (I.8)$$

the Vlasov equation (I.7) becomes

$$\frac{\partial\psi}{\partial t} + \omega_\beta\frac{\partial\psi}{\partial\varphi_x} + \omega_s\frac{\partial\psi}{\partial\varphi_s} + F\frac{\partial\psi}{\partial\dot{x}} = 0 \qquad (I.9)$$

The linearized solution to this equation can be written as the sum of an unperturbed distribution and a first order perturbation term:

$$\psi = \underbrace{f_o(r_x)\, g_o(r_s)}_{\text{unperturbed distribution}} + \underbrace{f_1(r_x)\, e^{i\varphi_x} \cdot g_m(r_s)\, e^{im\varphi_s} \cdot e^{-i\Omega_m t}}_{\text{1-st order perturbation}} \qquad (I.10)$$

in which the unperturbed distribution f_o and g_o are assumed to be known but f_1, g_m and Ω_m are yet to be determined. Substituting (I.10) into (I.9) yields

$$-i(\Omega_m - \omega_\beta - m\omega_s)\, f_1\, g_m\, e^{i\varphi_x + im\varphi_s - i\Omega_m t} - \frac{F}{\omega_x} g_o f_o' \sin\varphi_x = 0 \qquad (I.11)$$

If we assume the mode frequency shift is small compared with the betatron frequency ω_β, the factor $\sin\varphi_x$ can be replaced by $e^{i\varphi_x}/2i$ to a good approximation. The solution for f_1 is then obtained by inspection

$$f_1 = -D\, f_o'(r_x) \qquad \text{(I.12)}$$

This distribution gives a dipole moment

$$\frac{\int x\, f_1\, e^{i\varphi_x}\, r_x dr_x d\varphi_x}{\int f_o r_x dr_x d\varphi_x} = D \qquad \text{(I.13)}$$

but we have assumed a dipole moment of D exp $\left[i(\frac{\xi}{\alpha} - \nu_m)\theta\right]$, so in order to be self-consistent we must take (*)

$$f_1 = -D\, f_o'(r_x)\; e^{i(\frac{\xi}{\alpha} - \nu_m)\theta} \qquad \text{(I.14)}$$

Substituting into (I.11), we obtain

$$-i(\Omega_m - \omega_\beta - m\omega_s)\, g_m\, e^{im\varphi_s} + \frac{ce}{2T_oE_o\nu_\beta} g_o \sum_p Z_1(p+\nu_m)\, \tilde{\rho}_m(p+\nu_m - \frac{\xi}{\alpha}) \times e^{i(p+\nu_m-\frac{\xi}{\alpha})\, r_s \cos\varphi_s} = 0 \qquad \text{(I.15)}$$

If we further assume the mode frequency shift is small compared with the synchrotron frequency ω_s, we can extract the relevant Fourier component by applying $\frac{1}{2\pi}\int_0^{2\pi} d\varphi_s\, e^{-im\varphi_s}$ to Eq. (I.15) and obtain

$$-i(\Omega_m - \omega_\beta - m\omega_s)\, g_m + \frac{ce}{2T_oE_o\nu_\beta} g_o \sum_p Z_1(p+\nu_m)\, \tilde{\rho}_m(p+\nu_m - \frac{\xi}{\alpha})\, i^m J_m\left[(p+\nu_m - \frac{\xi}{\alpha}) r_s\right] = 0 \qquad \text{(I.16)}$$

(*) This step is plausible but not rigorous. A rigorous treatment of the chromaticity effects seems to require a nonlinear theory.

where $J_m(x)$ is the Bessel function. To solve (I.16) for g_m, we need to express $\tilde{\rho}_m$ in terms of g_m. Since, by definition,

$$\rho_m(\theta) = \int d\dot{\theta} \; g_m(r_s)\, e^{im\varphi_s} \qquad (I.17)$$

we can use (I.3) to obtain the identity

$$\tilde{\rho}_m(q) = (-i)^m \omega_s \int_0^\infty r_s dr_s \; g_m(r_s) J_m(qr_s) \qquad (I.18)$$

Eq. (I.18) can be substituted into (I.16). The solution to the resulting equation is in general very difficult to find. In the following we assume a "hollow bunch" beam which has the unperturbed distribution in the synchrotron phase space

$$g_o(r_s) = \frac{Ne}{2\pi a\omega_s} \delta(r_s - a) \qquad (I.19)$$

where a is related to the rms bunch length σ_z by $a = \sqrt{2}\sigma_z/R$. The solution for this special case is easily found to be

$$g_m(r_s) \propto \delta(r_s - a) \qquad (I.20)$$

When substituted into Eq. (I.16), this solution yields the final expression for the complex mode frequency shift

$$\Omega_m - \omega_\beta - m\omega_s = -i \frac{N e^2 c}{4\pi T_o E_o \nu_\beta} \sum_p Z_\perp(p+\nu_m)\, J_m^2 \left[\left(p+\nu_m - \frac{\xi}{\alpha}\right) a\right] \qquad (I.21)$$

In practice, the ν_m's on the right hand side of Eq. (I.21) are replaced by its approximate value $\nu_\beta + m\nu_s$. The real part of $(\Omega_m - \omega_\beta - m\omega_s)$ gives the frequency shift of the mode under consideration, while the imaginary part gives the instability growth rate.

Appendix II

In Eq. (I.21), the mode frequency is given in terms of the transverse impedance in the frequency domain. In the following we will rewrite Eq. (I.21) in the time domain, using a transverse wake field $W(\theta)$ which is related to $Z_\perp(q)$ by a Fourier transformation:

$$W(\theta) = \frac{i\, e^2 \omega_o}{4\pi^2 R} \int_{-\infty}^{\infty} e^{iq\theta}\, dq\; Z_\perp(q) \tag{II.1}$$

This wake field is nonvanishing only if $\theta < 0$. Using (II.1), Eq. (I.21) becomes

$$\Omega_m - \omega_\beta - m\omega_s = - \frac{N R c}{4\pi E_o \nu_\beta} \sum_p J_m^2 \left[(p + \nu_m - \frac{\xi}{\alpha})\, a\right] \int_{-\infty}^{o} d\theta\, W(\theta)\, e^{-i(p + \nu_m)\theta} \tag{II.2}$$

If we now (i) exchange the order of summation and integration in (II.2); (ii) split the integral over θ into integrals in steps of 2π's and sum over all the steps, i.e.

$$\int_{-\infty}^{o} d\theta \longrightarrow \sum_{k=o}^{\infty} \int_{(-\pi-2\pi k\ )}^{(0,\ \pi-2\pi k)} d\theta$$

where the integration involving k=0 is from $-\pi$ to 0 and all the other integrals are from $-\pi-2\pi k$ to $\pi-2\pi k$ and (iii) make a change of variable $\theta=-2\pi(k + x)$, we find

$$\Omega - \omega_\beta - m\omega_s = - \frac{NRc}{2E_o\nu_\beta} \sum_{k=0}^{\infty} \int_{(0,-\frac{1}{2})}^{\frac{1}{2}} dx\; W(-2\pi k-2\pi x)\, e^{i2\pi\nu_m(k+x)}\; B(x) \tag{II.3}$$

where we have introduced for abbreviation a function

$$B(x) = \sum_p J_m^2 \left[(p + \nu_m - \frac{\xi}{\alpha})\, a\right] e^{i2\pi px} \tag{II.4}$$

If we assume the bunch is very short compared with the radius of the machine, i.e. a << 1, the summation over p can be replaced by an integral, i.e.

$$B(x) = \int_{-\infty}^{\infty} dp \; J_m^2 \left[(p + \nu_m - \frac{\xi}{\alpha}) a \right] e^{i2\pi px} \qquad \text{(II.5)}$$

which, after some algebra, is equivalent to

$$B(x) = \frac{2}{a} e^{i2\pi(\frac{\xi}{\alpha} - \nu_m)x} \int_0^{\infty} du \; \cos\left(2 \frac{\pi x}{a} u\right) \; J_m^2 (u) \qquad \text{(II.6)}$$

The integration that appears in (II.6) vanishes when $|x| > a/\pi$.

Combining (II.6) and (II.3), we obtain

$$\Omega_m - \omega_\beta - m\omega_s = - \frac{NRc}{E_o \nu_\beta a} \sum_{k=o}^{\infty} \int_{(o,-\frac{a}{\pi})}^{a/\pi} dx \; W(-2\pi k - 2\pi x) \; e^{i2\pi\nu_m k}$$
$$\times \; e^{i2\pi \frac{\xi}{\alpha} x} \int_0^{\infty} du \; \cos(2 \frac{\pi x}{a} u) \; J_m^2 (u) \qquad \text{(II.7)}$$

Although it looks more complicated than Eq. (I.21), this expression is more convenient if we want to separate the instability effect into contributions from individual revolutions. The integration over u in (II.7) can be written in closed form in terms of some Legendre function if desired.

Appendix III

The transverse impedance of a resistive wall of a circular vacuum chamber of radius b and conductivity σ is

$$Z_{\perp}\left(\frac{\omega}{\omega_o}\right) = \frac{2R}{b^3}\sqrt{\frac{2\pi}{\sigma|\omega|}}\left(\mathrm{sgn}\ (\omega) - i\right) \tag{III.1}$$

The wake field produced by the resistive wall is, using (II.1), given by

$$W(\theta) = \begin{cases} 0 & \text{if } \theta > 0 \\ \dfrac{2e^2}{\pi b^3}\sqrt{\dfrac{\omega_o}{\sigma|\theta|}} & \text{if } \theta < 0 \end{cases} \tag{III.2}$$

We separate the single-turn and the multi-turn effects of this wake field by dividing Eq. (II.7) into two terms: the k=0 term and the term involving the sum over k from k=1 to ∞. The first (k=0) term can be written as

$$\left(\Omega_m-\omega_\beta-m\omega_s\right)_{\substack{\text{single turn}\\ k=0}} = -\frac{2Ne^2Rc}{E_o\nu_\beta ab^3\pi\sqrt{\sigma T_o}}\int_0^{a/\pi}dx\,\frac{e^{i2\pi\frac{\xi}{\alpha}x}}{\sqrt{x}}\int_0^{\infty}du\,\cos\left(2\,\frac{\pi x}{a}-u\right)J_m^2(u) \tag{III.3}$$

The double integral on the right hand side of (III.3) can be shown to be identically equal to

$$\frac{1}{2\pi}\sqrt{\frac{a}{\pi}}\int_0^{\pi/2}\frac{du}{\sqrt{\cos u}}\int_0^{\pi}d\upsilon\,\frac{\cos m\upsilon}{\sqrt{\sin\frac{\upsilon}{2}}}\,e^{i2a\frac{\xi}{\alpha}\sin\frac{\upsilon}{2}\cos u} \tag{III.4}$$

This expression (III.4) becomes, if $a\frac{\xi}{\alpha} \ll 1$ as is the case for PEP,

$$\frac{1}{2\pi}\sqrt{\frac{a}{\pi}}\left\{\left[\int_0^{\pi/2}\frac{du}{\sqrt{\cos u}}\right]\left[\int_0^{\pi}d\upsilon\,\frac{\cos m\upsilon}{\sqrt{\sin\frac{\upsilon}{2}}}\right] + i2a\frac{\xi}{\alpha}\left[\int_0^{\pi/2}du\sqrt{\cos u}\right]\left[\int d\upsilon\,\cos m\upsilon\sqrt{\sin\frac{\upsilon}{2}}\right]\right\} \tag{III.5}$$

The imaginary term that is proportional to ξ gives the instability growth rate for small values of ξ. These results are a factor of 2 different from the results of Ref. 2.

The second ($k\neq 0$) term which gives the multi-turn effects is, assuming $a \ll 1$ but arbitrary ξ,

$$\left(\Omega_m-\omega_\beta-m\omega_s\right)_{\substack{\text{multi-turn}\\ k\neq 0}} = -\frac{2N e^2 Rc}{E_o \nu_\beta a b^3 \pi \sqrt{\sigma T_o}} \sum_{k=1}^{\infty} \frac{e^{i2\pi k \nu_m}}{\sqrt{k}} \qquad \text{(III.6)}$$

$$\times \int_{-a/\pi}^{a/\pi} dx \ \cos\left(2\pi \frac{\xi}{\alpha} x\right) \int_0^\infty du \ \cos\left(2 \frac{\pi x}{a} u\right) J_m^2(u)$$

The double integral on the right hand side can be shown to be equal to

$\frac{a}{2} J_m^2\left(\frac{\xi}{\alpha} a\right)$. Thus,

$$\left(\Omega_m-\omega_\beta-m\omega_s\right)_{\substack{\text{multi-turn}\\ k\neq 0}} = -\frac{Ne^2 Rc}{E_o \nu_\beta b^3 \pi \sqrt{\sigma T_o}} J_m^2\left(\frac{\xi}{\alpha} a\right) \sum_{k=1}^{\infty} \frac{e^{i2\pi k \nu_m}}{\sqrt{k}} \qquad \text{(III.7)}$$

Due to the factor $J_m(\xi\ a/\alpha)$, we find that for the case $\xi=0$, only the rigid dipole mode $m=0$ is affected by the resistive wall. On the other hand, the factor that involves the sum over k determines whether the beam is stable; it has been extensively studied in Ref. 3. In particular, the stability condition $\mathrm{Im}(\Omega_m) < 0$ is satified if $n < \nu_m < n + \frac{1}{2}$, or

$$n < \nu_\beta + m\nu_s < n + \frac{1}{2} \qquad \text{(III.8)}$$

for some integer n. For $m=0$, the stability criterion reduces to the Courant-Sessler result $n < \nu_\beta < n + \frac{1}{2}$.

For PEP, if we take the single-turn

$2\pi R = 2200$ m

$N = 2.3\times 10^{11}$ (5 mA/bunch)

$E_o = 4$ GeV

$\nu_\beta = 20$

$$a = 1.1\times10^{-4} \quad (\sigma_z = 2.7 \text{ cm})$$

$$b = 5 \text{ cm}$$

$$\sigma = 3\times10^{17} \text{ sec}^{-1} \quad \text{(aluminum)}$$

$$\alpha = 0.003$$

we find

ξ	m	growth rate
-1.0	0	2.7 sec^{-1}
1.0	1	0.53 sec^{-1}

These resistive wall growth rates are smaller than the radiation damping rate 5 sec^{-1} at 4 GeV.

Appendix IV

We will derive Eq. (1) in this appendix. Let the three bunches in PEP be specified by $\ell = 0$ for the reference bunch, $\ell = 1$ for the bunch in front and $\ell = -1$ for the bunch behind. There are three modes of oscillation for a given longitudinal mode m; let the three modes be specified by the additional mode number μ which has three possible values $0, \pm 1$. The mode oscillation phase from the $\ell = 0$ bunch to the bunch ℓ is equal to $\frac{2\pi}{3}\ell\mu$.

The (dipole moment) x (beam current) seen by the transverse impedance is

$$D\,\omega_o \sum_{\ell=-1}^{1} e^{-i\left(\Omega_{\mu m} t + \frac{2\pi}{3}\mu\ell\right)} \sum_k \rho_m(\theta)\, e^{i\left(\frac{\xi}{\alpha} - \nu_{\mu m}\right)\theta} \qquad \text{(IV.1)}$$

$$\left(\theta = -\omega_o t + 2\pi k + \frac{2\pi}{3}\ell\right)$$

where we are summing over the bunches ℓ and the revolutions k. In terms of the Fourier transform defined in Eq. (I.3), Eq. (IV.1) can be written as

$$D\,\omega_o \sum_{\ell} e^{-i\left(\Omega_{\mu m} t + \frac{2\pi}{3}\mu\ell\right)} \sum_p \tilde{\rho}_m\left(p + \nu_{\mu m} - \frac{\xi}{\alpha}\right) e^{ip\left(-\omega_o t + \frac{2\pi}{3}\ell\right)} \qquad \text{(IV.2)}$$

For each p the summation over ℓ vanishes unless $p-\mu$ is an integral multiple of 3. Let $p-\mu=3k$, we have

$$3D\,\omega_o \sum_k \tilde{\rho}_m\left(3k + \mu + \nu_{\mu m} - \frac{\xi}{\alpha}\right) e^{-i(3k + \mu + \nu_{\mu m})\,\omega_o t} \qquad \text{(IV.3)}$$

By definition of the transverse impedance, this gives a transverse kick (to a particle located at θ) given by

$$e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)_{\perp} = \frac{ie}{2\pi R}\ 3\ D\ \omega_o\ e^{-i\Omega_{\mu m} t} \tag{IV.4}$$

$$\times \sum_k Z_{\perp}\left(3k + \mu + \nu_{\mu m}\right)\ \tilde{\rho}_m\left(3k + \mu + \nu_{\mu m} - \frac{\xi}{\alpha}\right) e^{i(3k + \mu)\theta}$$

Same procedure as the single bunch case is then followed as is done in Appendix I. This gives the final expression for the complex mode frequency shift

$$\left(\Omega_{\mu m} - \omega_\beta - m\omega_s\right) = -i\ \frac{3\ N\ e^2\ c}{4\pi T_o E_o \nu_\beta} \sum_k Z_{\perp}\left(3k + \mu + \nu_{\mu m}\right) J_m^2\left[\left(3k + \mu + \nu_{\mu m} - \frac{\xi}{\alpha}\right) a\right] \tag{IV.5}$$

In practice we replace the $\nu_{\mu m}$'s on the right hand side by its approximate value $\nu_\beta + m\nu_s$. The total number of particles in this 3-bunch-beam is 3N. The imaginary part of Eq. (IV.5) gives finally Eq. (1).

In Eq. (IV.5), the frequencies at which the impedance is evaluated are not integral multiples of the revolution frequency because $\nu_{\mu m}$ is in general not an integer. A similar situation that happens in the longitudinal case gives rise to the well-known Robinson damping effect[7,8,9]. In the present case, therefore, this might be referred to as a "transverse Robinson effect". (The main difference is that $\nu_{\mu m}$ is usually far away from an integer and the growth rate involves the algebraic difference between the impedances at two frequencies that are, in contrast to the longitudinal case, rather far apart). The Courant-Sessler resistive wall instability is a direct consequence of this transverse Robinson effect. This effect is pronounced if the impedance is narrow-band in frequency (long range wake field).

The bunch mode spectrum, which in this example is given by the Bessel function in Eq. (IV.5), is evaluated at a frequency that depends on the chromaticity. This "head-tail effect" will be pronounced if the impedance is broad-band in frequency (short range wake field).

Table 1

m_o = electron rest mass

e = electron charge

c = speed of light

N = number of particles per bunch

E_o = particle energy

γ = E_o/m_oc^2

ξ = $\Delta\nu_\beta/(\Delta P/P)$ = chromaticity

α = momentum compaction factor

R = mean radius of machine

T_o = $2\pi R/c$ = revolution period

ω_o = c/R = revolution angular frequency

θ = z/R= azimuthal coordinate (head of bunch has $\theta > 0$)

m = mode number that specifies the longitudinal bunch structure

μ = mode number that specifies the relative motion of the 3 bunches

Ω_m, $\Omega_{\mu m}$ = $\nu_m\omega_o$, $\nu_{\mu m}\omega_o$ = mode frequencies

$\tau_{\mu m}^{-1}$ = $-\,\mathrm{Im}(\Omega_{\mu m})$ = damping rate of the mode (μ,m)

ω_β = $\nu_\beta\omega_o$ = betatron frequency

ω_s = $\nu_s\omega_o$ = synchrotron frequency

$\rho_{\mu m}(\theta)$, $\rho_m(\theta)$ = longitudinal bunch structure

$Z_\perp(\omega/\omega_o)$ = transverse impedance at frequency ω

$R_\perp(\omega/\omega_o)$ = real part of $Z_\perp$

$\psi(x,\dot{x},\theta,\dot{\theta},t)$ = distribution function of the reference bunch in phase space

a = (maximum longitudinal excursion)/R for a hollow-bunch model; see Eq. (I.19)

References

1. F. Sacherer, CERN/SI-BR/72-5 (1972)

   F. Sacherer, 9th Intern. Conf. High Energy Accel., Stanford, 347 (1974)

2. M. Sands, SLAC-TN-69-10 (1969)

3. E. D. Courant and A. M. Sessler, Rev. Scien. Instr., 37, 1579 (1966)

4. C. Pellegrini and M. Sands, PEP-258 (1977)

5. P. Wilson, private communications, to appear as PEP Note

6. J.L. Pellegrin, private communication (1979)

7. K. Robinson, CEAL-1010 (1964)

8. C. Pellegrini and M. Sands, PEP-258 (1977)

9. A. W. Chao, PEP-309 (1979)

10. A. Hofman and J. R. Maidment, LEP-168 (1979)