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ABSTRACT

Shell model Hamiltonians which have a set of eigenstates made up of only monopole and quadrupole pairs are presented. These Hamiltonians have many other types of states as well. The subspace of states made up of monopole and quadrupole pairs have a one-to-one correspondence to states made up of monopole and quadrupole bosons. Some properties of these Hamiltonians are discussed.

FERMION HAMILTONIANS WITH MONOPOLE AND QUADRUPOLE PAIRING*

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INTRODUCTION

In this workshop we have seen evidence that the interacting boson model (IBM) seems to provide a unified phenomenological model of vibrational, transitional, and rotational nuclei. The underlying concept of this model is that the low-lying collective states of heavy nuclei are made up of monopole and quadrupole Bosons $s^\dagger$ and $d_\mu^\dagger$ bosons where $\mu = -2,-1,0,1,2$. These bosons can be interpreted to represent correlated pairs of valence nucleons outside the closed core. Hence a natural view of the IBM is that it is an approximation to the complicated shell model description of these heavy nuclei. That is, the valence nucleons move in an average field and the residual interactions produce correlated monopole, $S^\dagger$, and quadrupole, $D_\mu^\dagger$, pairs of valence nucleons. The collective low-lying states are then comprised mostly of these pairs, and the effect of the many other more complicated states is mainly to renormalize the shell model Hamiltonian and transition operators.

Given this viewpoint it is natural to ask the question: Are there shell model Hamiltonians which will have a class of eigenstates made up only of monopole and quadrupole pairs? The answer to the question is yes, and I shall discuss examples of such Hamiltonians in this talk. Of course these Hamiltonians will be model Hamiltonians in the same sense that the famous pairing Hamiltonian with degenerate single-particle energies is a model Hamiltonian. Nevertheless these models can be very instructive for understanding the microscopic structure of the IBM and also may provide insight on how more realistic shell model Hamiltonians may provide eigen-

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states made up primarily of monopole and quadrupole pairs.

DISCUSSION

The monopole and quadrupole pair operators will be linear combinations of nucleon pairs in spherical shell model orbits labeled by angular momentum j and projection m:

$$S^{\dagger} = \tfrac{1}{2} \Sigma_j \alpha_j \sum_m (-1)^{j-m} A^{\dagger}_{jm} A^{\dagger}_{j-m} \tag{1a}$$

$$D^{\dagger}_{\mu} = \Sigma_{j,j'} \beta_{jj'} \left[A^{\dagger}_j A^{\dagger}_{j'} \right]^2_{\mu} \tag{1b}$$

where $A^{\dagger}_{jm}$ creates a nucleon in orbit (j,m) and the brackets $[\ \ ]^J_{\mu}$ mean that these operators are coupled to angular momentum J and projection μ,

$$\left[A^{\dagger}_j A^{\dagger}_{j'} \right]^J_{\mu} \equiv \sum_{m,m'} (jmj'm' | jj'J\mu) A^{\dagger}_{jm} A^{\dagger}_{j'm'} \tag{1c}$$

where $(jmj'm'|jj'J\mu)$ is a Clebsch-Gordon coefficient.

We consider a shell model Hamiltonian H which has a spherical average one-body field h and a two-body interaction υ between valence nucleons:

$$H = \sum_{i=1}^{n} h_i + \sum_{i<j}^{n} \upsilon_{ij} \tag{2}$$

where n is the number of valence nucleons. We are interested in such Hamiltonians which have a class of eigenstates which are linear combination of the states made up of monopole and quadrupole nucleon pairs:

$$|N\, N_d \gamma JM\rangle = \left(S^{\dagger}\right)^{N-N_d} \left(D^{\dagger}\right)^{N_d}_{\gamma JM} |0\rangle \quad , \tag{3a}$$

where

$$N = \tfrac{1}{2} n \quad , \tag{3b}$$

and N_d is the number of quadrupole pairs. These quadrupole pairs are coupled to angular momentum J and projection M, and γ refers to any additional quantum numbers which may be necessary. We refer to the space spanned by the states in (3) by $(S^{\dagger},D^{\dagger})^N$ for convenience.

The necessary and sufficient conditions that a shell model Hamiltonian have a class of eigenstates in the $(S^{\dagger},D^{\dagger})^N$ space are an extension of those given by Talmi for generalized seniority[1,2]. One set of conditions is that the pair operators (1) create two-nucleon states which are eigenstates of H,

$$[H,S^\dagger]|0\rangle = E_0 S^\dagger |0\rangle \tag{4a}$$

$$[H,D^\dagger_\mu]|0\rangle = E_2 D^\dagger_\mu |0\rangle \tag{4b}$$

where $|0\rangle$ represents the core of A-n nucleons, where A is the total number of nucleons in the nucleus.

Another set of conditions is that the double commentators of the pair operators give back the pair operators:

$$\left[[H,S^\dagger],S^\dagger\right] = (00|\Delta|00)S^\dagger S^\dagger + (00|\Delta|22)\, D^\dagger D^\dagger \tag{5a}$$

$$\left[[H,S^\dagger],D^\dagger_\mu\right] = (02|\Delta|02)S^\dagger D^\dagger_\mu + (02|\Delta|22)\left[D^\dagger D^\dagger\right]^2_\mu \tag{5b}$$

$$\begin{aligned}\left[[H,D^\dagger_\mu],D^\dagger_{\mu'}\right] = {} & \delta_{\mu,-\mu'}(-1)^\mu(22|\Delta|00)S^\dagger S^\dagger \\ & + 2\,(2\mu 2\mu'|22\;2,\mu+\mu')(22|\Delta|02)S^\dagger D^\dagger_{\mu+\mu'} \\ & + \sum_{J=0,2,4}(2\mu 2\mu'|22J,\mu+\mu')(22|\Delta_J|22)\left[D^\dagger D^\dagger\right]^J_{\mu+\mu'}\end{aligned} \tag{5c}$$

These conditions are very restrictive. One way to satisfy them is by means of group theory. Since the pair creation operators span a six-dimensional space we look for groups which have a six-dimensional irreducible representation which contain states with angular momentum zero and two. There are only three such groups, SU_3, SO_6 and SU_6. If we then include single-particle levels which fit into representations of one of these groups, we can then construct an H which is invariant with respect to the group, and hence the conditions (4) and (5) must be satisfied. However of the three groups, only one has representations which have angular momenta corresponding to single-nucleon angular momenta. This group is the SO_6 group. In a recent paper[3] we considered a particular example of this method which corresponded to valence nucleons filling the (1p, 0f) major shell. Other examples corresponding to other shells can be constructed as well.

This method restricts the possible single-particle angular momenta that are allowed. There is another method to obtain solutions which is less restrictive. We will not go into the details of this method, which will be explained in a forthcoming paper, but just give the solution.

For this solution the single-particle orbitals are degenerate in energy, as is the case for the exactly soluble pairing model. This feature may not be a severe handicap because for many valence nucleons the two-body interactions will be more important than the single-particle energies.

Furthermore for this solution the single-particle orbitals will come in groups with all j between the limits,

$$k + \frac{3}{2} \geq j \geq |k-\frac{3}{2}|, \tag{6}$$

being necessary. Here k is any integer. We note that for k=1, the orbitals are those of the (1s,0d) major shell, and for k=2, of the (1p,0f) major shell. Different combinations of orbitals may be taken, as long as the k's differ by four, so that a particular angular momentum does not occur more than once. We don't feel that this grouping has any deep physical meaning but is just a peculiarity of the model. What is significant is that, except for the trivial case of k=0 (j=3/2 only), many spherical orbitals are needed to have a decoupling of the $(S^\dagger,D^\dagger)^N$ space from the other complicated states.

A set of Hamiltonians which are solutions of equations (4) and (5) and hence have this decoupling feature is given by,

$$H = \overline{E}_o\, S^\dagger S + \overline{E}_2 D^\dagger \tilde{D} + \frac{1}{4} \sum_{r=0}^{3} b_r\, P^r \cdot P^r, \tag{7}$$

where $S^\dagger$ is the usual pairing mode,

$$S^\dagger = (4\Omega)^{-\frac{1}{2}} \sum_{jm} (-1)^{j-m} A^\dagger_{jm} A^\dagger_{j-m} \tag{8a}$$

the quadrupole pairing mode is,

$$D^\dagger_\mu = \sum_{j,j'} (-1)^{k+\frac{3}{2}+j'} \left[\frac{(2j+1)(2j'+1)}{\Omega}\right]^{\frac{1}{2}} \begin{Bmatrix} \frac{3}{2} & \frac{3}{2} & 2 \\ j & j' & k \end{Bmatrix} \left[A^\dagger_j A^\dagger_{j'}\right]^2_\mu , \tag{8b}$$

where $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ is the Wigner 6-j symbol and,

$$\tilde{D}_\mu \equiv (-1)^\mu D_{-\mu} , \tag{8c}$$

and Ω is one-half of the total occupancy of the shell

$$\Omega \equiv \tfrac{1}{2} \sum_j (2j+1) = 2(2k+1) . \tag{8d}$$

These pairing modes are normalized so that they create a normalized two-nucleon state,

$$\langle o|SS^\dagger|0\rangle = \langle o|D_\mu D^\dagger_\mu|0\rangle = 1 \tag{8e}$$

The multipole operators are given by

$$P^r_q = 2\sum_{j,j'} \left[(2j+1)(2j'+1)\right]^{\frac{1}{2}} (-1)^{k+r+\frac{3}{2}+j'} \begin{Bmatrix} \frac{3}{2} & \frac{3}{2} & r \\ j & j' & k \end{Bmatrix} \left[A^\dagger_j \tilde{A}_{j'}\right]^r_q \tag{9a}$$

with

$$\tilde{A}_{jm} \equiv (-1)^{j+m} A_{j-m} \tag{9b}$$

The integer k is the same as mentioned in (6). The multipole operator of zero rank is just the fermion valence number operator

$$P^{o} = n \tag{9c}$$

and, for the states within the $(S^{\dagger},D^{\dagger})^{N}$ space, the dipole operator is proportional to the angular momentum operator.

Because of the multipole interactions the energy difference for the two nucleon spectrum is given by

$$E_2 - E_0 = \overline{E}_2 - \overline{E}_0 + 4(b_3-b_2) + \frac{6(b_1-b_2)}{5} \quad . \tag{10}$$

These solutions have a very important feature. The set of states given in (3) have a one-to-one correspondence to the boson basis,

$$|N\ N_d\gamma JM\rangle_B = (s^{\dagger})^{N-N_d}\ (d^{\dagger})^{N_d}_{\gamma JM}\ |0\rangle_B \tag{11}$$

and all the states are Pauli allowed for $N_d \leq \frac{\Omega}{2}$. For $N_d > \frac{\Omega}{2}$, instead of valence particles, the one-to-one correspondence is between valence holes. This is consistent with the assumption of the IBM that for the number of valence nucleons greater than the half-filled shell the bosons represent the creation of a pair of holes in the full shell.

The eigenvalues of the Hamiltonian in (7) depend on the parameters of the Hamiltonian and in general need to be solved for numerically. However for special values of the parameters there are analytical expressions for the eigenvalues. One interesting case occurs when the quadrupole pairing and quadrupole-quadrupole interaction are related by

$$\overline{E}_2 = \Omega b_2 \tag{12a}$$

In that limit nucleon seniority, v, is a good quantum number and the eigenspectrum for states in the $(S^{\dagger},D^{\dagger})^{N}$ space is that of an anharmonic vibrator. The eigenspectrum is given by

$$\begin{aligned} E^{*}(Nv\tau JM) &\equiv E(Nv\tau JM) - E(Nv{=}0,\tau{=}0,J{=}0) \\ &= (E_2-E_0)\ \frac{v}{4\Omega}\ (2\Omega+2-v) \\ &+ (b_3-b_2)\left[\tau(\tau+3) - \frac{v}{\Omega}\ (2\Omega+2-v)\right] \end{aligned}$$

$$+\frac{(b_1-b_3)}{5}\left[J(J+1)-\frac{3v}{2\Omega}(2\Omega+2-v)\right] \tag{12b}$$

The allowed values of seniority are limited by the total number of valence nucleons:

$$v=0,2,\ldots,\ 2N\ ;\ N\le\tfrac{1}{2}\Omega \tag{13a}$$

$$v=0,2,\ldots,2(\Omega-N);\ N\ge\tfrac{1}{2}\Omega \tag{13b}$$

The first term in the eigenvalue spectrum (12) is just the pairing interaction and gives the main vibrational spectrum which is harmonic in v for low v. For large v this energy difference between levels decreases due to a Pauli correction.

The next terms in (12) split the degeneracy of the harmonic spectrum. The eigenvalue τ is the number of quadrupole operators which are not coupled to zero. The allowed values are

$$\tau=\tfrac{1}{2}v,\ \tfrac{1}{2}(v-4),\ldots,\ 1\text{ or }0. \tag{14}$$

For each τ the allowed angular momenta are determined by partitioning τ as

$$\tau=3n_\Delta+\lambda \tag{15}$$

where n_Δ and λ are any positive integers. The allowed angular momenta for a given τ and n_Δ is

$$J=\lambda,\lambda+1,\ldots,2\lambda-2,2\lambda. \tag{16}$$

Hence the allowed eigenvalues are the same as given by Arima and Iachello[4] for the vibrational limit of the IBM.

In Fig. 1 we show a schematic representation of the vibrational spectrum given in (12), for $v\le 6$ which illustrates the roles of the three terms in (12). The detailed ordering of the levels will depend on the details of the parameters and could differ from that given in Fig. 1.

Furthermore if we assume that the electric quadrupole operator is proportional to the quadrupole multipole operator given in (9a),

$$Q_q=\chi P_q^2 \tag{17}$$

then it follows that the matrix elements of Q_q between any states with the same seniority is exactly zero:

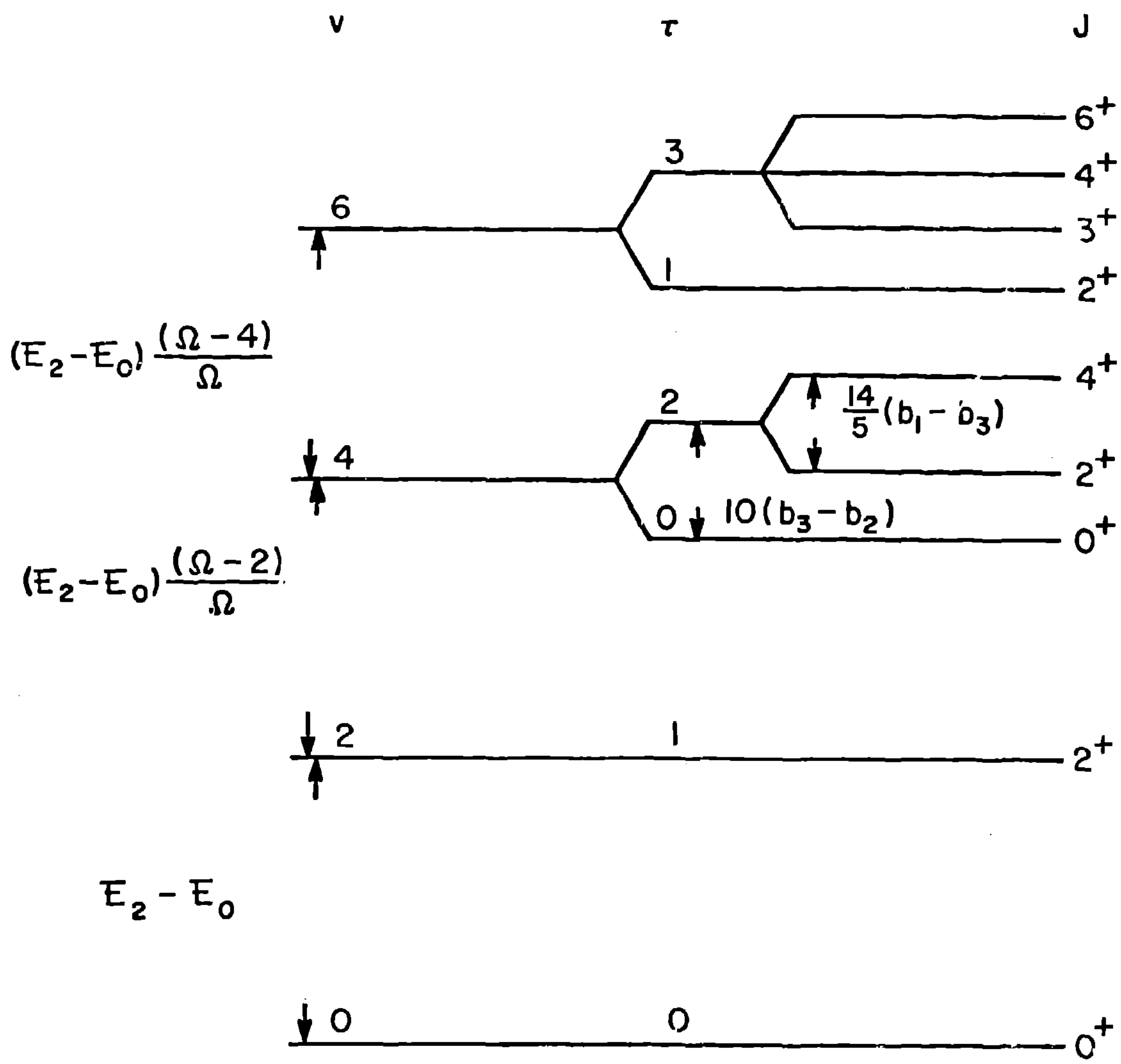

Fig. 1 A schematic representation of the vibrational spectrum of equation (12) for the lowest seniorities ($v \leq 6$).

$$(N v \tau' J' M' | Q_q | N v \tau J M) = 0 \quad . \tag{18}$$

This means that all quadrupole moments vanish and there are no BE2 transitions between states in the same seniority multiplet, only transitions between seniority multiplets. Hence this gives the selection rules of the extreme vibrational limit.

Another limit which has an analytical solution occurs when the monopole and quadrupole pairing are equal:

$$\overline{\mathcal{E}}_0 = \overline{\mathcal{E}}_2 \quad . \tag{19}$$

This gives a solution which has SO_6 symmetry. We note from (10) that this condition doesn't mean that the J=0 and J=2 two-nucleon states are degenerate. We will discuss this limit in a future paper.

The realistic case will be somewhere between the vibrational and SO_6 limits, and has to be solved numerically. However even for these cases τ, J, and M will be conserved quantum numbers. The seniority v will be the only quantum number broken by the Hamiltonian in (7).

We want to emphasize that the Hamiltonian in (7) has a large number of eigenstates not in the $(S^\dagger, D^\dagger)^N$ space. In fact this space is only a very tiny part of the entire shell model space. For n=2N nucleons, the total number of states is given by the binomial factor

$$\mathcal{D}_N = \binom{2\Omega}{2N} \tag{20}$$

However, the total number of states in the $(S^\dagger, D^\dagger)^N$ space is

$$\overline{\mathcal{D}}_N = \binom{\overline{N}+5}{5} \quad . \tag{21a}$$

where

$$\overline{N} = N \text{ for } N \leq \Omega/2 \tag{21b}$$

and

$$\overline{N} = \Omega - N \text{ for } N \geq \Omega/2 \tag{21c}$$

Therefore, for $\Omega, \overline{N}$ large, the ratio is

$$\frac{\overline{\mathcal{D}}_N}{\mathcal{D}_N} \approx e^{-2\overline{N}\ln 2(\Omega-\overline{N})+\frac{11}{2}\ln\frac{\overline{N}}{5}} \tag{22a}$$

which is very small for $\overline{N}$ large, since

$$2(\Omega-\overline{N}) \geq \overline{N} \tag{22b}$$

for all values of $\bar{N}$. Hence if only the $(S^\dagger,D^\dagger)^N$ space is playing a vital role in the collective low-lying states of real nuclei, the IBM does provide a very substantial reduction in the complexity of the shell model calculation of these states.

CONCLUSION

We have shown that there exists fermion Hamiltonians which decouple states containing monopole and quadrupole pairs from the rest of the spectrum. These Hamiltonians have both a vibrational (good seniority) and an SO_6 limit depending on the parameters of the Hamiltonian.

We have only discussed identical nucleons in this paper. Of course the properties of the transitional and rotational nuclei depend crucially on the interaction between protons and neutrons. We will be able to study this aspect as well by introducing proton-neutron interactions into the Hamiltonians of equation (7). We can also study the effect of the coupling of the other states on the spectra of the states in the $(S^\dagger,D^\dagger)^N$. By mapping H onto a boson Hamiltonian H_B which is active in the $(s^\dagger,d^\dagger)^N$ boson space, we can study the effect of the Pauli principle and the effect of the states not in the $(S^\dagger,D^\dagger)^N$ space on the renormalization of the boson Hamiltonian. We also will be able to study the coupling of a single-nucleon to the $(S^\dagger,D^\dagger)^N$ space and hence study odd nuclei as well. Thus by studying these, albeit schematic, fermion Hamiltonians we can perhaps understand the success of the phenomenological interacting boson model in describing real nuclei.

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