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THREE-DIMENSIONAL STELLARATOR EQUILIBRIA BY ITERATION

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ABSTRACT

The iterative method of evaluating plasma equilibria is especially simple in a magnetic coordinate representation. This method is particularly useful for clarifying the subtle constraints of three-dimensional equilibria and studying magnetic surface breakup at high plasma beta.

I. INTRODUCTION

The increased importance of the stellarator in the magnetic fusion program implies the need for efficient and reliable equilibrium solvers. That is, methods for solving

$$\vec{\nabla}p = \frac{1}{c}\vec{J} \times \vec{B} \tag{1}$$

are required. At present, there are several fully three-dimensional equilibrium codes: the Betancourt-Garabedian,[1] the Chodura-Schlüter,[2] and the Hender or Near code.[3] All three codes are based on the principle of minimizing the plasma energy under the assumptions of zero plasma resistivity and entropy production. In addition, there has been considerable study and application of the asymptotic theory of averaged equilibria[4-8] and analytic work using expansions about a magnetic axis.[9-11] None of these equilibrium solvers deals in a satisfactory way with the breakup of the magnetic surfaces at high plasma beta. Since this effect gives the equilibrium beta limit in a stellarator, its study is quite important. In addition to the energy principle, averaged equilibria, and expansion around the axis, there is a fourth method of finding equilibria. This method is based on a perturbation about a known magnetic field or equilibrium and is called the iterative method. It was discussed and used in the early years of the magnetic fusion program by Kruskal and Koenig,[12] and Grad and Rubin.[13] In recent years, it has been used by Yamagishi et al.[14] to study equilibria in the Ohte device.

The basic formulation of the iterative method is quite simple. Suppose we have a magnetic field $\vec{B}_k(\vec{x})$ at some stage of the iteration. Then the next stage is evaluated using the equations

$$\vec{\nabla} p_k = \frac{1}{c} \vec{j}_k \times \vec{B}_k \quad , \tag{2}$$

$$\vec{\nabla} \times \vec{B}_{k+1} = \frac{4\pi}{c} \vec{j}_k \quad , \tag{3}$$

$$\vec{\nabla} \cdot \vec{B}_{k+1} = 0 \quad . \tag{4}$$

There is an exact equilibrium if and only if $\vec{B}_{k+1}(\vec{x}) = \vec{B}_k(\vec{x})$. These equations have a number of attractive features. In particular, the subtle constraints on three-dimensional equilibria are explicit, and there is no assumption of perfect magnetic surfaces. In this paper, we will give the relatively simple mathematical procedure for evaluating equilibria iteratively if magnetic coordinates are employed.

II. MAGNETIC COORDINATES

Let $\vec{B}$ be a magnetic field associated with a scalar pressure equilibrium. In a region of space in which $\vec{\nabla} p \neq 0$ one can represent $\vec{B}$ in two forms[15] using the magnetic coordinates ψ, θ, ϕ:

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\phi \times \vec{\nabla}\psi_p(\psi) \quad , \tag{5}$$

$$\vec{B} = g(\psi)\vec{\nabla}\phi + I(\psi)\vec{\nabla}\theta + \beta_*(\psi,\theta,\phi)\vec{\nabla}\psi \quad . \tag{6}$$

The quantities ψ, θ, ϕ, $\psi_p(\psi)$, $g(\psi)$, and $I(\psi)$ which appear in these representations are defined in Fig. 1. The quantity $\beta_*(\psi,\theta,\phi)$ is closely related to the Pfirsch-Schlüter current but will not play a major role in this paper.

The demonstration that these two forms exist is simple. Let $\bar{\theta}$ and ϕ be any poloidal and toroidal angle which make $\vec{\nabla} p \cdot (\vec{\nabla}\bar{\theta} \times \vec{\nabla}\phi)$ finite. Then p, $\bar{\theta}$, ϕ can be used as coordinates. That is, they span the space and any vector can be written

$$\vec{B} = a\vec{\nabla} p \times \vec{\nabla}\bar{\theta} + b\vec{\nabla}\phi \times \vec{\nabla} p + c\vec{\nabla}\bar{\theta} \times \vec{\nabla}\phi \quad . \tag{7}$$

Since $\vec{B} \cdot \vec{\nabla} p = 0$, the coefficient $c = 0$. Using the fact that the divergence of crossed gradients is zero, $\vec{\nabla} \cdot \vec{B} = 0$ implies a and b can be written in the form

$$a = a_o(p) + a_o \frac{\partial\omega}{\partial\bar{\theta}} \quad , \quad b = b_o(p) - a_o \frac{\partial\omega}{\partial\phi} \tag{8}$$

with ω a function of p, $\bar{\theta}$, ϕ. Let $d\psi = a_o\, dp$, $d\psi_p = b_o\, dp$, and $\theta = \bar{\theta} + \omega$, then one finds Eq. (7) can be written in the form of Eq. (5).

To demonstrate the covariant form for $\vec{B}$, Eq. (6), write $\vec{B}$ in the general form

$$\vec{B} = \alpha\, \vec{\nabla}\phi + \gamma\vec{\nabla}\theta + \beta\vec{\nabla} p \quad . \tag{9}$$

Now $(\vec{\nabla} \times \vec{B}) \cdot \vec{\nabla} p = 0$ implies

$$\frac{\partial\alpha}{\partial\theta} - \frac{\partial\gamma}{\partial\phi} = 0 \quad . \tag{10}$$

This equation means one can write α and γ in the form

$$\alpha = g(p) + (g + \chi I)\frac{\partial\nu}{\partial\phi} \quad , \quad \gamma = I(p) + (g + \chi I)\frac{\partial\nu}{\partial\theta} \tag{11}$$

with $\chi = d\psi_p/d\psi$. Let $\theta_n = \theta + \chi\upsilon$ and $\phi_N = \phi + \nu$. The transformation from θ,ϕ to θ_n, ϕ_n does not change the representation of Eq. (5) but does imply $\vec{B}$ can be written in the form of Eq. (6).

The evaluation of the transformation equations, $\vec{X}(\psi,\theta,\phi)$, between the ordinary Cartesian coordinates x,y,z and magnetic coordinates ψ, θ, ϕ can be made with field line integrations.[16,17] Here we outline the procedure for the coordinate y. The periodicities of the torus imply

$$y(\psi, \theta, \phi) = \sum_{n,m} y_{nm} (\psi) \exp [i(n\phi - m\theta)] \quad . \tag{12}$$

Equation (5) implies a field line has ψ constant and $\theta_o = \theta - \chi\phi$ constant with $\chi = d\psi_p/d\psi$. This follows from the obvious implications of Eq. (5) that $\vec{B} \cdot \vec{\nabla}\psi = \vec{B} \cdot \vec{\nabla}\theta_o = 0$. Equation (6) implies that if χ is defined by $\chi = g(\psi)\phi + I(\psi)\theta$, then $d\chi = Bd\ell$ with $d\ell$ the differential distance along a line. If one chooses the starting point of the field line integration so $\theta = \phi = 0$, then χ and θ_o vanish there. As one performs the field line integration one can evaluate χ and hence $y(\chi)$. By Eq. (12), $y(\chi)$ must be of the form

$$y(\chi) = \Sigma \ y_{nm} \exp \left[i \frac{n - \chi m}{g + \chi I} \chi\right] \tag{13}$$

The Fourier decomposition of $y(\chi)$ has distinct peaks. The amplitudes of the peaks set the y_{nm} and the locations determine the n and m as well as χ.

If a finite volume is covered by a single field line, then the treatment must be altered.[18] The field lines, in such regions, are said to be stochastic. In cases of practical interest, one can still evaluate $\vec{X}(\psi, \theta, \phi)$ and $\chi(\psi)$, as described above, over much of the plasma volume. The stochastic regions can be included by smoothly interpolating $\chi(\psi)$ and the Fourier coefficients $\vec{X}_{nm}(\psi)$ across these regions. Given $\vec{X}(\psi, \theta, \phi)$ and $\chi(\psi)$ a unique magnetic field with perfect surfaces is defined by Eq. (5). Consequently the construction of $\vec{X}(\psi, \theta, \phi)$ and $\chi(\psi)$ divides the magnetic field into a field with perfect surfaces and a perturbation.

III. PLASMA CURRENT

Let $\vec{B}$ be the magnetic field at some stage of the iteration. We wish to find the relation between the presssure and the current using

$$\vec{\nabla} p = \frac{1}{c} \vec{j} \times \vec{B} \quad . \tag{14}$$

In the next section we will find the new magnetic field given the current. For simplicity of discussion, we assume $\vec{B}$ has perfect surfaces. The procedure to be used when $\vec{B}$ has stochastic regions will be discussed at the end of the section.

Since $\vec{B}$ is near a scalar pressure equilibrium, it can be accurately represented in the forms of Eqs. (5) and (6). Equilibrium, Eq. (14), implies the current perpendicular to $\vec{B}$ is

$$\vec{j}_\perp = \frac{c}{B^2} \vec{B} \times \vec{\nabla} p \quad . \tag{15}$$

Using Eq. (6) and $p = p(\psi)$,

$$\vec{j}_\perp = \frac{c}{B^2} \left(g(\psi) \vec{\nabla}\phi \times \vec{\nabla}\psi - I(\psi) \vec{\nabla}\psi \times \vec{\nabla}\theta \right) \frac{dp}{d\psi} \quad . \tag{16}$$

Of course, the current must be divergence-free since it is the curl of the magnetic field used in the next interation step. This implies

$$\vec{B}\cdot\vec{\nabla}\,\frac{j_\parallel}{B} = -\vec{\nabla}\cdot\vec{j}_\perp \quad . \tag{17}$$

The divergence of $\vec{j}_\perp$ can be simply taken. We assume B^2 has the form

$$\frac{1}{B^2} = \frac{1}{B_o^2}\left(1 + {\sum_{n,m}}'\,\delta_{nm}\cos(n\phi - m\theta)\right) \tag{18}$$

with the prime implying the term n = 0, m = 0 is omitted from the sum. Then the solution to Eq. (18) is[15]

$$\frac{j_\parallel}{B} = \frac{c}{4\pi}\,\mu(\psi) + \frac{c}{B_o^2}\frac{dp}{d\psi}\,{\sum}'\,\frac{mg+nI}{n-\chi m}\,\delta_{nm}\cos(n\phi - m\theta) \tag{19}$$

with $c\mu(\psi)/4\pi$ the general homogeneous solution, the so-called force-free current. In principle one should also have terms in $\sin(n\phi - m\theta)$, but most stellarator designs have the appropriate symmetry so that these terms can be eliminated by a proper choice of the position $\theta = 0$, $\phi = 0$. The total plasma current is obtained by adding $\vec{j}_\perp$ and $(j_\parallel/B)\vec{B}$,

$$\vec{j} = \left[\frac{c}{4\pi}\frac{dI_p}{d\psi} + (g + \chi I)\frac{c}{B_o^2}\frac{dp}{d\psi}\,{\sum_{n,m}}'\,\frac{m}{n-\chi m}\,\delta_{nm}\cos(n\phi - m\theta)\right](\vec{\nabla}\psi \times \vec{\nabla}\theta)$$

$$+\left[-\frac{c}{4\pi}\frac{dg_p}{d\psi} + (g + \chi I)\frac{c}{B_o^2}\frac{dp}{d\psi}\,{\sum_{n,m}}'\,\frac{n}{n-\chi m}\,\delta_{nm}\cos(n\phi - m\theta)\right](\vec{\nabla}\phi \times \vec{\nabla}\psi) \quad , \tag{20}$$

$$\frac{dI_p}{d\psi} = \mu - \frac{4\pi}{B_o^2}\frac{dp}{d\psi}\,I \quad , \tag{21}$$

$$\frac{dg_p}{d\psi} = -\chi\mu - \frac{4\pi}{B_o^2}\frac{dp}{d\psi}\,g \quad . \tag{22}$$

One can show using Eq. (6) that in equilibrium $I_p = I$ and $g_p = g$.

The singular form of the equation for the parallel current, Eq. (19), requires comment. This equation implies the parallel current will have non-integrable singularities on every surface in which the transform is a rational number unless all the δ_{nm}'s associated with that rational number are zero, or $dp/d\psi$ vanishes there. The vanishing of the associated δ's is equivalent to $\oint d\ell/B$ being identical on every field line of the rational surface. Actually the parallel current, Eq. (19), gives a constraint on the pressure gradient if we apply the physical requirement that there be no singular sources or sinks of particles. One can show the Pfirsch-Schlüter diffusion coefficient[15]

$$D \propto \sum \left(\frac{nI+mg}{n-\iota m}\,\delta_{nm}\right)^2 \quad . \tag{23}$$

The particle flux $\Gamma(\psi)$ crossing a surface is given by the sources of particles inside that surface. The pressure gradient satisfies

$$\frac{dp}{d\psi} = -\Gamma(\psi)/D(\psi) \tag{24}$$

and therefore vanishes wherever $D(\psi)$ is singular. If all the δ_{nm} were finite, the pressure gradient would have a quadratic zero near each rational surface and the pressure driven or Pfirsch-Schlüter part of the parallel current would have a linear zero. The physical constraint that the pressure gradient vanish at the rational surfaces on which the resonant δ's do not vanish is manifest in the iterative scheme and more subtly obscured, although present, in other methods of finding equilibria.

If one uses the simple Ohm's law, with $\eta_\parallel$ the parallel resistivity, the force-free current is simply related[15] to the loop voltage V:

$$\mu = \frac{2}{c\eta_\parallel} \frac{V}{g + \iota I} \quad . \tag{25}$$

If the magnetic field $\vec{B}$ does not have perfect surfaces, one must make μ and p constant over regions in which $\vec{B}$ is stochastic. This follows from the constraints $\vec{B} \cdot \vec{\nabla}\mu = 0$ and $\vec{B} \cdot \vec{\nabla}p = 0$.

IV. MAGNETIC FIELD DUE TO THE PLASMA

The plasma current at each point in space $\vec{j}(\vec{x})$ is determined by Eq. (20) and the transformation equation $\vec{X}(\psi,\theta,\phi)$. The magnetic field due to the plasma $\vec{b}(\vec{x})$ obeys the equations

$$\vec{\nabla} \times \vec{b} = \frac{4\pi}{c} \vec{j} \tag{26}$$

and

$$\vec{\nabla} \cdot \vec{b} = 0 \quad . \tag{27}$$

The physical boundary condition is normally $\vec{b}(\vec{x} \rightarrow \infty) = 0$. Fixed boundary equilibria correspond to $\vec{b} \cdot \hat{n} = 0$ on the boundary with $\hat{n}$ the normal to the boundarv. If $\vec{j}$ is calculated using the kth iteration order magnetic field, then $\vec{b}$ is added to the vacuum field to obtain the k + 1 order magnetic field.

Although the evaluation of the magnetic field, given the current, is in principle straightforward, in practice it can be difficult. For plasmas without net current the evaluation of $\vec{b}$ can be reduced to the solution of Poisson's equation. With no net current, both $\mu(\psi)$ and $I(\psi)$ are zero, as shown in Eq. (21). The case with net plasma current is considered in the appendix.

To reduce the evaluation of $\vec{b}$ to Poisson's equation, divide the plasma field into two parts

$$\vec{b} = \vec{h} + \vec{\nabla} f \quad . \tag{28}$$

The field $\vec{h}$ can be made to vanish outside the plasma and have its curl equal the plasma current, Eq. (20), inside with the choice

$$\vec{h} = \left(\int_{\psi}^{\psi_a} \frac{4\pi g}{B_o^2} \frac{dp}{d\psi} d\psi\right)\vec{\nabla}\phi + \left(4\pi \frac{g}{B_o^2} \frac{dp}{d\psi} \sum{}' \frac{\delta_{nm}}{n-\iota m} \sin(n\phi - m\theta)\right)\vec{\nabla}\psi \tag{29}$$

with ψ_a the value of ψ at the plasma edge. The condition that $\vec{b}$ have zero divergence implies

$$\nabla^2 f = -\vec{\nabla}\cdot\vec{h} \tag{30}$$

which is Poisson's equation. Since $\vec{h}$ vanishes outside the plasma, the boundary conditions on $\vec{b}$ become boundary conditions on $\vec{\nabla} f$.

V. APPROXIMATE SOLUTION

If the magnetic surfaces were straight circular cylinders, the Poisson equation for f could be simply solved. In many equilibria the circularity assumption is an accurate approximation. Even in equilibria with significant noncircularity, the simple circular surface formulas may give a reasonable estimate of the surface shift and hence the beta limit. Although some analytic generalizations are possible, we consider only the simplest approximation to clarify the physics.

With circular surfaces of radius r,

$$\frac{d\psi}{dr} = B_o r \tag{31}$$

and the radial component of the $\vec{h}$ field, as shown in Eq. (29), is

$$h_r = \frac{4\pi q}{B_o^2} \frac{dp}{dr} \sum' \frac{\delta_{nm}}{n-\chi m} \sin(n\phi - m\theta) \quad . \tag{32}$$

If we assume $|n/R| << |m/r|$ with

$$\frac{1}{R} \equiv |\vec{\nabla} \phi| \quad , \tag{33}$$

then Poisson's equation, Eq. (30), becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = -\frac{1}{r}\frac{d}{dr}(rh_r) \quad . \tag{34}$$

Since the equation is linear, we can assume h_r has only one Fourier term and sum the various Fourier terms only at the end of the calculation. The simplest boundary condition, and the one that is used, is that f goes to zero for large r. The solution inside the plasma, $r \leq a$, is

$$f = \frac{r^{|m|}}{2}\left(\int_r^a \frac{h_r}{r^{|m|}} dr\right) - \frac{1}{2r^{|m|}}\left(\int_o^r r^{|m|} h_r \, dr\right) \quad . \tag{35}$$

Outside the plasma, $r > a$, only the term proportional to $1/r^{|m|}$ is present. The radial field is given by $b_r = h_r + \partial f/\partial r$ or

$$b_r = \frac{|m|}{2r}\left[r^{|m|}\left(\int_r^a \frac{h_r}{r^{|m|}} dr\right) + \frac{1}{r^{|m|}}\left(\int_o^r r^{|m|} h_r \, dr\right)\right] \quad . \tag{36}$$

It is, of course, the radial field which dominates the change in shape of the vacuum surfaces. Equations (32) and (36) give an easily evaluated expression for this field. The change in ι, at low beta, is dominated by the $\vec{\nabla}\phi$ component of $\vec{h}$, as given in Eq. (29). This component of $\vec{h}$ is divergence-free if $\vec{B}$ is vacuum magnetic field. One finds, with $\beta(\psi)$ the local beta value

$$\frac{\Delta\iota}{\iota} \simeq \frac{\beta(\psi)}{2} \quad . \tag{37}$$

Given the magnetic field $\vec{b}$ one can simply evaluate numerically the magnetic surfaces in the presence of the plasma. Of course the major features can be illustrated analytically. In analytic evaluations of the perturbed surfaces, one must first look for resonant surfaces. These are surfaces on which a Fourier term in b_r satisfies $n = \iota m$. Although b_r has a singular form near such surfaces, the consistency of the equilibria with finite transport implies b_r is actually finite, as shown in Sec. III. A finite b_r still breaks the topology of the magnetic surfaces by opening a magnetic island.[19] If b_r^* is the magnitude of the resonant terms in b_r, then the half-width of this island is

$$\Delta_i = \left| \frac{4}{m d\iota/dr} \frac{R\, b_r^*}{B_o} \right|^{1/2} \quad . \tag{38}$$

When neighboring islands overlap, the field lines stochastically cover a finite volume rather than lying in surfaces.

Away from resonant surfaces, one can evaluate the change in shape of the magnetic surfaces by perturbation theory. If we define Δ_{nm} by

$$b_r = -\frac{B_o}{R} \sum{}' (n - \iota m)\, \Delta_{nm} \sin(n\phi - m\theta) \quad , \tag{39}$$

then the local radius $r(r_o, \theta, \phi)$ of a surface which had an unperturbed radius r_o is

$$r = r_o + \sum{}' \Delta_{nm} \cos(n\phi - m\theta) \tag{40}$$

provided $r_o \gg |\Delta_{nm}|$. The $m = 1$ components of the $\Delta_{nm}(r_o)$ go to a constant as $r_o \to 0$ and shift the magnetic axis. The horizontal axis shift, along $\theta = 0$, is $\sum{}'\Delta_{n,1} \cos(n\phi)$ and the vertical shift, along $\theta = \pi/2$, is $\sum{}'\Delta_{n1} \sin(n\phi)$.

Although Eq. (32) for h_r and Eq. (36) for b_r can be easily integrated to obtain the finite pressure equilibria, it is illustrative to consider a simple approximation to h_r. Consider the approximate forms

$$\frac{\delta_{nm}}{n-\iota m} \propto r^{|m|} \quad \text{and} \quad p = p_o\left(1 - \frac{r^2}{a^2}\right) \quad . \tag{41}$$

We also assume $g = RB_o$, which is generally an accurate approximation. One then finds

$$\begin{aligned} \Delta_{nm} &= \frac{\beta_o}{4}\frac{R^2}{r}\frac{\delta_{nm}(r)}{(n-\iota m)^2}\, |m|\left[1 - \frac{|m|}{1+|m|}\left(\frac{r}{a}\right)^2\right] \quad \text{for } r \le a \\ \Delta_{nm} &= \frac{\beta_o}{4}\frac{R^2}{r}\frac{\delta_{nm}(a)}{[n-\iota(a)m]\,[n-\iota(r)m]}\,\frac{|m|}{1+|m|}\left(\frac{a}{r}\right)^m \quad \text{for } r > a \end{aligned} \tag{42}$$

$$\beta_o \equiv 8\pi\, p_o/B_o^2 \quad .$$

In this approximation the shift of the surfaces at the plasma edge is half the shift of the axis for $m = 1$ terms. This relative shift was derived many years ago.[12] One can summarize the most important feature of these results by defining β_{nm} by

$$\frac{\Delta_{nm}(a)}{a} = \frac{1}{2} \frac{|m|}{1+|m|} \frac{\beta_o}{\beta_{nm}} \tag{43}$$

which implies β_{nm} is given by

$$\beta_{nm} = 2\left(\frac{a}{R}\right)^2 \frac{(n-\iota m)^2}{\delta_{nm}} \quad . \tag{44}$$

As long as the plasma beta is small compared to all the β_{nm}, the shifts are small. It should be noted, however, that for $\Delta_{nm}/a \simeq 1/2m$ nonlinear terms become important in determining the plasma equilibrium by modifying the δ_{nm}'s.

V. SUMMARY

The iterative method for finding net-current-free stellarator equilibria can be simplified by the use of appropriate magnetic coordinates. At any stage of the iteration, a magnetic field $\vec{B}(\vec{x})$ is assumed known. The initial guess can be the vacuum field $\vec{B}_v$. Given the desired pressure function $p(\psi)$, with ψ the radial flux coordinate, the plasma current $\vec{j}$ can be explicitly given in contravariant form, Eq. (20), using appropriate magnetic coordinates. The magnetic field due to the plasma $\vec{b}$ is determined by the current $\vec{j}$. This field can be evaluated by solving only Poisson's equations. To do this, let $\vec{b} = \vec{h} + \vec{\nabla} f$. The vector $\vec{h}$ is chosen to solve Amphere's law in the plasma region but be zero outside. An analytic expression for $\vec{h}$ is given in Eq. (29). The field $\vec{h}$ is not divergence-free. However, $\vec{b}$ can be made divergence-free by choosing f to be a solution of Poisson's equation, Eq.

(30), with $\vec{\nabla} f$ having the same boundary conditions as $\vec{b}$. Once the plasma field $\vec{b}$ is added to the vacuum field $\vec{B}_v$, a new magnetic field is obtained for further iteration.

The most important quantity, which enters the evaluation of $\vec{b}$, is the magnetic field strength expressed in the appropriate magnetic coordinates, $B(\psi,\theta,\phi)$. If other geometric information is ignored, which can be justified if the surfaces are nearly circular, then the iterative method is almost trivial to implement. All that is required is that the field strength be obtained in Fourier-decomposed magnetic-coordinate form, for which codes exist,[16,17] and then one-dimensional integrals can be performed to obtain $\vec{b}$. The new magnetic surfaces can be obtained by the obvious numerical integration although much information can be obtained analytically. This approximation is discussed in Sec. V. With further approximations, a generalization to complex geometry of the tokamak beta limit estimate, ε/q^2, can be obtained, Eqs. (43) and (44).

The iterative method is simple and also makes the implicit contraints on the pressure function $p(\psi)$ explicit. These constraints are the constancy of $p(\psi)$ in a region of stochastic field lines and near a rational surface on which $\oint d\ell/B$ is not constant.

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APPENDIX Plasma With Net Current

The obvious generalization of Sec. V. to the case of finite plasma current is

$$\vec{h} = \left(g_p(\psi)-g_p(\psi_a)\right)\vec{\nabla}\phi+\left(I_p(\psi)-I_a\right)\vec{\nabla}\theta+\left(\frac{4\pi(g+\iota I)}{B_o^2}\frac{dp}{d\psi}\sum{}' \frac{\delta_{nm}}{n-\iota m}\sin(n\phi-m\theta)\right)\vec{\nabla}\psi \tag{A1}$$

with $I_a = I_p(\psi_a)$. This expression for $\vec{h}$ is zero outside the plasma and almost has the plasma current, Eq. (20), as its curl. We say almost because the curl of $\vec{\nabla}\theta$ does not vanish at the magnetic axis,

$$\vec{\nabla}\times(\vec{\nabla}\theta) = \delta(\psi)\vec{\nabla}\psi \times \vec{\nabla}\theta \quad . \tag{A2}$$

Let $\vec{b}_w$ be the field produced by a wire carrying a unit current along the magnetic axis. A formalism similar to that of Sec. V. follows if we let the field due to the plasma equal

$$\vec{b} = \vec{h} + I_a\vec{b}_w + \vec{\nabla}f \quad . \tag{A3}$$

The function f is then a single valued function of position. Of course, both $\vec{h}$ and $\vec{b}_w$ are singular near the axis but

$$\vec{h}_* = \vec{h} + I_a\vec{b}_w \tag{A4}$$

is finite there. The function f satisfies Poisson's equation

$$\nabla^2 f = -\vec{\nabla}\cdot\vec{h}_* \quad . \tag{A5}$$

One can equally well use $\vec{\nabla}\cdot\vec{h}$ since $\vec{\nabla}\cdot\vec{h}$ and $\vec{\nabla}\cdot\vec{h}_*$ are equal. The field $\vec{b}_w$ can be evaluated numerically by placing current along the magnetic axis. The boundary conditions on f remain relatively simple. If $\vec{b}$ vanishes at infinity, then so does $\vec{\nabla}f$. If $\vec{b}\cdot\hat{n}$ is fixed on a surface, then the normal derivative of f plus $I_a\vec{b}_w\cdot\hat{n}$ is fixed.

References

[1] F. Bauer, O. Betancourt, and P. Garabedian, A Computational Method in Plasma Physics, (Springer Verlag, New York, 1978).

[2] R. Chodura and A. Schlüter, J. Comput. Phys. 41, 68 (1981).

[3] T. C. Hender, B. A. Carreras, and V. E. Lynch, Bull. of Am. Phys. Soc. 27, 1020 (1982) (paper 5P14).

[4] J. M. Greene and J. L. Johnson, Phys. Fluids 4, 875 (1961).

[5] H. R. Strauss, Plasma Phys. 22, 733 (1980); H. R. Strauss and D. A. Monticello, Phys. Fluids 24, 1148 (1981).

[6] L. M. Kovrizhnykh and S. V. Shchepetov, Fiz. Plazmy 7, 965 (1981) [Sov. J. Plasma Phys. 7, 527 (1981)].

[7] M. I. Mikhailov and V. D. Shafranov, Plasma Phys. 24, 233 (1982).

[8] G. Anania, J. L. Johnson, and K. E. Weimer, Princeton Plasma Physics Laboratory Report PPPL-1945 (1982).

[9] C. Mercier, Nucl. Fusion 4, 213 (1964).

[10] L. S. Solovev and V. D. Shafranov in Reviews of Plasma Physics, edited by M. A. Leontovich (Consultants Bureau, New York, 1970), Vol. 5., p. 1.

[11] D. Lortz and J. Nührenberg, Z. Naturforsch., 34a, 167 (1979).

[12] H. R. Koenig, U. S. Atomic Energy Commission Report No. NYO-7310 (PM-S-20), (1956).

[13] H. Grad and H. Rubin in Proc. of the Second United Nations Int'l Conf. on the Peaceful Uses of Atomic Energy (United Nations, Geneva, 1958), Vol. 31, p. 190.

[14] T. Yamagishi, W. D. Bard, and M. S. Chu, Nucl. Fusion 22, 527 (1982).

[15] A. H. Boozer, Phys. Fluids 24, 1999 (1981).

[16] A. H. Boozer, Phys. Fluids 25, 520 (1982).

[17]G. Kuo-Petravic, A. H. Boozer, J. A. Rome, and R. H. Fowler, Princeton Plasma Physics Laboratory Report PPPL-1866-R (1982).

[18]A. H. Boozer, Princeton Plasma Physics Laboratory Report PPPL-1881 (1982).

[19]M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, and G. M. Zaslavski, Nucl. Fusion 6, 297 (1966).

Figure Caption

Fig. 1. Magnetic Coordinates

A topologically toroidal constant pressure surface has two other surfaces associated with it which are the domain of the toroidal and the poloidal area integrals. The toroidal flux function ψ and the "plasma current" I are defined on one such surface. The poloidal flux function ψ_p and the "coil current" g are defined on the other. The pressure, ψ_p, g, and I are all functions of ψ alone.

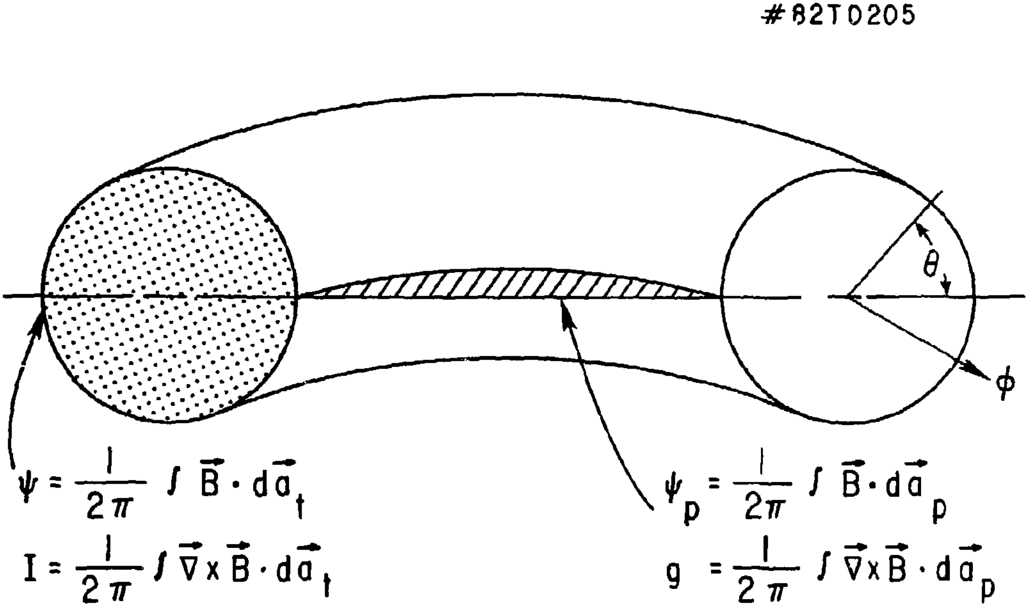
#82T0205
θ
ϕ
$\psi = \frac{1}{2\pi} \int \vec{B} \cdot d\vec{a}_t$
$I = \frac{1}{2\pi} \int \vec{\nabla} \times \vec{B} \cdot d\vec{a}_t$
$\psi_p = \frac{1}{2\pi} \int \vec{B} \cdot d\vec{a}_p$
$g = \frac{1}{2\pi} \int \vec{\nabla} \times \vec{B} \cdot d\vec{a}_p$

Fig. 1