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STABILITY OF LONGITUDINAL MODES IN A BUNCHED BEAM WITH MODE COUPLING*

K. Satoh†
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Introduction

In this paper we study a longitudinal coherent bunch instability in which the growth time is comparable to or less than the period of synchrotron oscillations. Both longitudinal and transverse bunch instabilities have been studied by many authors.[1-9] In most treatments, however, the coherent force is assumed to be small and is treated as a perturbation compared with the synchrotron force. This makes the problem simpler because an individual synchrotron mode is decoupled. As bunch current increases, the coherent force is no longer small and the mode frequency shift becomes significant compared with the synchrotron frequency. Therefore in this case it is necessary to include coupling of the synchrotron modes. Recently a fast blow-up instability which comes from mode coupling was studied in two papers.[10-11] Their method is to derive a dispersion relation for a bunched beam using the Vlasov equation and to analyze it as in a coasting beam. They showed that if mode

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†On leave from National Laboratory for High Energy Physics (KEK), Japan.

coupling is included the Vlasov equation predicts a fast microwave instability with a stability condition similar to that for a coasting beam.

In this paper we will partly follow their method and present a formalism which includes coupling between higher-order radial modes as well as coupling between synchrotron modes. The formalism is considered to be generalization of the Sacherer formalism without mode coupling. This theory predicts that instability is induced not only by coupling between different synchrotron modes but also by coupling between positive and negative modes,[12] since negative synchrotron modes are included in the theory in a natural manner. This formalism is to be used for a Gaussian bunch and a parabolic bunch, and is also useful for transverse problems.

Derivation of Formalism

We start from the following Vlasov equation[4]

$$\frac{\partial\psi}{\partial t} + \Omega_s\frac{\partial\psi}{\partial\phi} + \frac{eV(\theta,t)}{T}\,\frac{k_o}{\Omega_s}\,\sin\phi\frac{\partial\psi}{\partial r} = 0 \quad , \tag{1}$$

where

ψ = distribution function

Ω_s = synchrotron frequency

T = revolution period = $2\pi/\omega_o$

$k_o = \alpha\omega_o/E_o$

α = momentum compaction factor

θ = angular coordinate = z/R

ε = energy error,

and

$$r\cos\phi = \theta,\ r\sin\phi = \frac{k_o}{\Omega_s}\varepsilon \quad .$$

In Eq. (1), $eV(\theta,t)$ is an increase in energy per revolution for a particle at longitudinal position θ relative to the bunch center and does not contain the effect due to the stationary part of the distribution function. Let ψ be written as

$$\psi(r,\phi) = \psi_o(r) + f(r,\phi)e^{-i\Omega t} \quad , \tag{2}$$

where Ω is a mode frequency. The beam current is

$$i(t) = \omega_o \sum_{p=-\infty}^{\infty} \tilde{\rho}(p)e^{-i(p\omega_o + \Omega)t} \quad , \tag{3}$$

where the Fourier transform of the longitudinal charge distribution is given by

$$\tilde{\rho}(p) = \frac{1}{2\pi}\frac{\Omega_s}{k_o}\iint f(r,\phi)e^{-iprcos\phi}rdrd\phi \quad . \tag{4}$$

Since with impedance $Z(\omega)$

$$eV(\theta,t) = -e\omega_o \sum_p \tilde{\rho}(p)Z(p\omega_o + Re\Omega)e^{ip\theta}e^{-i\Omega t} \quad , \tag{5}$$

we obtain

$$\left(-i\Omega + \Omega_s \frac{\partial}{\partial\phi}\right)f(r,\phi) = \frac{e\omega_o}{T}\frac{k_o}{\Omega_s} \sin\phi\frac{d\psi_o}{dr} \sum_p \tilde{\rho}(p)Z(p\omega_o + Re\Omega)e^{iprcos\phi} \quad . \tag{6}$$

Taking into account the fact that $f(r,\phi)$ and the right-hand side in Eq. (6) have the periodicity of 2π, we integrate with ϕ and have

$$f(r,\phi) = \frac{e\omega_o k_o}{T\Omega_s^2}\frac{d\psi_o}{dr} e^{i\lambda\phi} \frac{1}{e^{-2\pi i\lambda} - 1} \sum_p \tilde{\rho}(p)Z(p\omega_o + Re\Omega)$$

$$\times \int_{\phi}^{\phi+2\pi} e^{-i\lambda\phi'} \sin\phi' \, e^{iprcos\phi'} \, d\phi' \quad , \tag{7}$$

where

$$\lambda = \frac{\Omega}{\Omega_s} \quad .$$

Performing the integral as

$$\int_{\phi}^{\phi+2\pi} e^{-i\lambda\phi'} \sin\phi' \, e^{ipr\cos\phi'} d\phi'$$

$$= i\, e^{-i\lambda\phi}(e^{-2\pi i\lambda} - 1) \frac{1}{pr} \sum_{m=-\infty}^{\infty} (i)^m \frac{m}{m-\lambda} J_m(pr) e^{im\phi} \quad , \tag{8}$$

we have

$$f(r,\phi) = i\frac{e\omega_o k_o}{T\Omega_s^2} \frac{1}{r} \frac{d\psi_o}{dr} \sum_p \frac{1}{p} Z(p\omega_o + Re\Omega)\, \tilde{\rho}(p)$$

$$\times \sum_m \frac{m}{m-\lambda} (i)^m J_m(pr) e^{im\phi} \quad . \tag{9}$$

Using Eq. (4) we obtain in the frequency domain

$$\tilde{\rho}(q) = i\frac{2\pi e}{\Omega_s T^2} \sum_p \frac{1}{p} Z(p\omega_o + Re\Omega)\, \tilde{\rho}(p) \sum_m \frac{m}{m-\lambda}$$

$$\times \int \frac{1}{r} \frac{d\psi_o}{dr} J_m(qr)\, J_m(pr) r dr \quad , \tag{10}$$

which already includes negative synchrotron modes. If we put $\lambda \sim m$, Eq. (10) becomes approximately

$$i(m\Omega_s - \Omega)\, \tilde{\rho}(q) = -\frac{2\pi m e}{T^2} \sum_p \frac{1}{p} Z(p\omega_o + m\Omega_s)\, \tilde{\rho}(p)$$

$$\times \int \frac{1}{r} \frac{d\psi_o}{dr} J_m(qr)\, J_m(pr) r dr \quad , \tag{11}$$

which is exactly equal to the Sacherer equation without mode coupling.

In what follows we will treat a Gaussian bunch and a parabolic bunch. For a Gaussian bunch

$$\psi_o(r) = \frac{Ne\alpha}{2\pi\nu_s E_o \sigma^2} \cdot \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad , \tag{12}$$

where

$$\iint \psi_o(r)d\varepsilon d\theta = N$$

N = number of particles in one bunch

σ = r.m.s. bunch length in units of $\theta = \sigma_z/R$

$\nu_s = \Omega_s/\omega_o$,

and Eq. (10) becomes

$$\tilde{\rho}(q) = -i\frac{Ne^2\alpha}{2\pi\nu_s^2E_oT\sigma^2} \sum_p \frac{1}{p} Z(p\omega_o + \mathrm{Re}\Omega)\ \tilde{\rho}(p) \sum_m \frac{m}{m-\lambda}$$

$$\times \int_0^\infty e^{-t^2/2}\ J_m(\sigma qt)\ J_m(\sigma pt)tdt \quad . \tag{13}$$

For the integral we have the following sum[9]

$$\int_0^\infty e^{-t^2/2}\ J_m(\sigma qt)\ J_m(\sigma pt)tdt = \sum_{k=0}^\infty C_{mk}(\sigma q)\ C_{mk}(\sigma p) \quad , \tag{14}$$

where

$$C_{mk}(\sigma p) = \frac{1}{\sqrt{(m+k)!k!}}\quad e^{-p^2\sigma^2/2}\left(\frac{p\sigma}{\sqrt{2}}\right)^{m+2k} \quad . \tag{15}$$

Using $C_{mk}(\sigma p)$ which corresponds to the frequency spectrum of the k-th radial mode in the m-th synchrotron mode, we obtain

$$\tilde{\rho}(q) = -i\ K \sum_p g(p,\mathrm{Re}\lambda)\ \tilde{\rho}(p) \sum_{m=1}^\infty \frac{m^2}{m^2-\lambda^2} \sum_{k=0}^\infty C_{mk}(\sigma q)\ C_{mk}(\sigma p) \quad , \tag{16}$$

where

$$K = \frac{Ne^2\alpha Z_o}{\pi\nu_s^2E_oT\sigma^2} \tag{17}$$

$$Z_og(p,\mathrm{Re}\lambda) = \frac{1}{p} Z(p\omega_o + \mathrm{Re}\Omega) \tag{18}$$

and Z_o is a real impedance for normalization. Since $C_{mk}(\sigma p)$ has a special relation

$$C_{mk}(\sigma p) = \sqrt{\binom{m+2k}{k}} \cdot C_{m+2k,o}(\sigma p) \quad , \tag{19}$$

we can transform the double summation with m and k in Eq. (16) into a single summation as

$$\tilde{\rho}(q) = -\,i\,K \sum_{p} g(p,\mathrm{Re}\lambda)\;\tilde{\rho}(p) \sum_{k=1}^{\infty} \beta_k(\lambda^2)\;F_k(\sigma q)\;F_k(\sigma p) \quad , \tag{20}$$

where

$$F_k(\sigma p) = C_{ko}(\sigma p) \quad , \tag{21}$$

and

$$\begin{aligned}
\beta_1(x) &= \frac{1}{1-x} \quad , \\
\beta_2(x) &= \frac{4}{4-x} \quad , \\
\beta_3(x) &= \frac{9}{9-x} + \frac{1}{1-x}\cdot 3 \quad , \\
\beta_4(x) &= \frac{16}{16-x} + \frac{4}{4-x}\cdot 4 \quad , \\
\beta_5(x) &= \frac{25}{25-x} + \frac{9}{9-x}\cdot 5 + \frac{1}{1-x}\cdot 10 \quad , \\
\beta_6(x) &= \frac{36}{36-x} + \frac{16}{16-x}\cdot 6 + \frac{4}{4-x}\cdot 15 \quad , \\
&\dots \; .
\end{aligned} \tag{22}$$

We look for an eigenmode freqency spectrum $\tilde{\rho}(q)$ given by

$$\tilde{\rho}(q) = \sum_{k=1}^{\infty} \alpha_k\;F_k(\sigma q) \quad . \tag{23}$$

This transformation leads to

$$\alpha_k = -i\,K\,\beta_k(\lambda^2) \sum_{\ell=1}^{\infty} M_{k\ell}(\mathrm{Re}\lambda)\alpha_\ell \quad , \tag{24}$$

where

$$M_{k\ell}(\mathrm{Re}\lambda) = \sum_{p} g(p,\mathrm{Re}\lambda)\;F_k(\sigma p)\;F_\ell(\sigma p) \quad . \tag{25}$$

We can now determine λ so that Eq. (24) has a non-trivial solution:

$$\det\left|\delta_{k\ell} + i\,K\,\beta_k(\lambda^2)\,M_{k\ell}(\mathrm{Re}\lambda)\right| = 0 \tag{26}$$

On the other hand, since

$$M_{k\ell}(-\mathrm{Re}\lambda) = (-1)^{k+\ell+1}\,M^*_{k\ell}(\mathrm{Re}\lambda) \quad , \tag{27}$$

$$\beta^*_k(\lambda^2) = \beta_k(\lambda^{*2}) \quad , \tag{28}$$

we can derive from Eq. (24)

$$(-1)^k\,\alpha^*_k = -i\,K\,\beta_k\left((-\lambda^*)^2\right)\sum_{\ell=1}^{\infty} M_{k\ell}(-\mathrm{Re}\lambda^*)(-1)^\ell\,\alpha^*_\ell \quad . \tag{29}$$

Therefore, if λ_n is a mode frequency with a mode spectrum

$$\tilde{\rho}_n(p) = \sum_{k=1}^{\infty} \alpha_k\,F_k(\sigma p) \quad , \tag{30}$$

we always have another mode frequency $-\lambda^*_n$ with a frequency spectrum

$$\sum_{k=1}^{\infty} (-1)^k\,\alpha^*_k\,F_k(\sigma p) = \tilde{\rho}^*_n(-p) \quad , \tag{31}$$

except in the case that $\lambda_n = -\lambda^*_n$. Returning to Eq. (3), the beam current for the mode $-\lambda^*_n$ is given by the complex conjugate of that for the mode λ_n:

$$\omega_o \sum_p \tilde{\rho}^*_n(-p)\,e^{-i(p\omega_o - \Omega^*)t} = \omega_o\left\{\sum_p \tilde{\rho}_n(p)\,e^{-i(p\omega_o + \Omega)t}\right\}^* \quad . \tag{32}$$

Therefore in solving Eq. (26) it is necessary only to pick up roots with a positive real part.

For a parabolic bunch

$$\psi_o(r) = \frac{Ne\alpha}{2\pi\nu_s E_o \sigma^2}\cdot 2(\mu+2)\left(1-\left(\frac{r}{\sigma}\right)^2\right)^{\mu+1} \quad \mu > -1 \quad , \tag{33}$$

and Eq. (10) becomes

$$\tilde{\rho}(q) = -i\, 4(\mu + 2)(\mu + 1)\, K \sum_p g(p, \mathrm{Re}\lambda)\, \tilde{\rho}(p) \sum_{m=1}^{\infty} \frac{m^2}{m^2 - \lambda^2}$$

$$\times \int_0^1 (1 - t^2)^{\mu}\, J_m(\sigma q t)\, J_m(\sigma p t)\, t\, dt \quad . \tag{34}$$

We can also expand the above integral as[9]

$$\int_0^1 (1 - t^2)^{\mu}\, J_m(\sigma q t)\, J_m(\sigma p t)\, t\, dt = \sum_{k=0}^{\infty} C_{mk}(\sigma q)\, C_{mk}(\sigma p) \quad , \tag{35}$$

where

$$C_{mk}(\sigma p) = \left(\frac{(m + \mu + 2k + 1)\, \Gamma(\mu + k + 1)\, \Gamma(m + \mu + k + 1)}{2 \cdot k!\, (m + k)!} \right)^{\frac{1}{2}}$$

$$\times \left(\frac{2}{\sigma p} \right)^{\mu+1} J_{m+\mu+2k+1}(\sigma p) \quad . \tag{36}$$

Since we have again a special relation for a parabolic bunch,

$$C_{mk}(\sigma p) = \left(\binom{m + 2k}{k} \cdot \frac{\Gamma(\mu + k + 1)\, \Gamma(m + \mu + k + 1)}{\Gamma(\mu + 1)\, \Gamma(m + \mu + 2k + 1)} \right)^{\frac{1}{2}}$$

$$\times \quad C_{m+2k,o}(\sigma p) \quad , \tag{37}$$

we can follow exactly the formalism for a Gaussian bunch and derive an equation similar to Eq. (26), which determines the mode frequencies.

For a general bunch distribution it is not known whether special relations as in Eqs. (19) and (37) exist or not. Therefore this formalism is useful so far only for Gaussian and parabolic bunches.

Discussion

We will consider how this theory predicts instabilities. We treat here only a wide-band impedance, which means that the frequency shift can be neglected in the impedance function. That is,

$$Z(p\omega_0 + \mathrm{Re}\Omega) \approx Z(p\omega_0) \quad . \tag{38}$$

Therefore $M_{k\ell}$ in Eq. (26) does not depend on λ.

As a first example, we take only the first term in Eq. (30). The mode frequency is determined by

$$(1 - \lambda^2) + i\, K\, M_{11} = 0 \quad , \tag{39}$$

where

$$M_{11} = i \sum_p \{\mathrm{Im}\ g(p)\}\{F_1(\sigma p)\}^2 \quad . \tag{40}$$

Since M_{11} is pure imaginary, Eq. (39) can have an unstable solution which comes from coupling between $m = \pm 1$ modes. The threshold is given by

$$K \cdot M_{11} = i \quad , \tag{41}$$

and is determined only by the imaginary impedance. Recently B. Zotter has also pointed out that by including negative mode numbers, even an imaginary impedance can lead to instability.[12]

Next we take the first two terms in Eq. (30). The mode frequency is determined by

$$\begin{vmatrix} (1 - \lambda^2) + i\, K\, M_{11} & i\, K\, M_{12} \\ i\, K\, M_{12} & \tfrac{1}{4}(4 - \lambda^2) + i\, K\, M_{22} \end{vmatrix} = D(\lambda^2) = 0 \quad , \tag{42}$$

where

$$M_{12} = \sum_p \{\mathrm{Re}\ g(p)\}\, F_1(\sigma p)\ F_2(\sigma p) \quad , \tag{43}$$

$$M_{22} = i \sum_p \{\mathrm{Im}\ g(p)\}\{F_2(\sigma p)\}^2 \quad . \tag{44}$$

We analyze the mode frequency by plotting the parabolic function D(x) with changing K. If D(x) = 0 has a complex root or a negative root, instability occurs. If K is small, D(x) is shown in Fig. 1a and mode frequency shifts are small. As K or bunch current increases, two cases are possible. In the first case D(x) = 0 has complex roots as in Fig. 1b. This instability is induced by coupling between the m = 1 and m = 2 modes. In the second case D(x) = 0 has a negative root as in Fig. 1c. This case comes from coupling between m = ±1 modes, as in the first example. Which case first occurs when bunch current increases depends on the value of M_{ij}.

If we take more terms in Eq. (30), this theory also predicts that each synchrotron mode splits into several modes because higher-order radial modes are included. For example by taking the first five terms, we have one mode for m = 5 and m = 4, two modes for m = 3 and m = 2, and three m = 1 modes.

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References

1. F. Sacherer, CERN/SI-BR/72-5 (1972); CERN/PS/BR/77-5 (1977).
2. J. L. Laclare, LNS-003, Saclay (1977); LNS-021 (1978); LNS-056 (1981).
3. J. L. Laclare, Proc. 11th Int. Conf. On High Energy Accelerators (Birkhäuser Verlag, Basel, 1980), p. 526.
4. A. W. Chao, PEP-Note 309, SLAC, Stanford (1979).
5. A. W. Chao and C. Y. Yao, PEP-Note 321, SLAC, Stanford (1979).
6. G. Besnier, Nucl. Instrum. Methods 164, 235 (1979).

7. R. D. Kohaupt, DESY 80/22 (1980).

8. C. Y. Yao and A. W. Chao, IEEE Trans. Nucl. Sci. NS-28, No. 3 2571 (1981).

9. K. Satoh, PEP-Note 357, SLAC, Stanford (1981).

10. J. M. Wang and C. Pellegrini, Proc. 11th Int. Conf. on High Energy Accelerators (Birkhäuser Verlag, Basel, 1980), p. 554.

11. R. D. Ruth and J. M. Wang, IEEE Trans. Nucl. Sci. NS-28, No. 3, 2405 (1981).

12. B. Zotter, IEEE Trans. Nucl. Sci. NS-28, No. 3, 2602 (1981).

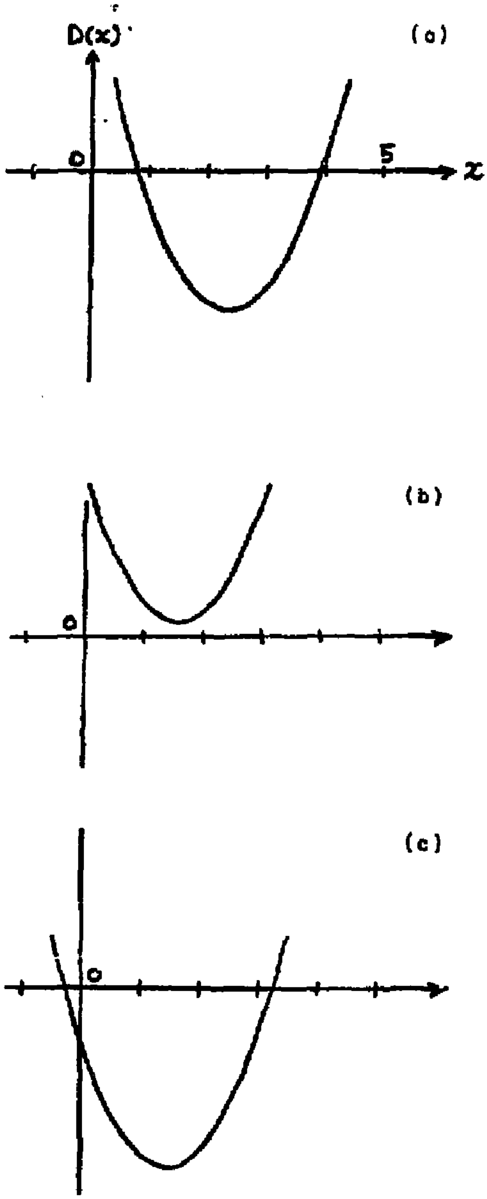

Fig. 1 The determinant D(x) in Eq. (42) for three values of the parameter K.