OITS - 93
August, 1978

RLO-225075-224

DOE/ER/70004-224

WHAT DOES THE EUCLIEAN PSEUDOPARTICLE DO IN MINKOWSKI SPACE?*

Ilmun Ju
Institute of Theoretical Science
University of Oregon
Eugene, Oregon 97403

Abstract

Self dual pseudoparticle solutions for the classical Yang-Mills field equation with finite action have been constructed in Minkowski space. It is shown that the topological structures apparent in Euclidean space are no longer present in Minkowski space. Topolcgical charges become fractional leading to the unquantized axial charge violaticn in the process involving fermions.

17 references

* Research supported in part by U.S. Department of Energy under contract No. AT (45-1)-2230. By acceptance of this article, the publisher and/or recepient acknowledges the U.S. Government's right to retain a non-exclusive, royalty-free license in and to any copyright covering this paper.



## I. INTRODUCTION

Recently the existance of the finite action pseudoparticle solutions for the classical Yang-Mills theory attracted a lot of attention due to its ramifications to the quark confinement and violation of parity and time reversal invariance as well as the possibility of the existance of Axion.[1] Furthermore the pathology of the present formulation of quantum field theory manifested by the non-uniqueness of the gauge conditions employed so far as pointed out by Gribov[2] indicates the need for more fundamental understanding of the Yang-Mills theories and its classical solutions. Much work[3] has been done by many authors to find the classical solutions in the framework of many different gauge groups in Euclidean space simply because functional formulation of quantum field theory is based on the Euclidian functional integrals. However it may be most interesting to find the solutions in Minkowski space that render finite action, finite non-zero topological charge. Several solutions exist in the literature.[4] However, all the Minkowski space solutions known so far lead to the vanishing topological charges. The purpose of this paper is to present infinite number of Minkowski space solutions which are regular everywhere and give finite non-zero actions and topological charges. The method is in the analytic continuation of the finite Euclidean space pseudoparticle solutions prescribed by changing all the fourth component of four vectors to be imaginary, i.e.,

$$A_i^{a(M)}(\vec{x}, x_0) = A_i^{a(E)}(\vec{x}, ix_0)$$

$$A_0^{a(M)}(\vec{x}, x_0) = i A_4^{a(E)}(\vec{x}, ix_0)$$

## DISCLAIMER

$$F_{ij}^{a(M)}(\vec{x}, x_0) = F_{ij}^{a(E)}(\vec{x}, ix_0)$$

$$F_{0j}^{a(M)}(\vec{x}, x_0) = i F_{4j}^{a(E)}(\vec{x}, ix_0)$$

$$b_i^{(M)} = b_i^{(E)}, \quad b_0^{(M)} = i b_4^{(E)} \tag{1}$$

where $b_\mu$ is an arbitrary four vector in the solution. The Minkowski space solutions obtained this way are in the most cases imaginary and singular. For example, the topological charge density for the pseudoparticle solution due to Belavin et al is

$$q(x) = \frac{6}{\pi^2} \frac{\lambda^4}{(r^2 - x_0^2 + \lambda^2)^4} \tag{2}$$

in Minkowski space, which is singular at $x_0 = \pm\sqrt{r^2 + \lambda^2}$.

One of the objections for the complex solution is that, in general, isotopic spin and Hamiltonian are complex. However, if the Minkowski space solution is self dual, i.e.,

$$\tilde{F}_{\mu\nu} = \pm i F_{\mu\nu}$$

$$\tilde{F}_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \tag{3}$$

where

energy momentum tensor is identically zero, which is manifest from the form of it written as,

$$\theta_{\mu\nu}^{(E)} = \frac{1}{4}\{(\tilde{F}_{\mu\alpha}^a - F_{\mu\alpha}^a)(\tilde{F}_{\nu\alpha}^a + F_{\nu\alpha}^a) + (\mu \rightleftarrows \nu)\}$$



$$\theta_{\mu\nu}^{(M)} = \frac{1}{4}\{(i\tilde{F}_{\mu\alpha}^a - F_{\mu\alpha}^a)(i\tilde{F}_\nu^{a\alpha} + F_\nu^{a\alpha}) + (\mu \rightleftarrows \nu)\}$$

$$\theta_{\mu\nu} = -\frac{1}{4} g_{\mu\nu} F_{\alpha\beta}^a F^{a\alpha\beta} - F_{\mu\alpha}^a F_\nu^{a\alpha} \tag{4}$$

Thus the self dual solutions in either Minkowski or Euclidean space must be closely related to the properties of Yang-Mills vacuum, while the other finite energy solutions in Minkowski space correspond to the particles in the theory.

## II. FINITE TOPOLOGICAL CHARGE SOLUTIONS IN MINKOWSKI SPACE

Most of the self dual solutions in Euclidean space when they were analytically continued to Minkowski space become singular, thus resulting in the diverging actions and topological charges. However, we find that following Minkowski space solution derived from the following $\Phi$ leads to the regular solution.[4]

$$A_\mu^{(E)} = -\frac{1}{g} \eta_{\mu\nu}^a \partial_\nu \log \Phi \tag{5}$$

where

$$\phi = \sum_{m=0}^{N}\left[ -\frac{\lambda_m^2}{a_m^2} + \frac{\lambda_m^2}{(x-b_m-ia_m)^2} + \frac{\lambda_m^2}{(x-b_m+ia_m)^2} \right] \tag{6}$$

$$a_m = (0, 0, 0, \beta_m), \quad a_m^2 = +\beta_m^2$$

$$b_m = (\alpha_{m1}, \alpha_{m2}, \alpha_{m3}, \alpha_{m4}), \; b_m^2 = \vec{\alpha}_m^{\,2} + \alpha_{m4}^2$$

The finite action gauge field analytically continued from Eq. (5) is[5]

$$A_\mu^{a(M)} = \frac{2}{g_0}\eta^a_{\mu\nu} \frac{\displaystyle\sum_{n=1}^{N/2}\left[ \lambda_n^2 \frac{2X_\nu\{(\beta_n^2 - t_{(+)n}t_{(-)n})^2 - \beta_n^2(t_{(+)n}-t_{(-)n})^2\} - 4\delta_{\nu 0}\beta_n^2 B}{(\beta_n^2 + t_{(+)n}^2)(\beta_n^2 + t_{(-)n}^2)} \right]}{\displaystyle\sum_{m=1}^{N/2}\left\{ \frac{\lambda_m^2}{\beta_m^2} + \frac{2\lambda_m^2(\beta_m^2 - t_{(+)m}t_{(-)m})}{(\beta_m^2 + t_{(+)m}^2)(\beta_m^2 + t_{(-)m}^2)} \right\}} \tag{7}$$

where

$$B \equiv (t_{(+)n} - t_{(-)n})^2(\beta_n^2 - t_{(+)n}t_{(-)n})$$

$$\eta_{ij}^{a(M)} = \epsilon_{aij}$$

$$\eta_{0j}^{a(M)} = i\delta_{aj} = -\eta_{j0}^{a(M)}$$

$$t_{(\pm)n} \equiv (r - |\vec{b}_n|) \pm (x_0 - b_{n0})$$

$\lambda_m$ and $\beta_m$ are arbitrary non-zero real numbers and N is an arbitrary integer.



Notice that

$$\frac{\lambda_m^2}{\beta_m^2} + \frac{2\lambda_m^2(\beta_m^2 - t_{(+)n}t_{(-)n})}{(\beta_m^2 + t_{(+)m}^2)(\beta_m^2 + t_{(-)m}^2)}$$

$$= \frac{\lambda_m^2}{\beta_m^2}\frac{[3\beta_m^4 + \beta_m^2(t_{(+)m}^2 + t_{(-)m}^2) + \beta_m^2(t_{(+)m} - t_{(-)m})^2]}{(\beta_m^2 + t_{(+)m}^2)(\beta_m^2 + t_{(-)m}^2)} > 0 \tag{9}$$

for every m.

Therefore $A_\mu^{a(M)}$ is regular for arbitrary space time points in Minkowski space and integer N.

An interesting feature is that topological charge given by Eq. (6) in Euclidean space is 2N, i.e., we have to combine two pseudoparticles at the same space but separated by time only in order to produce the regularity of the solutions (7).[6]

III. ACTION AND TOPOLOGICAL CHARGE FOR A MINKOWSKI SPACE SOLUTION

The action integrals though finite are not known for arbitrary N. However, we present the result of the topological charge calculation for N = 2 in Eq. (7). For the self dual solution in Minkowski space the following relation holds as in Euclidean Space.

$$q = \frac{g_0^2}{8\pi^2} S \tag{10}$$

where

$$S = -\frac{1}{8\pi^2}\int d^4x \, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \tag{11}$$

For simplicity, if we put $\beta_1 = a$, $\vec{b}_1 = b_{10} = 0$, the action density is given by

$$S(x) = \frac{\pi^2}{32} \frac{16a^4}{g_0^2} \frac{A}{B} \tag{12}$$

where

$$A = 12x^8 + 16a^2x^6 - 8a^4x^4 + 48a^6x^2 + 18a^8 - 128a^2x^2x_0^2 - 448a^4x^2x_0^2 - 384a^6x_0^2 + 64a^4x_c^4$$

$$B = \{(x^2+a^2)^2 + 4a^2x_0^2 + 2a^2(x^2+a^2)\}^4 \tag{13}$$

$$x^2 = r^2 - x_c^2$$

In order to evaluate the action integral analytically we change the variables such that

$$\int_0^\infty dr \int_{-\infty}^{\infty} dx_c \longrightarrow \frac{1}{4} \int_{-\infty}^{\infty} dt_{(-)} \int_{-\infty}^{\infty} dt_{(+)} \tag{14}$$

where $t_{(+)} = r + x_c$ , $t_- = r - x_0$

Then for N = 2, topological charge and action integral turn out to be finite [see Appendix for detailed calculations],

$$q \approx -0.85$$

$$S \approx 0.85 \frac{8\pi^2}{g_0^2} \tag{15}$$

Notice that for the corresponding Euclidean space solution $q = 2$.

As was expected, the topological characters of Euclidean space solutions are no longer present and become fractional. However, the action implied by these complex self dual solutions is finite.



Furthermore, the existance of classical pseudoparticle solution in SU(N) Yang-Mills field theory with fermion implies the violation of axial charge given by

$$\partial^\mu J_\mu^5 = -iN \frac{g_c^2}{16\pi^2} F_{\mu\nu}^a(x) \tilde{F}^{a\mu\nu}(x) = 2Nq(x)$$

$$\Delta Q^5 = 2Nq \tag{16}$$

Thus these complex solutions are similar to the complex potential in the classical quantum mechanics which absorb electric charge. The fractional topological charge should not be a surprise since the mapping defined by $g_i$ is not from $S^3$ to $S^3$, but to the Riemannian manifold[7] where $g_i$ is the SU(2) group matrix that appears when we rewrite Eq. (5) in Minkowski space, i.e.,

$$A_\mu^{(M)} = \sum_i f_i \, g_i^\dagger \partial_\mu g_i \tag{17}$$

In conclusion, we see that the solutions (5) violate axial charge in real time, while they are responsible for the tunneling of vacuum in imaginary time.

Finally I deeply appreciate Professors Paul Csonka and Michael Moravcsik for their hospitality and Professor John Leahy for his kindness extended to me.

## FOOTNOTES AND REFERENCES

(1) A.M. Polyakov, Phys. Lett. 59B, 82(1975);
C. Callan, R. Dashen, and D. Gross, Phys. Rev. D17, 2717(1978);
S. Weinberg, Phys. Rev. Lett. 40, 279(1978);
F. Wilczek, Phys. Rev. Lett. 40, 279(1978).

(2) V.N. Gribov, Lectures at the 12th Winter School of the Leningrad Nuclear Physics Institute, 1977. SLAC-TRANS-176(1977).

(3) A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. 59B, 85(1975);
E. Corrigan and D.B. Fairlie, Phys. Lett. 67B, 69(1977);
G. 'tHooft, Unpublished;
E. Witten, Phys. Rev. Lett. 38, 121(1977);
V. de Alfaro, S. Fubini, and G. Furlan, Phys. Lett. 72B, 203(1977), 73B, 463(1978);
J. Glimm and A. Jaffe, Phys. Lett. 73B, 167(1978).

(4) V. de Alfaro, S. Fubini, and G. Furlan, Phys. Lett. 65B, 163(1976);
B.M. Schechter, Phys. Rev. D16, 3015(1977);
M. Lüscher, Phys. Lett. 70B, 321(1977);
C. Rebbi, Brookhaven preprint (1977).

(5) Notice that $\phi$ contains many peculiar complex space time translations. However, this is not a fundamental feature because one can obtain exactly same gauge fields starting from

$$\phi = \sum_m \left[ \frac{\lambda_m^2}{a_m^2} + \frac{\lambda_m^2}{(x - b_m - a_m)^2} + \frac{\lambda_m^2}{(x - b_m + a_m)^2} \right]$$

expanding four vector squared $(x - bm - am)^2$ in Euclidian space first and treat $b_m (= a_{m4})$ as a real number rather than the fourth component of a four vector when we analytically continue it, while we treat bm as a four vector both in Euclidian and Minkowski space as Alfaro et al in Reference (4).

(6) Another form

$$A_i^a = \frac{2}{g_0} \frac{\sum_{n=1}^{N/2} \lambda_n^2 \left[ \frac{2\epsilon_{ai\alpha} x^\alpha \{(x^2 - a_n^2)^2 - 4(a_n \cdot x)^2\} + 2B}{\{(x^2 - a_n^2)^2 + 4(x \cdot a_n)^2\}^2} \right]}{\sum_{m=1}^{N/2} \left[ -\frac{\lambda_m^2}{a_m^2} + \frac{2\lambda_m^2 (x^2 - a_m^2)}{(x^2 - a_m^2)^2 + 4(x \cdot a_m)^2} \right]}$$



$$A_0^a = i \frac{2}{g_0} \frac{-2x_a \sum_{n=1}^{N/2} \left[ \lambda_n^2 \frac{(x^2 - a_n^2)^2 - 4(a_n \cdot x)^2}{(x^2 - a_n^2)^2 + 4(a_n \cdot x)^2} \right]}{\sum_{m=1}^{N/2} \left[ -\frac{\lambda_m^2}{a_m^2} + \frac{2\lambda_m^2 (x^2 - a_m^2)}{(x^2 - a_m^2)^2 + 4(x \cdot a_m)^2} \right]}$$

where

$$B = i \delta_{ia} \{ x_0 (x^2 - a_n^2)^2 - 4(a_n \cdot x)^2 + 4 a_n (x \cdot a_n)(x^2 - a_n^2) \}$$

The above solution is translation invarient, thus $A_\mu^a (x + b_m)$ is also solution.

(7) In the case of Alfaro et al in Reference (4) this combination resulted in the vanishing topological charge density, thus zero topological charge automatically. This is due to the fact the meron charge density is confined at the singularities only.

## APPENDIX. EVALUATION OF THE INTEGRAL

I. Procedure to follow.

Step 1. After angle integration and contour integration of $t_{(+)}$ variable defined in Eq. (14), let $t \equiv t_{(-)}$.

Step 2. Use the expansion

$$t^{2n} = \sum_{m,k} (1+t^2)^m (t^4+3t^2+3)^k \tag{A-1}$$

to obtain the integral of the forms

$$\int \frac{(a+bt^2)}{(\sqrt{t^4+3t^2+3})^m} dt \tag{A-2}$$

and

$$\int \frac{dt}{(1+t^2)^n (\sqrt{t^4+3t^2+3})^{2m+1}} \tag{A-3}$$

Step 3. For the integral of the form Eq. (A-a), use following formula

$$\int_0^\infty \frac{t^2\,dt}{(\sqrt{t^4+3t^2+3})^3} = \frac{1}{18} B\left(\frac{1}{6}, \frac{1}{2}\right) + \frac{2}{9} B\left(\frac{5}{6}, \frac{1}{2}\right) - \frac{4}{9} B\left(\frac{7}{6}, \frac{1}{2}\right)$$

$$\int_0^\infty \frac{t^2\,dt}{(\sqrt{t^4+3t^2+3})^5} = \frac{1}{27} B\left(\frac{1}{6}, \frac{1}{2}\right) + \frac{5}{27} B\left(\frac{5}{6}, \frac{1}{2}\right) - \frac{10}{27} B\left(\frac{7}{6}, \frac{1}{2}\right)$$

$$\int_0^\infty \frac{t^2\,dt}{(\sqrt{t^4+3t^2+3})^7} = \frac{1}{405} \left\{ 14 B\left(\frac{1}{6}, \frac{1}{2}\right) + 70 B\left(\frac{1}{6}, \frac{1}{2}\right) - 140 B\left(\frac{7}{6}, \frac{1}{2}\right) + 16 B\left(\frac{11}{6}, \frac{1}{2}\right) - 20 B\left(\frac{13}{6}, \frac{1}{2}\right) \right\}$$



where use has been made of the well known formula,

$$\int \frac{x^2\,dx}{(a+bx^2+cx^4)^p} = \frac{2cx^3+bx}{2(p-1)(4ac-b^2)\Phi^{p-1}} + \frac{c(4p-7)}{(p-1)(4ac-b^2)} \int \frac{x^2\,dx}{\Phi^{p-1}} - \frac{b}{2(p-1)(4ac-b^2)} \int \frac{dx}{\Phi^{p-1}}.$$

$$\Phi = a + bx^2 + cx^4.$$

Step 4. For the integral of the form Eq. (A-3), the change of variable

$$x = \frac{1}{1+t^2}$$

leads to

$$\int_0^\infty \frac{dt}{(1+t^2)^n (\sqrt{t^4+3t^2+3})^{2m+1}} = \int_0^1 \frac{x^{2(m+n)} (1-x^2)^m}{(\sqrt{1-x^6})^{2m+1}} dx$$

then use

$$\int_0^x \frac{x^m\,dx}{(a+bx^n)^{p/q}} = \frac{q\,x^{m+1}}{an(p-q)\Phi^{p/q-1}} \Bigg|_0^x + \frac{np-q(m+n+1)}{an(p-q)} \int_0^x \frac{x^m\,dx}{\Phi^{p/q-1}}$$

and

$$\int_0^1 \frac{x^m}{\sqrt{1-x^6}} dx = \frac{1}{6} B\left(\frac{m+1}{6}, \frac{1}{2}\right)$$

where

$$\Phi \equiv a + bx^n$$

Step 5. Use

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

Then we need

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} = 1.772453851$$

$$\Gamma(\tfrac{1}{3}) = 2.6789385347$$

$$\Gamma(\tfrac{2}{3}) = 1.3541179394$$

II. An example.

$$I \equiv \int \frac{x^2\, d^4x}{\left[(x^2+1)^2 + 4x_0^2 + 2(x^2+1)\right]^4}$$

$$I = \frac{65}{27} B(\tfrac{1}{6}, \tfrac{1}{2}) + \frac{334}{27} B(\tfrac{5}{6}, \tfrac{1}{2}) - \frac{668}{27} B(\tfrac{7}{6}, \tfrac{1}{2})$$

$$- \frac{64}{27} B(\tfrac{11}{6}, \tfrac{1}{2}) - \frac{40}{9} B(\tfrac{13}{6}, \tfrac{1}{2}) - \frac{224}{9} B(\tfrac{17}{6}, \tfrac{1}{2})$$

$$+ \frac{640}{27} B(\tfrac{19}{6}, \tfrac{1}{2}).$$

$$I = -0.005118242.$$