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# LONGITUDINAL INSTABILITIES WITH A NON-HARMONIC RF POTENTIAL*

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## Abstract

We consider the longitudinal instabilities of a bunched beam subject to a non-harmonic RF potential. Assuming the unperturbed bunch to be described by a Maxwell-Boltzmann distribution, our treatment is based upon the linearized Vlasov equation. The formalism developed is exact, and in particular, correctly describes the effect of the dependence on amplitude of the synchrotron oscillation frequency. We discuss the fast blowup limit, and extend Wang and Pellegrini's[1] treatment of the microwave instability to include the case of a non-Gaussian bunch. Next, within the short bunch approximation, we derive the dispersion relation describing the Landau damping of the coupled bunch modes, resulting from the use of a Landau cavity.

## Equations of Motion

The azimuthal position of a circulating particle relative to a stationary observer is denoted by angle $\theta$, and $\dot{\theta}$ is the instantaneous value of the angular velocity. Relative to a synchronous particle of energy $E_o$ and angular velocity $\omega_o$, the azimuthal position is $\phi = \theta - \omega_o t$ and the energy is $\varepsilon = E - E_o$. Assuming the energy $E_o$ to be large compared to the rest mass, the equations of motion describing the synchrotron oscillations are:[2]

$$\dot{\phi} = -\alpha\omega_o \varepsilon / E_o \tag{1}$$

$$\dot{\varepsilon} = \frac{e\omega_o}{2\pi}\left[V_{RF}(\phi) + V_i(\phi,t)\right] \quad . \tag{2}$$

Here, e is the particle's electric charge, $\alpha$ the momentum compaction, $V_{RF}(\phi)$ the RF potential, and $V_i(\phi,t)$ the induced potential resulting from the impedance of the ring.

In the absence of the induced potential, the equations of motion are derived from the Hamiltonian:

$$H_o = \frac{1}{2}p^2 + U_o(\phi) \ , \tag{3}$$

with

$$U_o(\phi) = \frac{\alpha\omega_o}{E_o}\frac{e\omega_o}{2\pi}\int_0^{\phi} d\phi' \ V_{RF}(\phi') \quad . \tag{4}$$

Under the canonical transformation[3] from $p,\phi$ to action-angle variables $J,\theta$, the element of phase space area is invariant, $dpd\phi = dJd\theta$, and the transformed Hamiltonian is a function only of the action variable

$$J = \frac{1}{2\pi}\oint pd\phi \quad . \tag{5}$$

The new equations of motion are $\dot{J} = 0$ and $\dot{\theta} = \omega_s(J)$, where $\omega_s(J) = dH_o/dJ$ is the angular synchrotron oscillation frequency. The azimuthal position $\phi$ is determined as a function of J and $\theta$ by

$$\phi = \phi_o(J,\theta) \tag{6}$$

with periodic $\phi_o$ satisfying $\phi_o(J,\theta+2\pi) = \phi_o(J,\theta)$.

To describe the effect of the induced potential, we introduce the distribution $\psi(p,\phi,t)$, normalized to the total number N of particles in the ring,

$$\int dpd\phi \ \psi(p,\phi,t) = Ne \ , \tag{7}$$

and the line charge density

$$\lambda(\phi,t) = \int dp\psi(p,\phi,t) \ . \tag{8}$$

Our goal is to determine the conditions required for the line charge density to exhibit a coherent oscillation with frequency $\Omega$, i.e.

$$\lambda(\phi,t) = \rho_o(\phi) + \rho(\phi)\exp(-i\Omega t), \tag{9}$$

corresponding to an induced potential of the form $V_i(\phi,t) = V_o(\phi) + V_i(\phi)\exp(-i\Omega t)$. We shall ignore the time independent potential well distortion, $V_o(\phi)$, and concentrate our attention upon the coherent term, which is related to the ring impedance $Z_n(\omega)$ and the Fourier transform of the perturbed line charge density $\rho_n$ by

$$V_i(\phi) = -\omega_o \sum_n \rho_n \ Z_n(n\omega_o+\Omega)\exp(in\phi). \tag{10}$$

The Vlasov equation for the distribution $\psi$ is

$$\frac{\partial\psi}{\partial t} + [\psi,H] = 0 \ , \tag{11}$$

where $[\psi,H]$ is the Poisson-Bracket between $\psi$ and the full Hamiltonian:

$$H = H_o(J) + \frac{\alpha\omega_o}{E_o}\frac{e\omega_o}{2\pi}\int_0^{\phi_o(J,\theta)} d\phi' \ V_i(\phi')\exp(-i\Omega t). \tag{12}$$

We look for coherently oscillating solutions of Eq. (11), having the form,[4]

$$\psi = \psi_o(J) + \psi_1(J,\theta)\exp(-i\Omega t) \ , \tag{13}$$

and we shall assume the equilibrium distribution $\psi_o(J)$ to be Maxwell-Boltzmann:[5]

$$\psi_o(J) = A\exp(-H_o(J)/\sigma^2) \ . \tag{14}$$

In terms of the variables p and $\phi$, clearly, $\psi_o(p,\phi) \propto \exp(-p^2/2\sigma^2 - U_o(\phi)/\sigma^2)$, showing that $\sigma$ represents the one-standard deviation spread in revolution frequency among particles in the bunch. From Eq. (2) it follows that $\sigma$ is related to the energy spread $\sigma_\varepsilon$ via

$$\sigma = \frac{\alpha\omega_o}{E_o}\sigma_\varepsilon \ . \tag{15}$$

The constant A in Eq. (14) is determined from the normalization condition of Eq. (7).

To proceed we now insert Eqs. (12) and (13) into the Vlasov Eq. (11), and linearize the result dropping terms second-order in $\psi_1$, obtaining

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$$-i\Omega\psi_1 + \omega_s(J)\,\frac{\partial\psi_1}{\partial\theta} = \chi(J,\theta), \qquad (16)$$

where $\chi(J,\theta) = \chi(J,\theta+2\pi)$ is defined by

$$\chi(J,\theta) = \frac{\alpha\omega_o}{E_o}\,\frac{e\omega_o}{2\pi}\,\psi_o'(J)V_1[\phi_o(J,\theta)]\,\frac{\partial\phi_o(J,\theta)}{\partial\theta}\,, \qquad (17)$$

and $\psi_o'(J) = d\psi_o/dJ$. The periodic solution of Eq. (16) can be written:

$$\psi_1(J,\theta) = \frac{\exp(iQ(J)\theta)}{\omega_s(J)[1-\exp(2\pi iQ(J))]}\int_{\theta-2\pi}^{\theta} d\theta'\,\chi(J,\theta')\exp(-iQ(J)\theta') \qquad (18)$$

where we have defined

$$Q(J) = \Omega/\omega_s(J)\ . \qquad (19)$$

Eq. (18) has the important property that it expresses the perturbed distribution $\psi_1(J,\theta)$ in terms of the Fourier coefficients $\rho_n$ of the line charge density. Since, on the other hand, the line charge density is derivable from the distribution according to Eq. (8), one is led to the following infinite set of linear equations determining the Fourier coefficients of the perturbed line charge density:

$$\rho_m = \sum_{n=-\infty}^{\infty} T_{mn}\rho_n\,, \qquad (20)$$

where the matrix $T_{mn}$ is given by

$$T_{mn} = i\kappa\,\frac{Z_n}{n}\int_0^{\infty}\frac{dJ}{\omega_s(J)}\,\frac{\psi_o'(J)}{1-\exp(2\pi iQ(J))}\int_{-2\pi}^{0} d\theta'\exp(-iQ(J)\theta')$$

$$* \frac{\partial}{\partial\theta'}\int_{-2\pi}^{0} d\theta\exp(in\phi_o(J,\theta+\theta')-im\phi_o(J,\theta))\,, \qquad (21)$$

with $2\pi\kappa = e\alpha\omega_o^3/2\pi E_o$ and $Z_n = Z_n(n\omega_o+\Omega)$.

The coherent frequency $\Omega$ is determined by the condition that the matrix T have an eigenvalue equal to unity, and the coherent perturbation to the line charge density is the corresponding eigenvector. A representation of T in terms of synchrotron modes follows upon defining the coefficients $F_\mu(n,J)$ by

$$\exp(in\phi_o(J,\theta)) = \sum_{\mu=-\infty}^{\infty} F_\mu(n,J)\exp(i\mu\theta)\,, \qquad (22)$$

and applying Eq. (22) to Eq. (21), yielding

$$T_{mn} = -2\pi i\kappa\,\frac{Z_n}{n}\sum_{\mu=-\infty}^{\infty}\mu\int_0^{\infty} dJ\,\psi_o'(J)\,\frac{F_\mu^*(m,J)F_\mu(n,J)}{\Omega-\mu\omega_s(J)}\,. \qquad (23)$$

This expansion is useful when one synchrotron mode dominates, however, when many synchrotron modes contribute the representation of Eq. (21) is preferrable.

## Microwave Instability

We can now extend Wang and Pellegrini's[1] treatment of the microwave instability to a Non-Gaussian bunch interacting with a non-harmonic RF potential. The microwave instability is characterized by coherent modes with growth rates large compared to the synchrotron oscillation frequency, driven by perturbing electromagnetic fields having wavelengths short compared to the bunch length. Let L denote the rms bunch length measured in radians. We assume a broad-band high frequency impedance satisfying $Z_n = Z_{n_o}$ for $|n-n_o|<\Delta$, where $n_o \gg \Delta \gg 1/L$. The bandwidth $\Delta$ is of the order of the inverse range of the wake field.

Assuming $\mathrm{Im}\,Q(J) \gg 1$ in Eq. (21), we may expand the function $\phi_o(J,\theta+\theta')$ in a Taylor expansion about $\theta'=0$. Then performing the change of integration variables: $(J,\theta) \to (\dot\phi,\phi)$ and $\theta' \to \omega_s(J)\xi$, we derive the following asymptotic representation for $T_{mn}$, valid for $n_o - \Delta < m,n < n_o + \Delta$:

$$T_{mn} \approx \frac{-ieI_o}{2\pi E_o\alpha(\sigma_\varepsilon/E_o)^2}\,\frac{Z_{n_o}}{n_o}\,h\left(\frac{\Omega}{|n_o|\sigma}\right)\beta(m-n)\,, \qquad (24)$$

where

$$h(x) = \int_0^{\infty}\xi d\xi\exp(-\xi^2/2 + ix\xi)\ . \qquad (25)$$

and $\beta(n)$ is the normalized Fourier coefficient of the unperturbed bunch density, $\rho_o(\phi) = \exp(-U_o(\phi)/\sigma^2)$, i.e.

$$\beta(n)\int_{-\infty}^{\infty} d\phi\rho_o(\phi) = \int_{-\infty}^{\infty} d\phi\rho_o(\phi)\exp(-in\phi)\ . \qquad (26)$$

In Eq. (24) we have used Eq. (15) to express $\sigma$ in terms of $\sigma_\varepsilon$, and we have denoted the average bunch current by $I_o$.

When all eigenvalues of $T_{mn}$ have magnitude less than unity, there can be no solution to Eq. (20), and this fact allows the derivation of a stability criterion. Since $\beta(m-n)$ is sharply peaked about $m=n$, the width of the peak being of the order $1/L \ll \Delta$, the largest eigenvalues of $T_{mn}$ should be well approximated by extending Eq. (24) to the entire range $-\infty<m,n<\infty$. Within this approximation, the threshold of the instability is determined by the largest eigenvalue, $\Lambda_{max}$, of the matrix $\beta(m-n)$. Using the fact that $|h(x)| \le 1$ for $\mathrm{Im}\,x \ge 0$, we see that there exists no coherent frequency with $\mathrm{Im}\,\omega > 0$ as long as

$$\frac{eI_o\Lambda_{max}}{2\pi E_o\alpha(\sigma_\varepsilon/E_o)^2}\left|\frac{Z_{n_o}}{n_o}\right| \le 1\ . \qquad (27)$$

When the bunch length is short compared to the circumference of the ring, $L \ll 1$, we find

$$\Lambda_{max} \approx \frac{2\pi}{\int d\phi\exp(-U_o(\phi)/\sigma^2)} = \frac{I_{peak}}{I_o}\ . \qquad (28)$$

Therefore, in Eq. (27), we replace $I_o\Lambda_{max}$ by the peak current $I_{peak}$ of the bunch, obtaining the Boussard[6] stability criterion, derived originally on the basis of an intuitive physical argument. Boussard noted that when the perturbing electromagnetic fields have wavelengths short compared to the bunch length, the bunch looks like a coasting beam with current $I_{peak}$.

## Landau Cavity

A Landau cavity[4,7] operates at a multiple of the fundamental RF frequency, with its voltage and phase chosen such that for small amplitude oscillations the RF "potential energy" defined in Eq. (4) becomes $U_0(\phi) = b\phi^4/4$, with $b>0$. The use of such a cavity results in a non-Gaussian bunch density, $\rho_0(\phi) \propto \exp(-U_0(\phi)/\sigma^2)$, and an increase of the rms bunch length. Hence, the use of the Landau cavity reduces the peak current and allows the threshold (expressed in terms of average current) of the microwave instability to be increased. Also, because of the nonlinear restoring force, the Landau cavity produces a large spread of synchrotron oscillation frequencies within the bunch. This provides stability, via Landau damping, against coupled bunch instabilities.

Neglecting the effect of the ring's impedance, the synchrotron oscillations in the presence of the Landau cavity are described by [see Eq. (6)]:

$$\phi_o(J,\theta) = r\ \mathrm{cn}(2K\theta/\pi)\ , \tag{29}$$

where cn(x) is the Jacobi elliptic function of modulus $k = 1/\sqrt{2}$, and $K = K(1/\sqrt{2})$ is the elliptic integral of the first kind. The amplitude r is related to the action-variable by $J = 2K\sqrt{b}\ r^3/3\pi$, and the distribution in oscillation amplitude of the unperturbed bunch is $\psi_o(r) = A\exp(-r^4/r_o^4)$. The normalization constant A is determined by Eq. (7), and $r_o^4 = 4\sigma^2/b$. Since the equation of motion is nonlinear, the synchrotron frequency varies with oscillation amplitude,

$$\omega_s(r) = \Delta\omega_s \frac{r}{r_o} \text{ with } \Delta\omega_s = \frac{\pi}{2K}\sqrt{b}\ r_o\ . \tag{30}$$

Let us suppose the wavelengths of the perturbing electromagnetic fields are long compared to the bunch length. We use the synchrotron mode expansion of Eq. (23), and assuming $nr<<1$, we approximate

$$F_\mu(n,J) \approx inr\phi_\mu\ , \quad (\mu\neq o) \tag{31}$$

with $\pi/\phi_\mu = \sqrt{2}\,K\cosh[(\mu-1/2)\pi]$ for $\mu$ odd, and $\phi_\mu = o$ for $\mu$ even. When Eq. (31) is used, the matrix T becomes of rank one, and we can derive the dispersion relation

$$1 = \frac{4.30\omega_o^2 eI_{av}Z_{eff}(\Omega)}{2\pi E_o(\Delta\omega_s)^2}\,G^{-1}(\Omega/\Delta\omega_s) \tag{32}$$

with

$$i\,G^{-1}(q) = \int_0^\infty dx\,\frac{x^6e^{-x^4}}{x^2-q^2}\ . \tag{33}$$

The neglect in Eq. (32) of synchrotron modes with $|\mu|>1$ is justified due to the rapid decrease of $\phi_\mu$ for increasing $\mu$. In the case of M equally spaced bunches each containing N/M particles, we define the effective impedance $Z_{eff}(\Omega)$ corresponding to fixed multibunch mode number s = 0,1,2...,M-1 by $Z_{eff}(\Omega) = \sum nZ(n\omega_o+\Omega)$, where the sum is restricted to n = Mj+s (with j integer), and we consider the sum to be cut off at $n \approx 1/L$. The average current in the ring is denoted $I_{av} = Ne\omega_o/2\pi$. In Fig. 1, we plot Im G(q) against ReG(q) at threshold (Imq = 0+) and above (Imq = 0.1).

As evidence that the neglect of higher-order synchrotron modes is justified in the derivation of Eq. (32), consider the limit $|\Omega|>>\Delta\omega_s$. In this case, when all higher-order modes are retained, one obtains

$$\left(\frac{\Omega}{\omega_o}\right)^2 = \frac{i\alpha eI_{av}Z_{eff}}{2\pi E_o}\ , \tag{34}$$

which has the form of a superposition of coasting beam dispersion relations.[8] One can check that in the same limit, Eq. (32) agrees with Eq. (34) to 1% accuracy.

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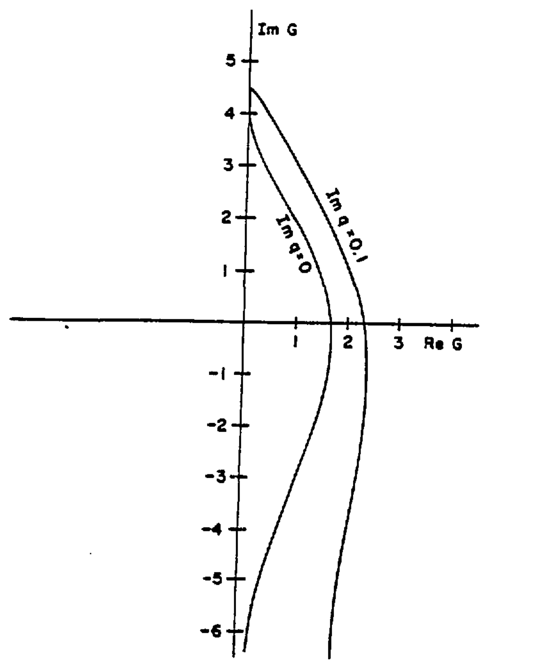


Fig. 1. Stability boundaries.