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NUMERICAL SOLUTION OF THE MULTIDIMENSIONAL
BUCKLEY-LEVERETT EQUATION BY A SAMPLING METHOD

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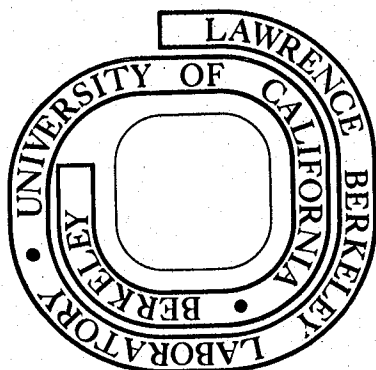
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ABSTRACT

A method developed earlier for solving numerically the one-dimensional Buckley-Leverett equation for two phase immiscible flow in a porous medium is extended to the case of non-uniform flow in two space dimensions. The method has the feature of tracking solution discontinuities sharply for purely hyperbolic problems, without requiring devices such as the introduction of artificial dissipation. It is found that the method is computationally efficient for solving a numerical example for the five-spot configuration of water flooding of a petroleum reservoir.

INTRODUCTION

In the study of the simultaneous flow of two incompressible, immiscible fluids through a porous medium, one is led to a first-order nonlinear partial differential equation, the Buckley-Leverett equation [1, pp. 19-22]. When capillary pressure effects are small or absent, this equation is hyperbolic in nature. For the purely hyperbolic case (zero capillary pressure) it is well known that, in general, solutions develop discontinuities in finite time, even for smooth initial data. These discontinuities correspond to propagating fronts between the two fluids. If a small, but non-zero, amount of capillary pressure is present then the fronts that are developed will not be perfectly sharp, but will correspond to a large change in fluid saturation over a small, but non-zero, distance.

The representation of such discontinuities, or near discontinuities, usually causes difficulty for conventional numerical methods, which are based on differencing or other discretizations, because of underlying assumptions on the smoothness of solutions [2,3]. These methods often require devices such as the introduction of artificial dissipation.

The purpose of our study is the development of a numerical method that is inherently capable of tracking sharp fronts in porous flow problems. This

References and figures are at the end of the paper.

development is based on extending to the porous flow case the Chorin-Glimm random choice method, which was formulated originally for tracking solution discontinuities for the equations of gas-dynamics [4]. The first stages of the extension are initiated in [5] and are continued in the present paper.

THE SATURATION EQUATION

The Buckley-Leverett saturation equation for the flow of two immiscible, incompressible liquids through a homogenous porous medium in the absence of capillary pressure and gravitational effects, in a region free of sources or sinks, is

$$\frac{\partial s}{\partial t} + \underline{v} \cdot \nabla [f(s)] = 0, \quad (1)$$

where $\underline{v} = (v_x, v_y)$ is the total velocity, and $s(\underline{x}; t)$ and $f(s)$ are respectively the saturation and fractional flow of the wetting liquid. (For convenience, the porosity (a constant) has been absorbed into the other variables and does not appear explicitly in (1).)

Typically $f(s)$ has the S-shape shown in Figure 1. This shape, which consists of a convex and concave part joined by an inflection point, is fundamental in determining the behavior of solutions of (1). This behavior is qualitatively more complicated than that of solutions of the hyperbolic equations occurring in gas dynamics, for which there is no inflection in the function corresponding to $f(s)$. The presence of an inflection permits the possibility of a juxtaposition of propagating discontinuities and expansion waves [5]. Related consequences are reported in [3], where it is observed that some numerical methods may not give the correct solution of the limiting purely hyperbolic problem as the capillary pressure approaches zero.

BASIC NUMERICAL PROCEDURE

In [5] the Chorin-Glimm random choice method is applied to solving (1) numerically for the case of one space dimension (for which $\underline{v} = \text{const.}$) and of

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two space dimensions with $v_x = v_y = \text{const.}$ For these cases (1) is, except for a constant multiplier, in conservation-law form. The following are among the features of the method:

1. The representation of discontinuities (sharp fronts) is based not on differencing or other discretizations but on local Riemann solutions and a sampling procedure.
2. Discontinuities are propagated sharply.
3. Devices such as artificial dissipation (capillary pressure) are not required.
4. For a purely hyperbolic problem the correct solution is obtained, corresponding to the limiting solution of parabolic problems as the dissipation approaches zero.

The method advances a solution one step in time by (a) approximating the solution at the initial time by a piecewise-constant function on a spatial grid, (b) solving analytically with the piecewise-constant initial data, (c) sampling this analytic solution to obtain values for a piecewise-constant approximation at the new time.

The time increments are chosen sufficiently small so that the waves propagating from initial discontinuities at the spatial grid points do not interact. The analytic solution may then be obtained by joining together the separate solutions to the Riemann problems for the propagation from each discontinuity. (A Riemann problem is one that has initial data that is constant to the left and to the right of a single step-discontinuity.)

The reader is referred to [5] and to the references therein for a complete description of the numerical method and for the background material relevant to the discussion in the next sections. Subsequently we shall refer to the method as the Piecewise Sampling Method (PSM).

NUMERICAL PROCEDURE FOR THE MULTIDIMENSIONAL CASE

As discussed in [5], a multidimensional problem can be advanced in time by PSM by splitting into a sequence of one-dimensional problems. In that paper velocities that were constant were the only ones considered. Here we discuss the splitting for the case in which $\underline{v}$ may be a function of x and y . (Other approaches that do not involve splitting for multidimensional problems are possible, but we defer their consideration to a subsequent study.)

The solution of (1) is advanced one time increment (half a complete time step, as described in [5]) by first solving

$$\frac{\partial s}{\partial t} + v_x \frac{\partial}{\partial x} [f(s)] = 0 \quad (2)$$

for each line of constant y in the spatial grid and then by solving

$$\frac{\partial s}{\partial t} + v_y \frac{\partial}{\partial y} [f(s)] = 0 \quad (3)$$

for each line of constant x . Since v_x and v_y are functions of x and y , the characteristics of (2) or (3) generally are not straight lines in the x - t or y - t planes, as they are for the constant- $\underline{v}$ case. Thus, obtaining for $y = y_j$ the solution of the Riemann problems (2) with initial condition

$$s(x, y_j; 0) = \begin{cases} s_L, & x < x_1 \\ s_R, & x > x_1 \end{cases}, \quad (4)$$

and obtaining corresponding solutions for (3), can be a more complicated task than for the cases considered in [5]. In (4) s_L and s_R are the values of s to the left and right, respectively, of the discontinuity at x_1 on the line $y = y_j$.

We obtain the approximate solution of the Riemann problems (2), (4) along the line $y = y_j$ by using in (2) the average of $v_x(x_1, y_j)$ and $v_x(\bar{x}, y_j)$, where $\bar{x}$ is the sampled value of x at the new time level in the interval surrounding x_1 . This permits direct use of the Riemann problem solutions that are given in [5] based on straight-line characteristics. We solve successively the Riemann problems corresponding to (2) and then those corresponding to (3), followed in the next complete time step by splitting in the reverse order, (3) then (2).

IMPROVED SAMPLING PROCEDURE

It has been shown [4,6,7] that if an equidistributed sampling procedure is used in PSM, better solutions can be expected than if a procedure containing a random element, such as the one in [5], is used. In numerical experiments we have compared the van der Corput equidistributed sampling procedure of [7] with the procedure used in [5] and have found the former sampling generally to give more satisfactory results. The procedure in [7] is therefore used for the numerical experiments reported below.

TEST PROBLEM

For our test problem with which to investigate the behavior of PSM, we select the total velocity $\underline{v}$ in (1) to be the potential flow for the standard five-spot configuration with unit sources at $x = 2p$, $y = 2q$ and sinks at $x = 2p-1$, $y = 2q-1$, $p, q = \dots, -2, -1, 0, 1, 2, \dots$. That is, for the quarter configuration of the unit square (see Figure 2), we take

$$\underline{v} = \text{grad } \psi$$

where ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \delta(x)\delta(y) - \delta(x-1)\delta(y-1) \quad (5)$$

with no-flow conditions

$$\partial \psi / \partial n = 0 \quad (6)$$

on the edges of the square. This y is the actual velocity in a five-spot configuration for the case of constant saturation.

Correspondingly, we take the boundary conditions on s for (1) to be that $\partial s / \partial n = 0$ on the edges of the unit square and $s = 1$ at the source $(0,0)$. Initially we take $s \equiv 0$ everywhere except at the source. The function $f(s)$ is the same one used in [3] and [5], with a ratio of water to oil viscosities of $1/2$,

$$f(s) = \frac{s^2}{s^2 + \frac{1}{2}(1-s)^2}$$

(see Figure 1).

For this test problem it is possible to obtain a solution analytically by means of a coordinate transformation, for comparison with the numerical solution obtained by PSM. Consider the orthogonal coordinates (ψ, η) , where $\psi(x, y) = \text{const.}$ are the velocity equipotentials and $\eta(x, y) = \text{const.}$ are the flow lines. In these coordinates (1) becomes

$$\frac{\partial s}{\partial t} + v^2 \frac{\partial}{\partial \psi} [f(s)] = 0,$$

where now $s = s(\psi, \eta; t)$ and $v = v(\psi, \eta)$.

We define

$$\xi(\psi, \eta) = \int_{-\infty}^{\psi} \frac{d\sigma}{v^2(\sigma, \eta)}.$$

If we transform to (ξ, η) coordinates, the saturation equation simplifies further to

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial \xi} [f(s)] = 0. \quad (7)$$

For each value of η , the characteristics of (7) are straight lines in the ξ - t plane.

For our test problem, with the initial conditions described above, (7) can be solved explicitly as a Riemann problem (case (IIa) of [5]). For each value of η , the solution $s(\xi, \eta; t)$ consists of a discontinuity propagating with constant speed, followed by an expansion wave (see Figure 5 of [5]). We have, for our $f(s)$ and $t > 0$,

$$s(\xi, \eta; t) = \begin{cases} 0, & \xi > at \\ 3^{-1/2} \approx 0.577, & \xi = at \\ \text{expansion wave,} & 0 \leq \xi < at \end{cases},$$

where a , the propagation speed (in the ξ - η plane) of the discontinuity, is given by

$$a = \frac{1}{2}(1 + \sqrt{3}) \approx 1.366.$$

The solution $s(\xi, \eta; t)$ was computed at times $t = 0.5, 1.0, 1.5, 2.0, 2.2$, and 2.5 . The contours on the x - y plane of $s = 0.577, 0.6, 0.7$, and 0.8 for these times are shown in Figures 4(a)-(f). The contour for $s = 0.577$ depicts the position of the discontinuity, in front of which $s \equiv 0$. The contours were drawn using subroutine CONREC from the National Center for Atmospheric Research (NCAR) graphics package. (The small oscillations of some of the contours are due to the effects of interpolation from a discrete grid.)

RESULTS

For solving the test problem numerically by PSM, a 41×41 uniform mesh parallel to the x - and y -axes was used for the representation of s on the square. The time increment k at each time step was chosen essentially to be as large as possible consistent with the Courant condition

$$\frac{k}{h} \max \left| v \frac{df(s)}{ds} \right| < \frac{1}{2}$$

being satisfied in the interior, where h is the spatial mesh increment. The velocity y was calculated by differencing the numerical solution of (5), (6), as obtained using subroutine PWSCRT of the NCAR elliptic partial differential equation package.

Contours $s(x, y; t) = 0.4(0.1)0.8$ of the numerical solution obtained with PSM are plotted at times $t = 0.5, 1.0, 1.5, 2.0, 2.2$, and 2.5 in Figures 3(a)-(f), respectively. These times are the same as those for Figures 4(a)-(f), which depict the analytically derived solution. Since the NCAR plotting package displaces contours that should lie on top of each other into separate ones, the contours for $s = 0.1, 0.2$, and 0.3 were chosen not to be drawn in Figures 3(a)-(f). In essentially all cases these should lie underneath the 0.4 contour, which in turn should lie under the 0.5 contour -- rather than separated from it as shown.

One observes from the figures that PSM is able to track the advancing front sharply in a satisfactorily accurate manner. Fluctuations of the order of one grid spacing, natural to PSM, occur in the contours, arising partly because splitting is used. These are stable, however, and do not grow with time. In a problem with dissipation (capillary pressure) these fluctuations would, of course, be smoothed.

Figure 3(e), corresponding to $t = 2.2$, depicts the solution obtained at the earliest time for which breakthrough of the displacing liquid into the sink had occurred. For the previous time, $t = 2.1$, at which the solution was computed (not shown in Figure 3) breakthrough had not yet occurred. These times are in good accord with the time of breakthrough $t = 2.132$ obtained for the analytically derived solution.

The average computer time required to advance the solution one time-step with our program on the CDC 7600 was approximately 0.10 seconds for the 41×41 mesh. A total of 157 steps were required to reach breakthrough ($t = 2.2$). For an actual five-spot water flooding problem, y is not a fixed function of x and y but must be computed along with

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s as time is advanced. Approximately 50% additional computer time would be required to include this computation, assuming that the amount of extra work would be about equivalent to that of solving (5), (6) by PWSCRT at each time step.

CONCLUDING REMARKS

Our experiments have indicated that the Piecewise Sampling Method is able to obtain a numerical solution of the test problem efficiently, with an accurate estimate of the time of breakthrough. The method advances the front sharply without introducing numerical dissipation. Alternatives to splitting in PSM, now under study, hold promise of further reduction of fluctuations in multidimensional solutions. PSM is capable of being extended to include the effects of gravity and of capillary pressure.

NOMENCLATURE

(in general, variables are nondimensional)

- a - propagation speed of discontinuity
- f - fractional flow of wetting liquid
- h - spatial mesh increment
- k - time increment
- s - saturation of wetting liquid
- t - time
- $\underline{y}$ - total velocity
- v_x - x-component of total velocity
- v_y - y-component of total velocity
- x, y - Cartesian coordinates
- $\partial/\partial n$ - derivative with respect to outward pointing normal
- $\delta(x)$ - Dirac delta distribution
- η - flow-line coordinate
- ξ - transformed velocity-potential coordinate
- ψ - velocity potential

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

- { Figure 1. Fractional flow as a function of saturation.
- { Figure 2. Quarter configuration of the five-spot problem.

Figure 3. Saturation contours 0.4(0.1)0.8 of the numerical solution at times (a) 0.5, (b) 1.0, (c) 1.5, (d) 2.0, (e) 2.2, and (f) 2.5.

Figure 4. Saturation contours 0.577, 0.6, 0.7, and 0.8 of the analytically derived solution at times (a) 0.5, (b) 1.0, (c) 1.5, (d) 2.0, (e) 2.2, and (f) 2.5.

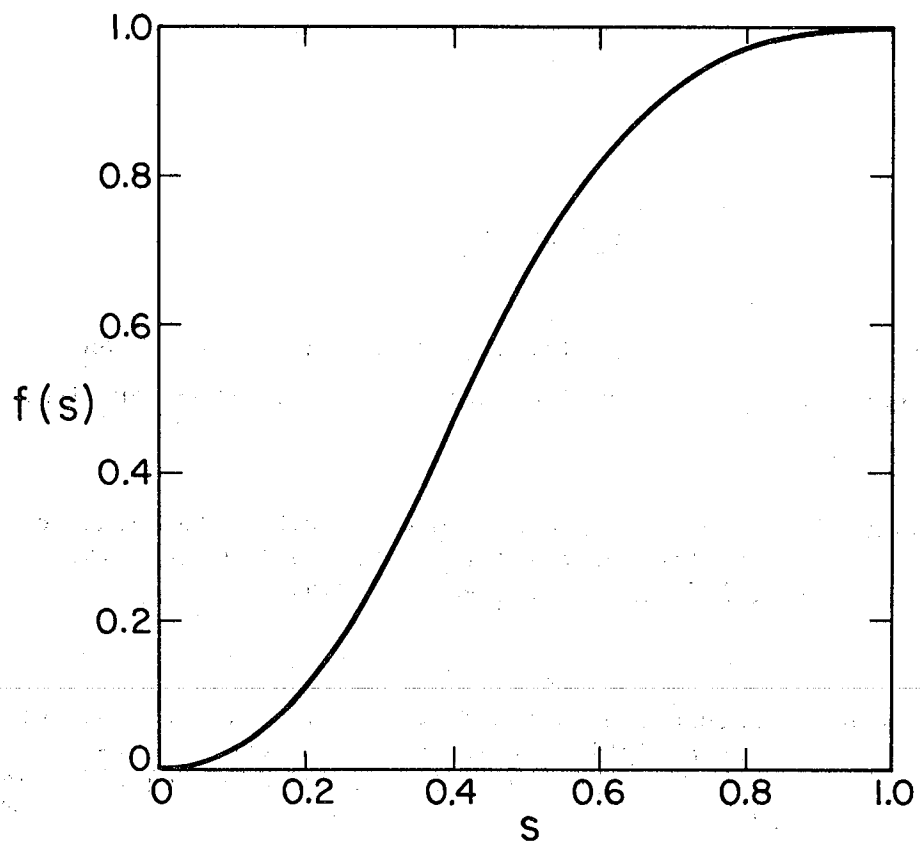
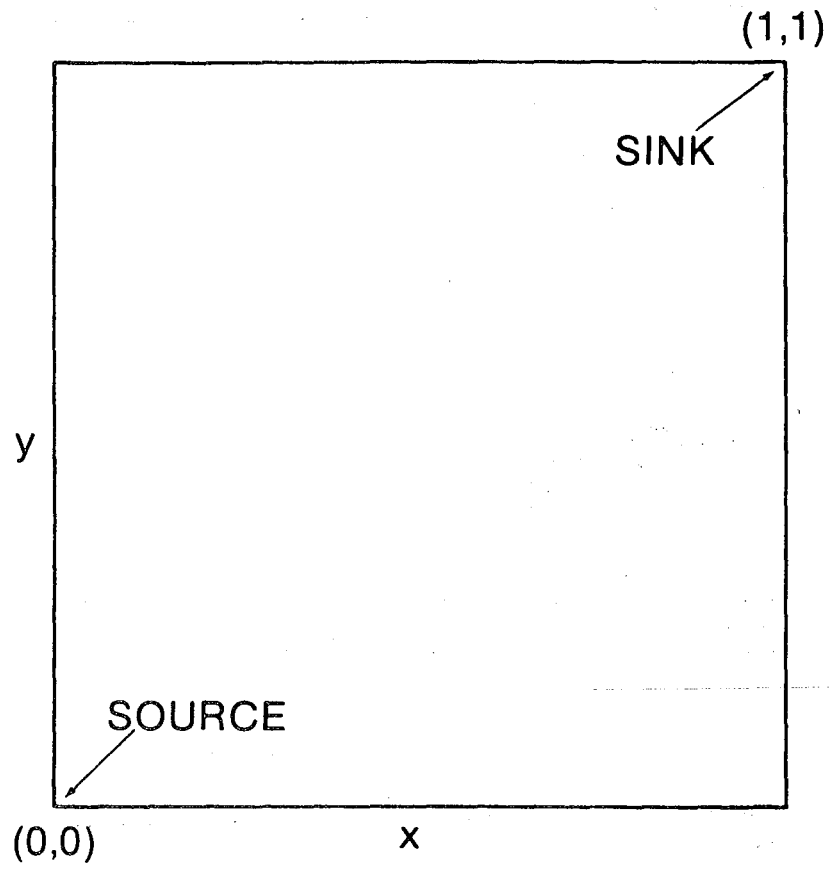


Figure 1



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Figure 2

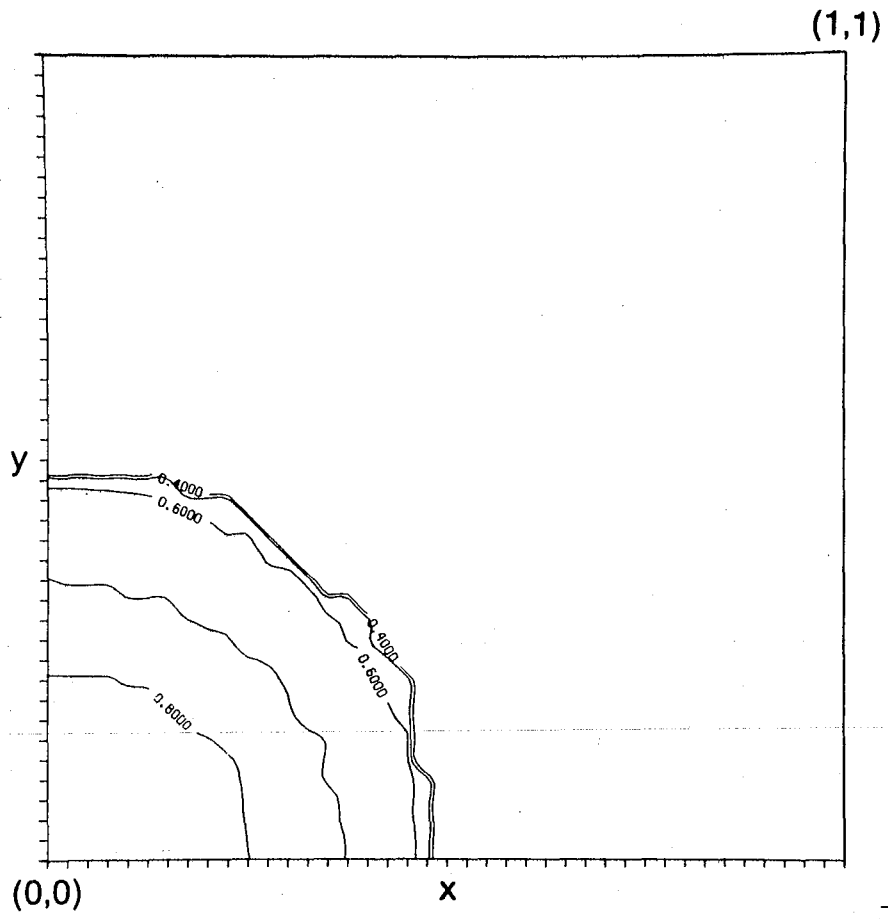


Figure 3a

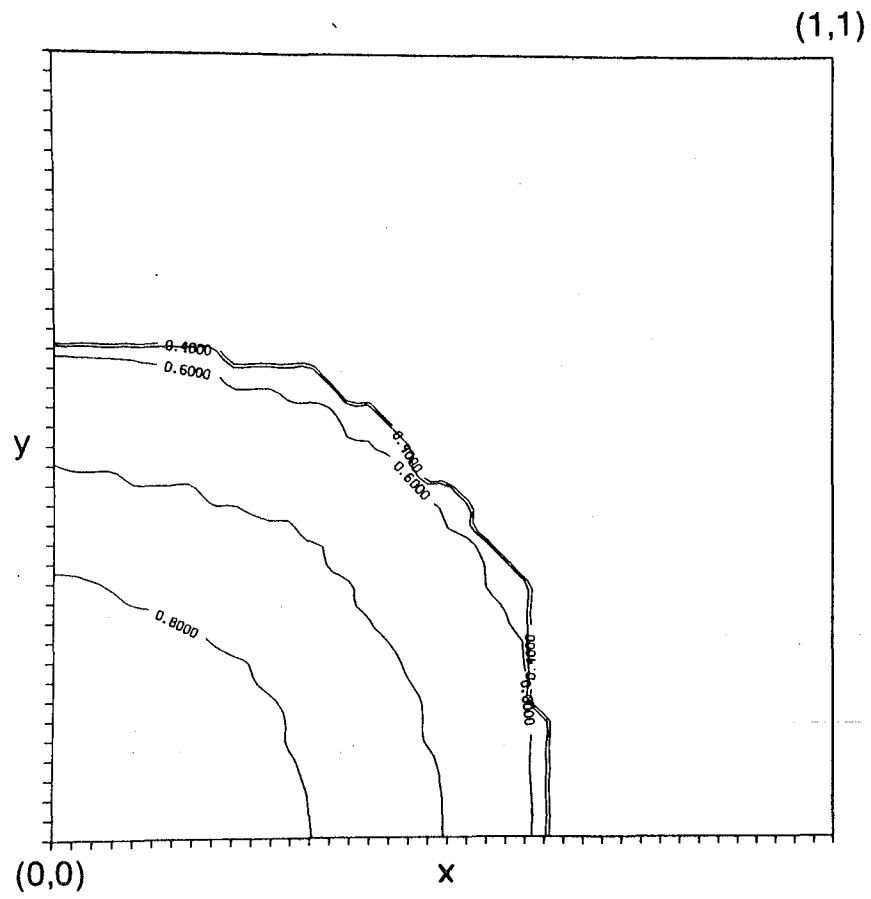


Figure 3b

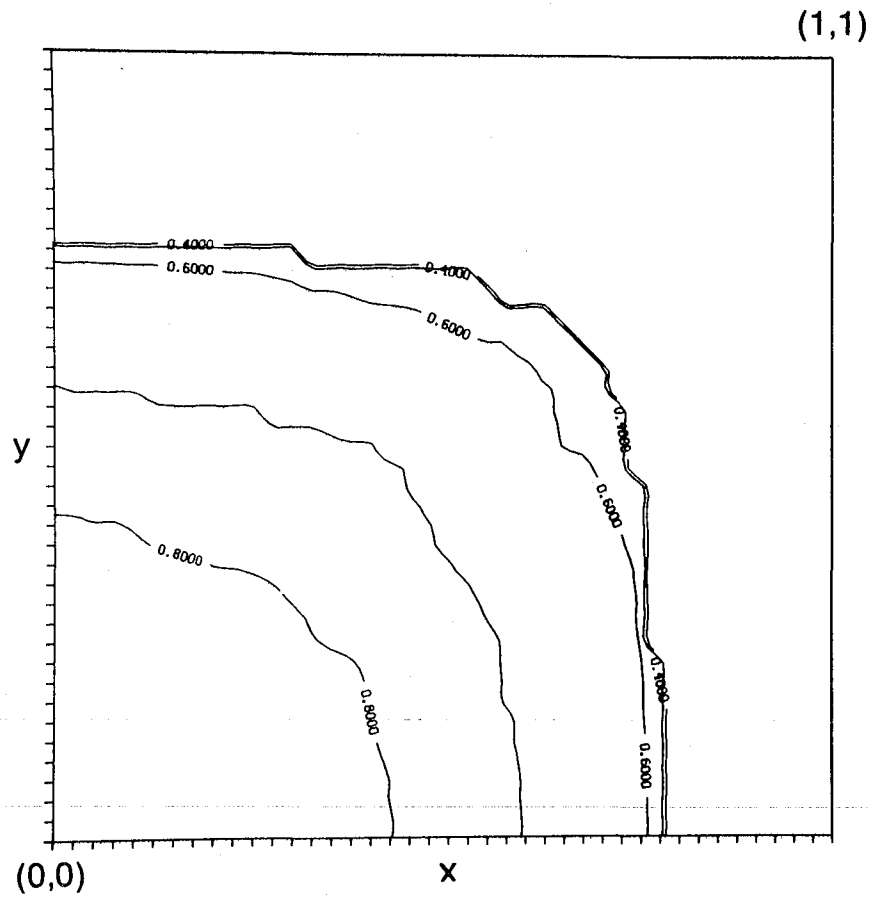


Figure 3c

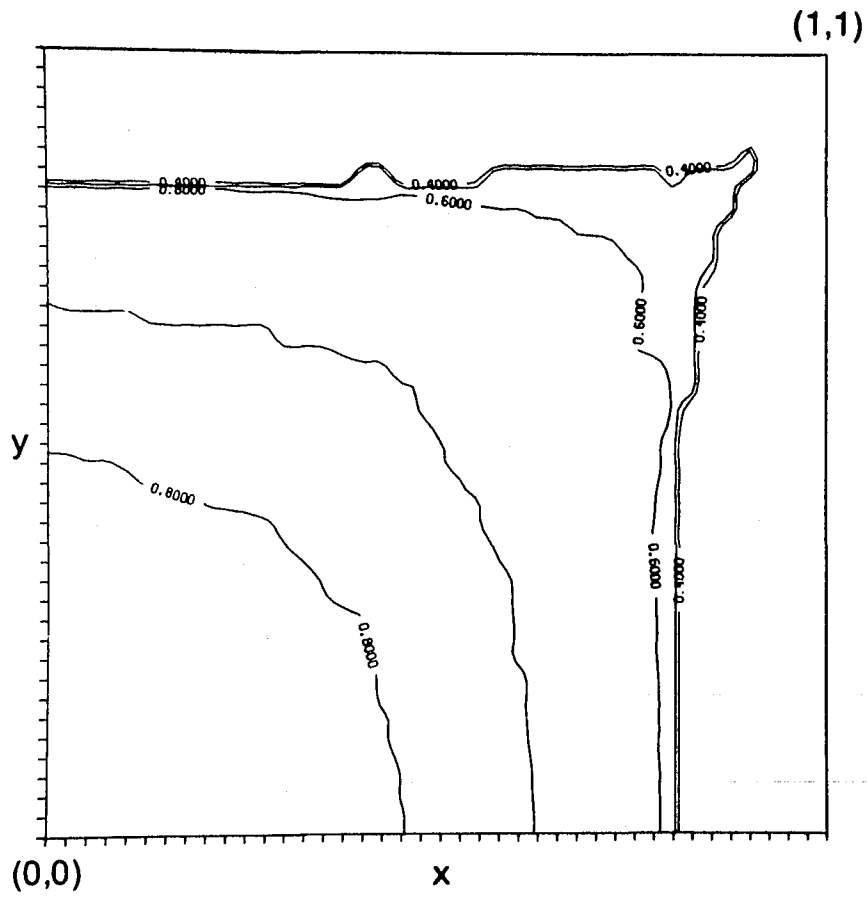


Figure 3d

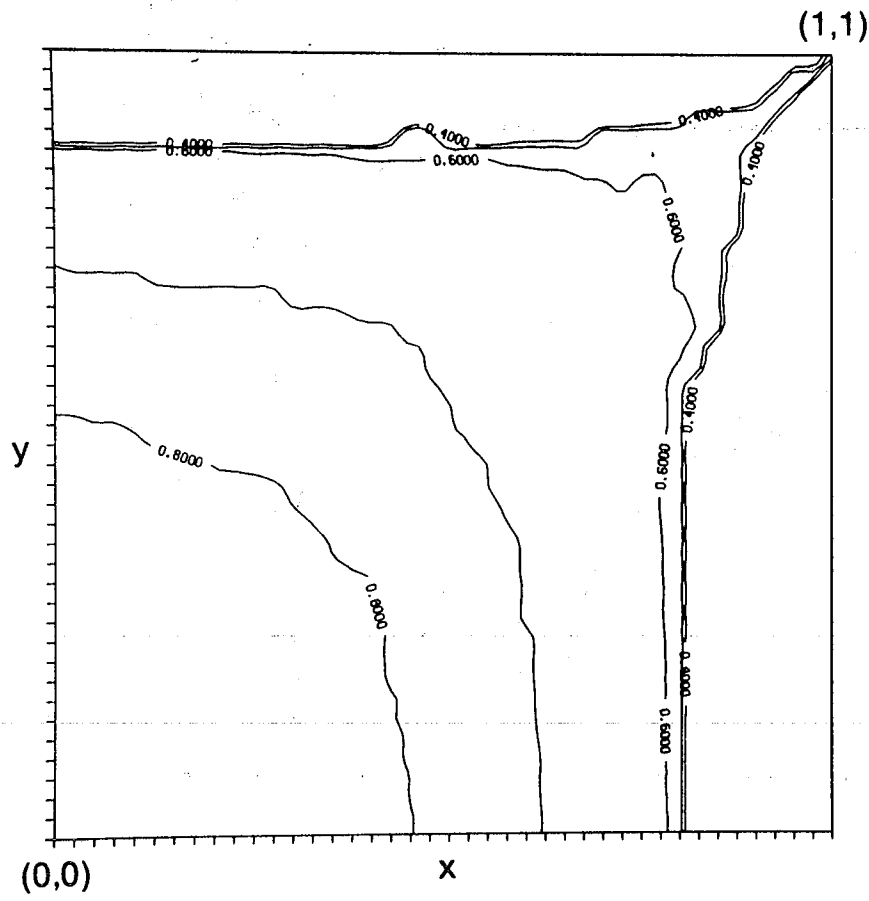


Figure 3e

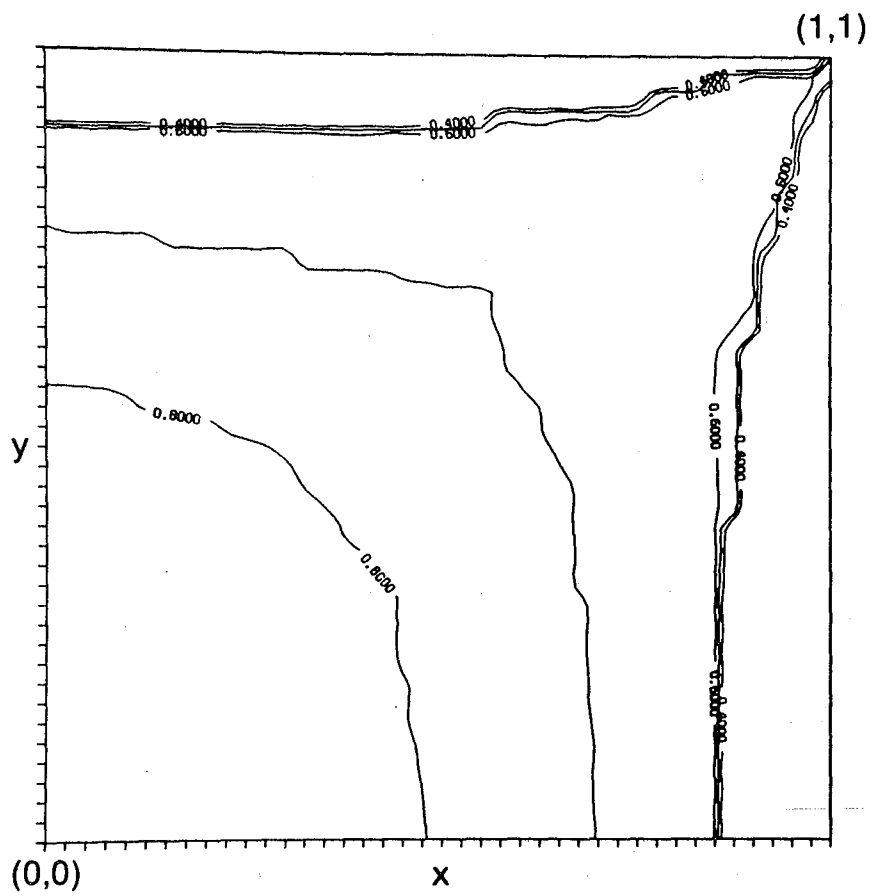
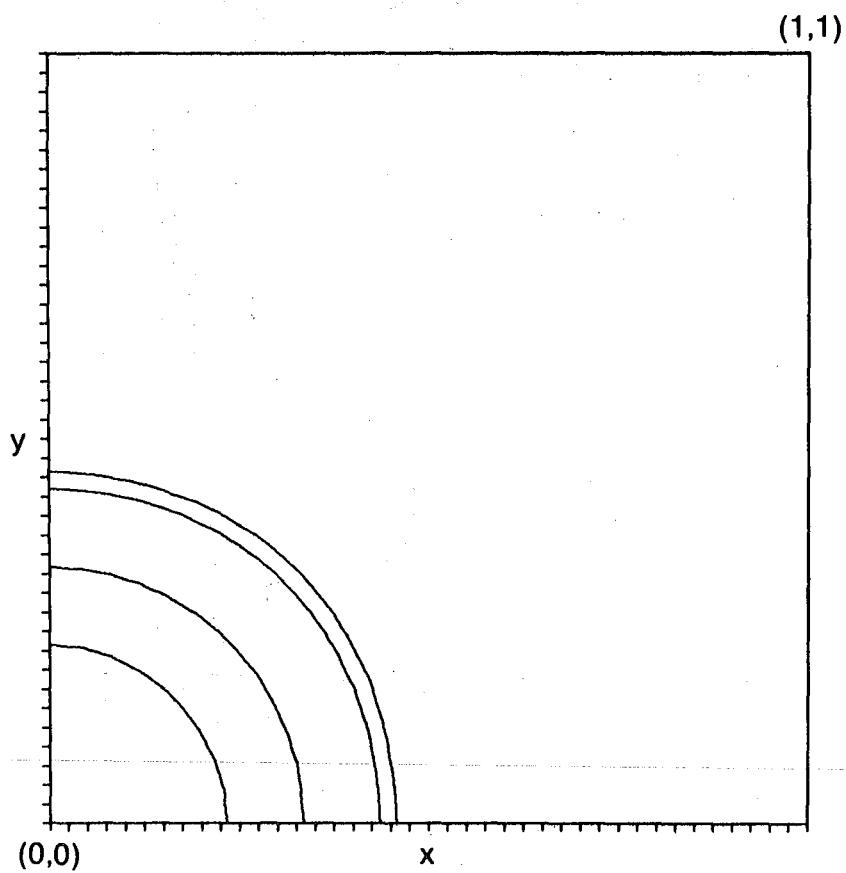
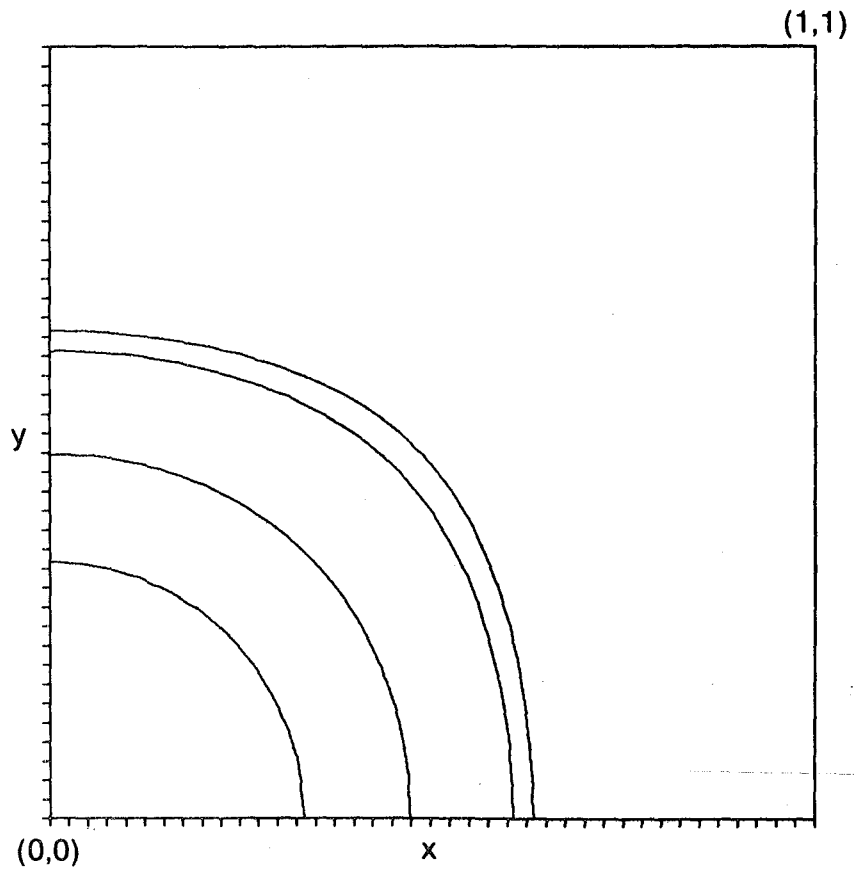


Figure 3f



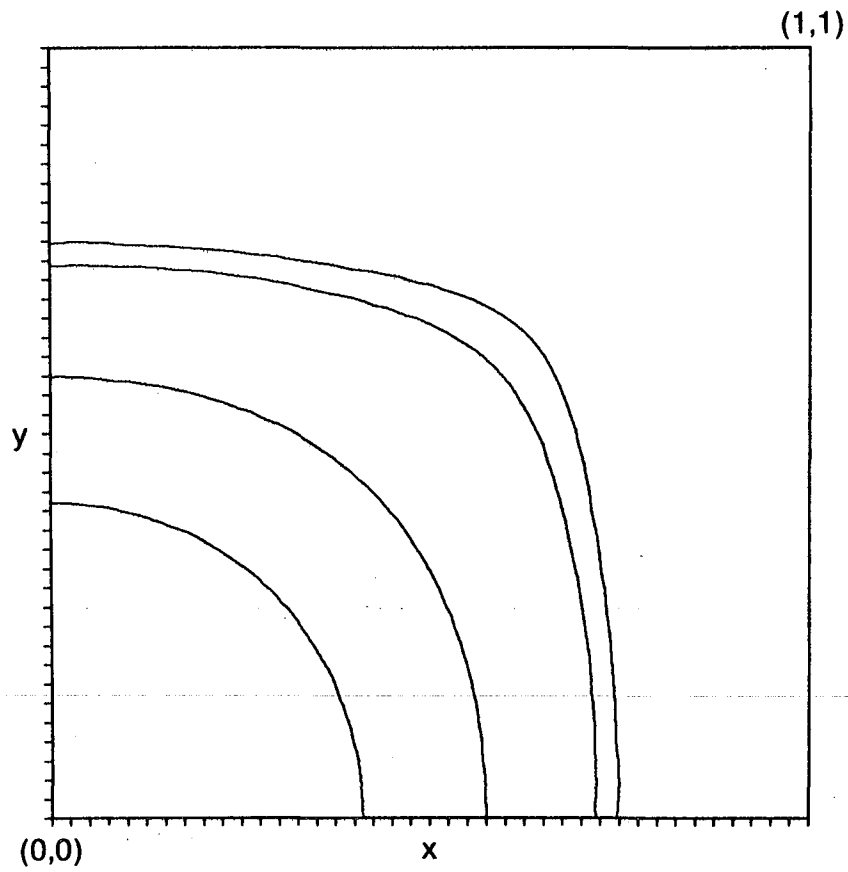
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Figure 4a



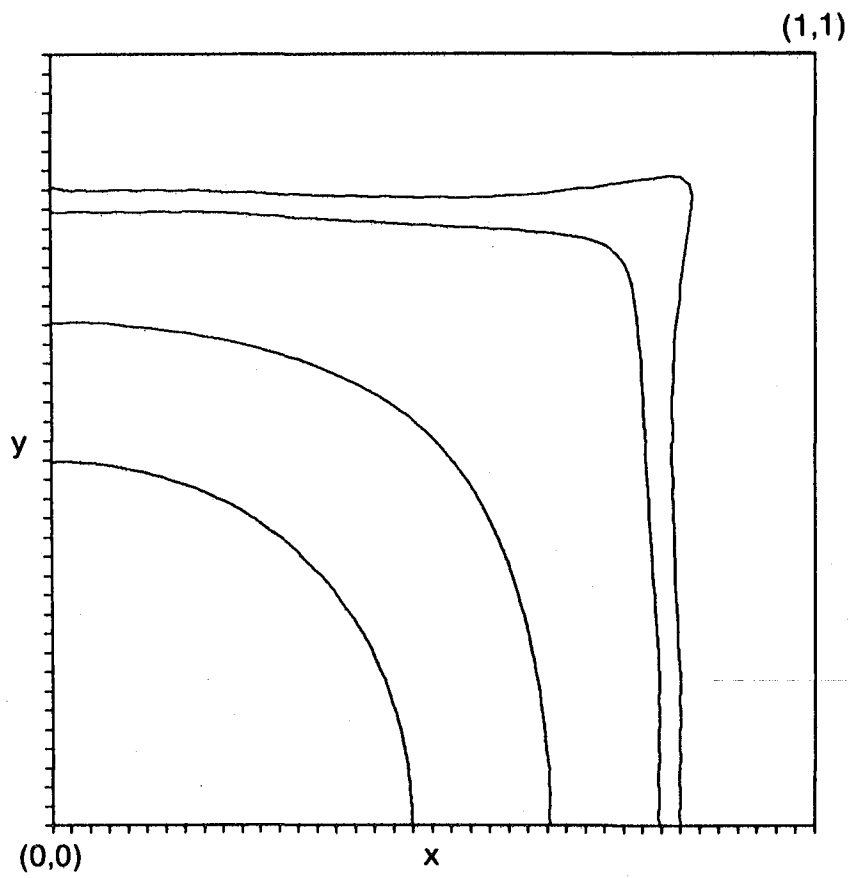
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Figure 4b



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Figure 4c



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Figure 4d

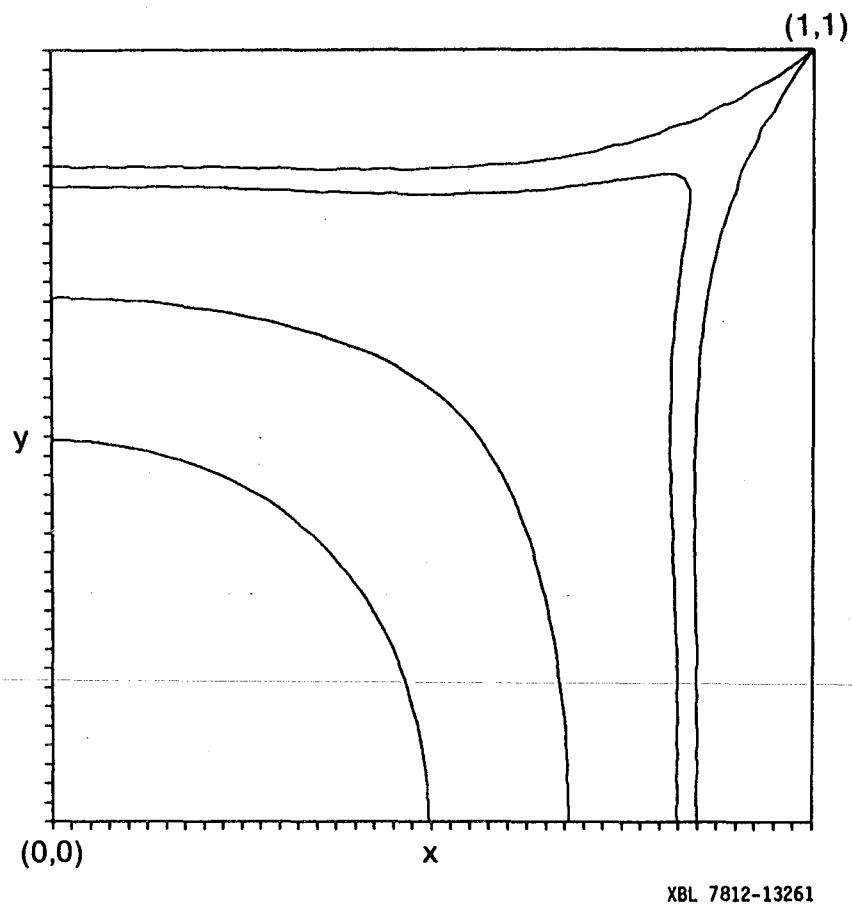
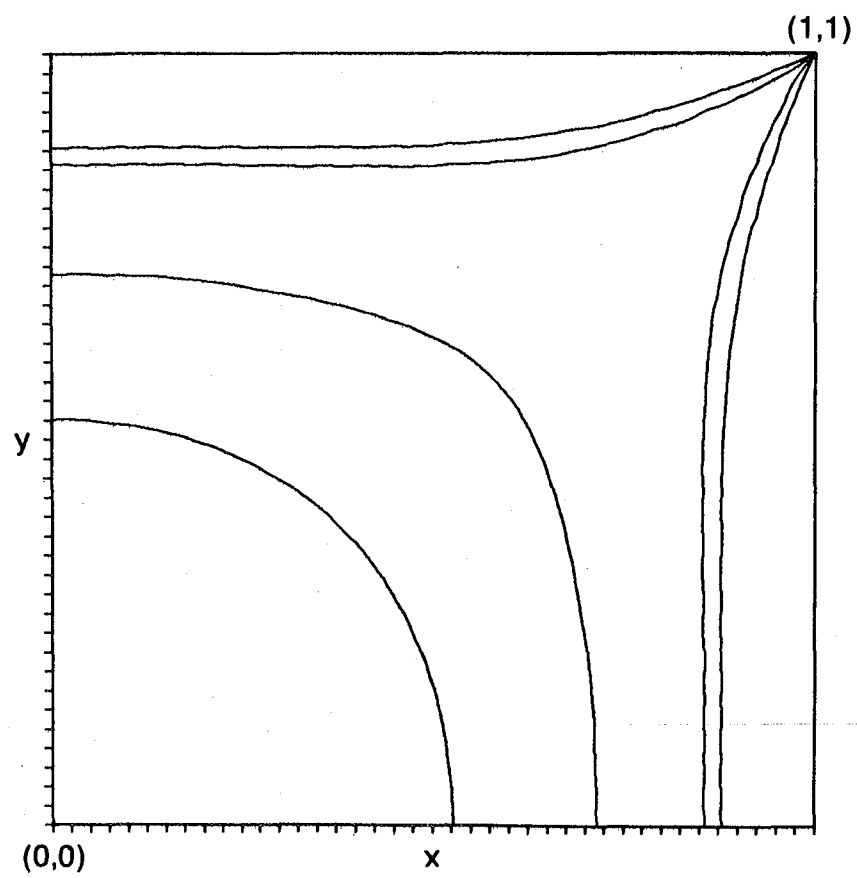


Figure 4e



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Figure 4f

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