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Introduction to String and Superstring Theory II

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1. Introduction

In the first half of this course, Michael Green set out the historical background of string theory and the basic principles of string mechanics. He described the various consistent string theories and displayed the spectrum of particles which they produce. These always include candidates for gravitons and gauge bosons; in the case of supersymmetric strings, one also finds candidates for approximately massless quarks and leptons. Thus, string theories provide a new basis for understanding the fundamental interactions, a generalized mechanics from which the dynamics of gauge-invariant fields can be derived as a low-energy limit.

Finding the correct particle spectrum is, of course, only the first step toward unravelling the full structure of this grand theory. One would like to understand the origin of the symmetries of the theory, and to realize solutions of the theory which display the observed interactions of particle physics. These deeper levels of analysis of the string theory are still not completed; indeed, they constitute one of the most exciting areas of research in mathematical physics. In this set of lectures I will continue the elucidation of string theories, in a direction that will shed some light on these large issues. The main topic of these lectures will be the formulation of manifestly Lorentz-covariant methods of calculation for string scattering amplitudes. We will find, though, that this study leads us directly to consideration of the gauge invariances of string theory, and to some tools which illuminate the construction of schemes of compactification.

The plan of these lectures is the following: In order to progress in understanding string theory, we must first retrace our steps a bit and review some elements of the quantum theory of massless fields in 2 dimensions. The string world sheet is, of course, a 2-dimensional surface, and the displacements of the string in space-time can well be viewed as (massless) fields on this surface. We have seen in Michael Green's lectures that the conformal invariance of 2-dimensionsal massless fields gives rise to important simplifications in the calculations of string amplitudes. In this second half of the course, I would like to elevate conformal invariance to a guiding principle for the construction of string theories. It will then be very useful to formulate 2-dimensional massless field theories in such a way that their conformal invariance is manifest. This was done in a very beautiful way by Belavin, Polyakov, and Zamolodchikov.[1] In Section 2, I will review the formalism which these authors have presented; this formalism will provide the basic language for our later arguments. Our use of conformal field theory methods in application to string theory follows the work of Friedan, Martinec, and Shenker;[2,3] Our general development, and especially our treatment of the critical dimensions and the formalism for the scattering amplitudes of strings and superstrings, follows their approach closely.

The discussion of string theory proper will begin in Section 3. Here, I will use conformal invariance to rederive the basic results on the embedding dimensionality for bosonic and fermionic strings. Section 4 will discuss the spectrum of the bosonic string and the computation of scattering amplitudes. In Section 5, I will extend this formalism to clarify the origin of Yang-Mills gauge invariance in the open bosonic string theory. Section 6 will address the question of the general-coordinate gauge invariance of string theory, presenting two disparate points of view on this question.

In Section 7, I will analyze the superstring theory from the viewpoint of 2-dimensional conformal invariance. I will rederive the basic results on the particle spectrum and present methods for the covariant calculation of superstring scattering amplitudes. In Section 8, I will discuss the 1-loop amplitudes of bosonic and supersymmetric string theories.

The last two sections will give a brief introduction to some of the deeper questions of the theory, especially the question of the reduction from the idealized string theory in 10 extended dimensions to more realistic solutions in which all but 4 of these dimensions are compactified. In Section 9, I will outline briefly what is known about the space-time supersymmetry of the superstring from the covariant viewpoint. I will then present a precis of the approach advanced by Candelas, Horowitz, Strominger, and Witten[4] for identifying possible 6-dimensional spaces which might represent the form of the compact dimensions. Section 10 will give a somewhat more detailed presentation of the *orbifold* scheme of compactification suggested by Dixon, Harvey, Vafa, and Witten.[5] This scheme has the advantage of allowing explicit calculations of many aspects of the conpactified theory, and we will find it illuminating to carry through a part of this analysis.

This full course of lectures is still far from comprising a complete summary of knowledge on string theory.* This set of lectures will certainly raise as many questions as it answers. I hope that you, the reader, will be intrigued to seek the

* A different, and more extensive, selection of topics is given in the new book of Green, Schwarz, and Witten.[6]

answers to these questions, and thus to join the quest for understanding of this most promising and mysterious branch of fundamental theory.

2. Conformal Field Theory

The main goal of this set of lectures will be to reconstruct the spectrum and interactions for bosonic and supersymmetric strings using as our primary tool the conformal symmetry of the dynamics on the string world sheet. It is therefore appropriate that we begin our discussion by setting out the formalism of 2-dimensional conformal invariance. Conformal symmetry is, exactly as it implies, the symmetry of conformal mappings of the 2-dimensional plane. One should then naturally expect that a conformally invariant theory will involve fields which are analytic functions of the 2-dimensional coordinates treated as a complex variable. Recently, Belavin, Polyakov, and Zamolodchikov,[1] (BPZ), building on results from the early period of string theories,[7–9] have shown how to write conformally invariant theories explicitly in terms of analytic fields. In this section, I will review this beautiful formalism, which will provide a natural language for our subsequent exploration of string theory.

2.1 CONFORMAL COORDINATES

To begin our study, we must define the basic coordinates. Throughout these lectures, I will describe both the string world surface and the spacetime in which it is embedded by their Euclidean continuations. On the string world surface, this continuation corresponds to

$$(\tau \pm \sigma) \ \rightarrow \ -i\,(\tau \pm i\sigma)\ . \tag{2.1}$$

Let us then define

$$w \ = \ \tau + i\sigma\ , \qquad w \ = \ \tau - i\sigma\ . \tag{2.2}$$

The decomposition of a string state into running waves moving to the left or to the right around the string becomes, after this continuation, a decomposition into analytic and anti-analytic functions on the 2-dimensional Euclidean surface. The Euclidean string covers only a finite interval of σ and therefore only a strip of the 2-dimensional plane, shown in Fig. 1(a). However, if we anticipate that the theory

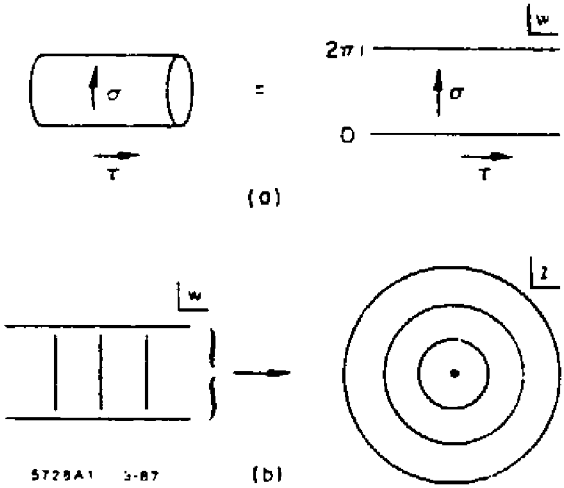

Figure 1. The string world sheet considered as a region of the complex plane (a) in the original variables τ and σ, (b) in the variable $z = \exp(\tau + i\sigma)$

we will construct will have complete symmetry under conformal transformations, we can map this region into the whole complex plane by the mapping

$$z = \exp(\tau + i\sigma) . \tag{2.3}$$

The form of this mapping is shown in Fig. 1(b). Lines of constant τ are mapped into circles on the z plane; the operation of time translation, $\tau \rightarrow \tau + a$, becomes the dilatation,

$$z \rightarrow e^{a} z . \tag{2.4}$$

The string coordinates $X^{\mu}(\tau, \sigma)$, separated into left- and right-moving excitations, correspond to analytic and anti-analytic fields in this complex z plane. We must address the question of whether a sensible Euclidean quantum theory of analytic fields can be defined. A part of this definition must include the identification of operators which implement conformal mappings of the z plane. Within the family of such operators, we should identify the generator of dilatations (2.4) with the Hamiltonian of the original string theory. Such a procedure of identifying dilatations with the Hamiltonian and circles about the origin with equal-time surfaces is called *radial quantization.*

Let us begin to piece together this formalism. To begin, assume that the z plane is a flat Euclidean space, so that its metric is $g_{\alpha\beta} = \delta_{\alpha\beta}$. In complex coordinates,

$$g_{z\bar{z}} = \frac{1}{2}\,, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0\,. \tag{2.5}$$

Assume that we can at least construct a theory on this plane which is scale invariant. Let us work out the consequences of this statement. Since the energy-momentum tensor $T_{\alpha\beta}$ generates local translations and a dilatation corresponds to a local motion $x^\alpha \to x^\alpha + \delta\lambda \cdot x^\alpha$, the dilatation current should be just

$$D_\alpha = T_{\alpha\beta} x^\beta\,. \tag{2.6}$$

Since the energy-momemtum tensor is conserved, the statement of scale invariance

$$\partial_\alpha D^\alpha = 0 \qquad \text{implies} \qquad T_\alpha{}^\alpha = 0\,. \tag{2.7}$$

Up to this point, these statements are true in any dimension. In 2 dimensions, however, when we use complex coordinates, (2.7) takes the following form:

$$T_{z\bar{z}} = 0\,. \tag{2.8}$$

But then we can use the equation of energy-momentum conservation

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \tag{2.9}$$

to prove that

$$\partial_{\bar{z}} T_{zz} = 0\,. \tag{2.10}$$

Thus, the field $T_{zz} \equiv T$ is an analytic function of z. Similarly, $T_{\bar{z}\bar{z}} \equiv \overline{T}$ is depends only on $\bar{z}$ and so is an anti-analytic field.

2.2. Conformal Transformations

It is not unreasonable to expect T and $\overline{T}$, as the remnants of the energy momentum tensor in these complex coordinates, to generate local conformal transformations. Let us try to formulate this conjecture more precisely. A natural form for the operator which generates the infinitesimal conformal mapping

$$z \rightarrow z + \epsilon^{z}(z) \tag{2.11}$$

would be $\int dz\, \epsilon^{z}(z)\, T_{zz}(z)$; the integral should be taken over an equal-time surface, that is, one of the circles shown in Fig. 1(b). Let us, then, define

$$T_{\epsilon} = \oint \frac{dz}{2\pi i}\, \epsilon(z)\, T(z) \ , \tag{2.12}$$

where the path of integration is a circle about the origin. Note that, since the integrand is an analytic function, we may move the integral from any such circle to any other by a contour deformation without changing its value; thus, T_{ϵ} is a conserved charge. We would expect that the transformation (2.11) would be implemented in a quantum theory in the form of the commutators of local fields with (2.12).

We can easily check this conjecture for the simple case of a free massless scalar field. Write the action for such a field as

$$\int \mathcal{L} = \frac{1}{2\pi} \int d^2z\, \partial_z X \partial_{\bar{z}} X \ . \tag{2.13}$$

The propagator of this field is the Green's function for the Laplace equation in 2 dimensions: $\langle X(z_1)X(z_2)\rangle = -2\log|z_1 - z_2|$. Split this into the pieces corresponding to the analytic and anti-analytic components. The propagator of the analytic field is then

$$\langle X(z_1)X(z_2)\rangle = -\log(z_1 - z_2) \ . \tag{2.14}$$

The analytic part of the energy-momentum tensor for this field is given by

$$T_{zz} = -\frac{1}{2} : (\partial_z X)^2 : \ . \tag{2.15}$$

Henceforth, unless a different convention is indicated explicitly, I will always consider products of analytic fields at the same point to be normal-ordered.

Now we are ready to compute the commutator of the operator (2.12) constructed from (2.15) with some local field operator $\mathcal{O}(z)$. $X(z)$ is actually not a good first choice, since this field can have logarithmic branch cuts (as seen in

(2.14)), so I will choose instead to compute $[T_\epsilon, \partial_z X(z)]$. The contour defining T_ϵ may be taken to be the equal-time circle containing z. We may make use of the analytic properties of $T(z)$ and $\partial_z X(z)$ to evaluate this commutator by relating it to expectation values on the plane. Consider, then, the correlation function defined by functional integration:

$$\langle T_\epsilon\, \partial_z X(z)\rangle \;=\; \frac{1}{Z}\int \mathcal{D}X\; e^{-S}\;\; T_\epsilon\, \partial_z X(z)\;. \tag{2.16}$$

The functional integral defines the operator product by setting the operators in *time order*. We may define the equal-time commutator of two operators, then, as the difference of two correlation functions of the form (2.16) in which the operator T_ϵ has been displaced slightly forward and backward in time (that is, in radial distance from the origin) with respect to the point z. Thus, we write

$$\begin{aligned}&\langle [T_\epsilon\,,\,\partial_z X(z)]\rangle \\ &\quad= \frac{1}{Z}\int \mathcal{D}X\; e^{-S}\;\;\big(T_\epsilon(r=|z|+\delta)\,\partial_z X(z) \;-\; T_\epsilon(r=|z|-\delta)\,\partial_z X(z)\big)\,.\end{aligned} \tag{2.17}$$

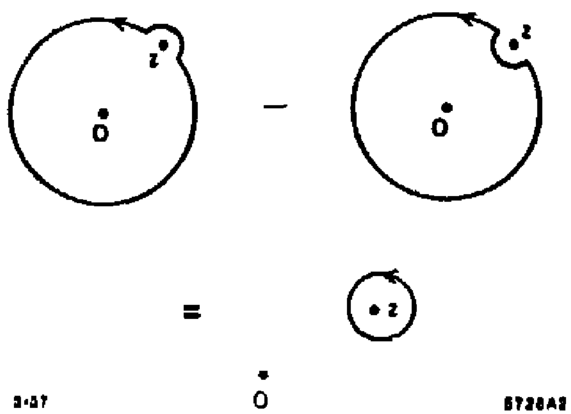

Figure 2. Computation of a commutator in terms of the poles of correlation functions.

The construction is illustrated in Fig. 2. As the figure indicates, T_ϵ is defined as a contour integral; since the integrand is analytic, we may deform the contour except where we encounter a singularity. Thus, by the cancellation shown, the commutator depends only on the singularity of the correlation function of $T(z')$ and $\partial_z X(z)$ in the limit $z' \to z$. This is easy to work out; letting $\langle X(z')X(z)\rangle$ denote the Wick contraction (2.14), we can compute

$$\begin{aligned} T(z')\,\partial_z X(z) &= -\frac{1}{2}\partial_z X\,\partial_z X(z')\quad \partial_z X(z) \\ &= -\frac{1}{2}\partial_z X\,\langle\partial_z X(z')\,\partial_z X(z)\rangle\cdot 2 \;+\; (\text{nonsingular}) \\ &= \quad \partial_z X(z')\,\frac{1}{(z'-z)^2} \;+\; (\text{nonsingular})\,. \end{aligned} \tag{2.18}$$

Taylor-expanding the field about $z' = z$ produces

$$T(z')\,\partial_z X(z) \;\sim\; \frac{1}{(z'-z)^2}\partial_z X(z) \;+\; \frac{1}{(z'-z)}\partial_z^2 X(z) \;+\; \cdots \tag{2.19}$$

This implies

$$\begin{aligned} \langle\, [T_\epsilon, \partial_z X(z)]\,\rangle &= \oint \frac{dz'}{2\pi i}\,\epsilon(z')\left[\frac{1}{(z'-z)^2}\partial_z X(z) \;+\; \frac{1}{(z'-z)}\partial_z^2 X(z) \;+\; \cdots\right]. \\ &= \partial_z\epsilon\cdot\partial_z X \;+\; \epsilon\cdot\partial_z^2 X\,. \end{aligned} \tag{2.20}$$

Is this a sensible result? Under the transformation $z \to z + \epsilon(z)$, we should expect

$$\begin{aligned} X(z) &\to X(z) \;+\; \epsilon\,\partial_z X(z)\,, \\ \partial_z X(z) &\to \partial_z X(z) \;+\; \partial_z\epsilon\,\partial_z X(z) \;+\epsilon\,\partial_z^2 X(z)\,. \end{aligned} \tag{2.21}$$

We may explain the extra term in the second line by noting that $\partial_z X$ is a tensor, of rank (-1). In general, a tensor transforms under reparametrizations according to

$$t^{\alpha\beta\cdots}(x) \;\to\; \frac{\partial \tilde{x}^\alpha}{\partial x^{\bar{\alpha}}}\cdots t^{\bar{\alpha}\bar{\beta}}\;\;(x(\tilde{x})). \tag{2.22}$$

If we specialize this equation to conformal transformations in 2 dimensions, and consider tensors with r analytic indices only, this transformation law becomes

$$t^{zz\cdots}(z) \to \left(\frac{dz}{d\bar{z}}\right)^{r} t^{zz\cdots}(\bar{z}(z)) . \tag{2.23}$$

For an infinitesimal transformation, $\bar{z} = z + \epsilon(z)$,

$$\delta t = -r\,\partial_z\epsilon \cdot t + \epsilon \cdot \partial_z t ; \tag{2.24}$$

this properly reproduces (2.21) and (2.20) for $r = -1$.

Under a dilatation $\bar{z} = \lambda z$, (2.23) takes the form

$$t \to \lambda^r t(\lambda^{-1}\bar{z}) , \tag{2.25}$$

so that the rank of an analytic tensor is also its scaling dimension. Normally in particle physics, we quote the dimension in mass units; thus we should write $d = -r$. Assembling the pieces of our analysis, we find that the commutator of the generator of a conformal transformation T_ϵ with a local tensor field of dimension d_t should be

$$[T_\epsilon, t(z)] = d_t \cdot \partial_z\epsilon \cdot \partial_z t(z) + \epsilon \cdot \partial_z^2 t(z) . \tag{2.26}$$

This commutator follows, by the manipulations described above, from the operator product expansion

$$T(w)\,t(z) \sim \frac{d_t}{(w-z)^2}\, t(z) + \frac{1}{w-z}\,\partial_z t(z) + \text{(nonsingular)} . \tag{2.27}$$

This operator product formula encodes the conformal transformation properties of fields in a way which will prove very convenient for mathematical analysis.

To gain a better understanding of the formula (2.27), it will be helpful to work through one more example. Consider the operator $e^{i\alpha\cdot X(z)}$ formed as the exponential of the free field. Let us compute the operator product of the free-field

$T(w)$ with this field:

$$-\frac{1}{2}(\partial_w X(w))^2 e^{i\alpha X(z)} \sim -\frac{1}{2}\left(\langle \partial_w X(w)\ i\alpha\cdot X(z)\rangle\right)^2 e^{i\alpha X(z)}$$
$$- \frac{1}{2}\cdot 2\,\partial_w X\left(\langle \partial_w X(w)\ i\alpha\cdot X(z)\rangle\right) e^{i\alpha X(z)}$$
$$\sim \frac{\alpha^2/2}{(w-z)^2}\, e^{i\alpha X(z)} + \frac{i\alpha\cdot\partial_w X}{w-z}\, e^{i\alpha X(z)} + \dots, \tag{2.28}$$

so that

$$-\frac{1}{2}(\partial_w X)^2\, e^{i\alpha X(z)} \sim \frac{\alpha^2/2}{(w-z)^2}\, e^{i\alpha X(z)} + \frac{1}{w-z}\,\partial_z\left(e^{i\alpha X}\right) + \dots \tag{2.29}$$

Thus, $e^{i\alpha X(z)}$ is a conformal tensor of dimension $\alpha^2/2$. This same scaling dimension can be read from the correlation function

$$\left\langle e^{i\alpha X(w)}\, e^{-i\alpha X(z)}\right\rangle = \exp[-\alpha^2 \log(w-z)] = (w-z)^{-\alpha^2}\,. \tag{2.30}$$

Not all fields are tensors. Derivatives of tensors, for example, have more complicated transformation laws. BPZ refer to fields with the above transformation laws as *primary fields*; their successive derivatives are called *secondary fields*. In general, the operator product of secondary fields with $T(z)$ has higher singularities than the double pole seen in (2.27).

In addition to analytic fields $t(z)$, transformed by the action of T_{zz}, a conformal field theory will have anti-analytic fields $\bar{t}(\bar{z})$, transformed by the action of $T_{\bar{z}\bar{z}}$. The simplest of these fields will be tensors with $\bar{r}$ anti-analytic indices; these are the primary anti-analytic fields of dimension $\bar{d}_t = -r$. The theory of these tensors and their transformations can be developed precisely in parallel to the discussion of analytic tensors which we have just completed. More generally, we should expect that some primary fields will transform with both analytic and anti-analytic indices; these will be characterized as tensors of dimension $(d_t, \bar{d}_t)$.

2.3. The Conformal Algebra

The elementary infinitesimal conformal transformations corresponding to $\epsilon(z) \sim z^{n+1}$ are generated by the Fourier components of $T(z)$ on the circle:

$$L_n = \oint \frac{dw}{2\pi i} w^{n+1} T(w) . \tag{2.31}$$

From Michael Green's lectures, you might expect that the L_n are the Virasoro operators.[10] To verify this, let us compute their algebra:

$$[L_n, L_m] = \left[\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\right] z^{n+1}T(z)\, w^{m+1}T(w) . \tag{2.32}$$

The change in the order of the z and w integrations represents a small displacement of the z contour outside and then inside the w contour, implementing the functional definition of the commutator that I have described earlier.

The difference of integrals is nonzero only by virtue of the singularity of the operator product as $z \to w$. In the free boson field theory, we can readily compute this singularity:

$$\begin{aligned} T(z)T(w) &= (-\tfrac{1}{2})^2 \cdot 2 \cdot (\langle \partial_z X \partial_w X \rangle)^2 + \\ &\quad + (-\tfrac{1}{2})^2 \cdot 4 \cdot \partial_z X \, \langle \partial_z X \partial_w X \rangle \, \partial_w X + \text{(nonsingular)} \\ &= \frac{1}{2}\left(\frac{-1}{(z-w)^2}\right)^2 + \partial_z X \frac{-1}{(z-w)^2} \partial_w X + \ldots \\ &= \frac{1}{2} \frac{1}{(z-w)^4} \\ &\quad + \frac{2}{(z-w)^2}\left[-\frac{1}{2}(\partial_w X)^2\right] + \frac{1}{(z-w)} \partial_w \left[-\frac{1}{2}(\partial_w X)^2\right] + \ldots \end{aligned} \tag{2.33}$$

The last two terms are precisely what would be required for T to transform itself as a tensor of dimension 2 under conformal transformations. (Note that the last term fixes the normalization of $T(z)$.) The first term is an extra c-number—generated as a purely quantum mechanical effect. The form of this term is determined by scale transformation properties: Since T has mass dimension 2, this term must be a pure power $(z-w)^{-4}$. However, the overall coefficient of this term is not fixed and can vary from system to system. In a general conformal field theory, then, we expect the $T-T$ operator product to take the form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T + \cdots \qquad (2.34)$$

where c is a fixed number.

We can now find the general form of the commutator of two L_n operators by inserting (2.34) into (2.32), and drawing the z contour tightly about the point w. This gives

$$\begin{aligned}
[L_n, L_m] &= \oint \frac{dw}{2\pi i}\, w^{m+1} \oint \frac{dz}{2\pi i}\, z^{n+1} \\
&\qquad \times \left[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T\right] \\
&= \oint \frac{dw}{2\pi i}\, w^{m+1}\Big\{(n+1)w^n \cdot 2T(w) + w^{n+1}\partial_w T \\
&\qquad + (c/2)\cdot\frac{w^{n-2}}{3!}(n+1)n(n-1)\Big\} \\
&= \oint \frac{dw}{2\pi i}\Big\{(2n+2)w^{n+m+1}T(w) - (m+n+2)w^{n+m+1}T(w) \\
&\qquad + \frac{c}{12}\, n(n+1)(n-1)w^{m+n-1}\Big\}.
\end{aligned} \qquad (2.35)$$

This is just

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} n(n+1)(n-1)\,\delta(m+n)\,, \qquad (2.36)$$

which is indeed the Virasoro algebra. The constant c is called the *central charge* of the the Virasoro algebra; this takes the value $c=1$ for one massless scalar field.

We can see from (2.34) that the constant c appears in the vacuum expectation value of the square of the operator T_{zz}. It can be argued on this basis that c must be positive if the underlying Hilbert space has a positive metric.[11] We may derive this result more explicitly as follows: In the next few sections, we will construct a state $|0\rangle$, such that $L_n|0\rangle = 0$ for $n \geq 0$, and an inner product such that $L_n^\dagger = L_{-n}$. Using these ingredients, we may write:

$$\frac{1}{2}c = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2 L_2^\dagger|0\rangle > 0 . \tag{2.37}$$

The most important of the L_n is L_0, the generator of dilatations. It should be noted, though, that the central charge term in (2.36) actually vanishes for the set of three generators L_{-1}, L_0, L_1. These operators generate the infinitesimal transformations

$$\delta z = \alpha + \beta z + \gamma z^2 \tag{2.38}$$

From (2.36), we see that these generators form a closed subalgebra. The subgroup of conformal transformations generated by this algebra is the group of fractional linear transformations

$$z \to z' = \left(\frac{az+b}{cz+d}\right) . \tag{2.39}$$

(Expanding (2.39) about $a = d = 1$, $b = c = 0$ reproduces (2.38).)

The complementary algebra of anti-analytic transformations is generated by a second set of Virasoro operators

$$\overline{L}_n = \oint \frac{d\overline{w}}{-2\pi i}\, \overline{w}^{n+1}\overline{T}(\overline{w}) . \tag{2.40}$$

Just analogously, these generators possess a closed subalgebra generated by $(\overline{L}_{-1}, \overline{L}_0, \overline{L}_1)$ with vanishing central charge. Together, the two sets of operators generate infinitesimal transformations of the form

$$\delta z = \alpha + \beta z + \gamma z^2 , \qquad \delta\overline{z} = \overline{\alpha} + \overline{\beta}\overline{z} + \overline{\gamma}\overline{z}^2 , \tag{2.41}$$

acting on all of the operators of the conformal field theory. This transformation is the infinitesimal form of a fractional linear transformation (2.39) with general complex coefficients.

When one composes two mappings of the form (2.39), the parameters of the product mapping are obtained by matrix multiplication of the original parameters. In particular, the determinants $(ad - bc)$ simply multiply. Since one of the four parameters of (2.39) is redundant, we may fix $(ad - bc) = 1$. Then the set of mappings (2.39) with complex coefficients, considered as a group under composition, is isomorphic to the group of 2×2 matrices with unit determinant, $SL(2, C)$. This subgroup of the full conformal group plays an important role in string theory, as we will see in Sections 4 and 5.

The definition of the L_n may be considered as a Fourier analysis of $T(z)$. It will be convenient to introduce a set of conventions for Fourier-analyzing a more general tensor field $t(z)$, of dimension d_t. Let us define

$$t_n = \oint \frac{dz}{2\pi i} z^{n+d_t-1} t(z) , \qquad t(z) = \sum_{n=-\infty}^{\infty} t_n z^{-n-d_t} . \tag{2.42}$$

To see the utility of this definition, compute the commutator of the Fourier component t_n, defined in this way, with L_0. We find

$$\begin{aligned} [L_0, t_n] &= \left[\oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} - \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \right] wT(w)\, z^{n+d_t-1} t(z) \\ &= \oint \frac{dz}{2\pi i} z^{n+d_t-1} \oint \frac{dw}{2\pi i} w \left[\frac{d_t}{(w-z)^2} t(z) + \frac{1}{(w-z)} \partial_z t \right] \\ &= \oint \frac{dz}{2\pi i} z^{n+d_t-1} [d_t\, t(z) + z\partial_z t(z)] \\ &= -n \oint \frac{dz}{2\pi i} z^{n+d_t-1} t(z) . \end{aligned} \tag{2.43}$$

Thus,

$$[L_0, t_n] = -n t_n ; \tag{2.44}$$

that is, t_n lowers L_0 by n units. In string theory, we will interpret the dilatation generator L_0 as the Hamiltonian of the single-string dynamics. Fourier components t_n will be annihilation operators for $n > 0$ and creation operators for $n < 0$. Ladder operators of anti-analytic tensors may be defined in an analogous way.

3. Critical Dimensions

In order to connect the conformal field theory on the complex plane defined in the previous section to the dynamics of coordinates $X^\mu(\tau,\sigma)$ on a string, one further condition must be satisfied. The program displayed in Fig. 1 requires that we replace functional integrals over fields on the string surface with functional integrals over fields on the plane. If the field theory of two-dimensional fields on the string is conformally invariant, it would seem that we could freely make this replacement. However, there is a subtlety which we must recognize and deal with. The usual criterion for conformal invariance is that expectation values of (scalar) operators are unchanged by conformal transformations. This leaves over the possibility that the functional integral over fields could change its normalization by a c-number factor when we make a conformal transformation. This factor would disappear when one computes a correlation function. However, in string theory, the string scattering amplitudes will be identified with the functional integrals themselves, and these possible c-number factors will appear explcitly as violations of conformal symmetry. We can take the transformation shown in Fig. 1 absolutely literally, then, only if we can identify and cancel this extra c-number term. In this section, I will discuss that cancellation, which, as we will see, implies that the string must live in a specific critical embedding dimension.

3.1. Conformal Transformations and Conformal Ghosts

We should begin by reviewing the route from a geometrically invariant formulation of the string dynamics to expressions of the form of eq. (2.13) in a fixed background metric. A geometrically invariant expression of the dynamics of the string coordinate field is given by writing[12]

$$Z = \int \mathcal{D}X\mathcal{D}g \, e^{-\int \mathcal{L}} , \tag{3.1}$$

where

$$\int \mathcal{L} = \frac{1}{8\pi} \int d^2\xi \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\mu \tag{3.2}$$

and the functional integral is taken over both the fields $X^\mu(\xi)$ and the metric $g_{\alpha\beta}(\xi)$ on the world surface. Since the action (3.2) is reparametrization invariant, we are free to change coordinates to simplify the form of the metric. On the plane, or on a region such as the strip of Fig. 1 with the topology of a plane, we may use 2-dimensional reparametrizations to remove two degrees of freedom from $g_{\alpha\beta}$

and convert it to the form

$$g_{\alpha\beta} = e^{\phi(\xi)} \cdot \delta_{\alpha\beta} . \tag{3.3}$$

The special case of (3.3) with $\phi(\xi) = 0$ is exactly the flat metric, eq. (2.5), used in the previous section.

If we insert (3.3) into the action (3.2), $\phi(\xi)$ cancels out. This is exactly the statement that (3.2) is classically conformally invariant. If the metric has the form (3.3) in one coordinate system, a conformal transformation $z \to \tilde{z}(z)$ carries

$$g_{z\bar{z}} \to \left|\frac{d\tilde{z}}{dz}\right|^2 g_{\bar{z}\bar{z}} , \tag{3.4}$$

so this tranformation causes simply a shift of $\phi(\xi)$. If $\phi(\xi)$ is irrelevant, then, we can move freely between different conformally related coordinate systems. This is what we need to view the strip and the plane shown in Fig. 1 as being completely equivalent.

If indeed the integrand of (3.1) is independent of $\phi(\xi)$, we can treat this variable as a gauge parameter and fix it as the same time that we fix the reparametrization freedom. In this case, the functional integral of eq. (3.1) should more properly be written as

$$Z = \frac{1}{V_{rep}V_{conf}} \int DXDg\, e^{-\int \mathcal{L}} , \tag{3.5}$$

where the two prefactors represent the volumes of the reparametrization and conformal groups. This expression may be evaluated by using the Fadde'ev-Popov procedure. If we gauge-fix to the coordinate system (2.5), shifting the conformal factor $\phi(\xi)$ simply shifts the diagonal component of the metric; this leads to a trivial Fadde'ev-Popov determinant. The off-diagonal terms in the metric are induced by reparametrizations

$$\delta g_{zz} = \partial_z(\delta\xi^{\bar{z}}) ; \tag{3.6}$$

setting these components equal to zero as a gauge condition leads to a nontrivial determinant

$$\Delta = \det(\partial_z) \cdot \det(\partial_{\bar{z}}) , \tag{3.7}$$

which may be represented by a functional integral over ghost fields. In all, we find that the gauge-fixing replaces

$$\frac{1}{V_{rep}V_{conf}} \int Dg \;=\; \int Db\, Dc\; e^{-\int (b_{zz}\partial_{\bar{z}}c^z + \bar{b}_{\bar{z}\bar{z}}\partial_z \bar{c}^{\bar{z}})} \,, \tag{3.8}$$

where b, c, $\bar{b}$, $\bar{c}$ are anticommuting fields. I have assigned the ghosts c, $\bar{c}$ the transformation properties of reparametrizations $\delta\xi^z$, $\delta\xi^{\bar{z}}$; the antighosts have the tranformation laws needed to make the action of (3.8) geometrically invariant.

Unfortunately, the formal property that the integrand of (3.1) does not depend on $\phi(\xi)$ is true only classically and (generally) disappears in the quantum theory. Polyakov[12] discovered that the regulation of the functional integral over $X^\mu(\xi)$ breaks the conformal symmetry and leads to a multiplicative c-number dependence on ϕ: Under a shift from $\phi = 0$ to a nonzero $\phi(\xi)$, $Z_X = \int DX \exp(-\int \mathcal{L})$ tranforms according to

$$Z_X \;\rightarrow\; Z_X \cdot \exp\left[-\frac{D}{96\pi}\int d^2\xi (\partial_\alpha \phi(\xi))^2\right] . \tag{3.9}$$

where D is the number of coordinate fields, or, equivalently, the dimension of the space-time in which the string is embedded. This violation of conformal invariance can be understood as a consequence of some considerations of the previous section. Since a nonzero $\phi(\xi)$ can be generated by a conformal transformation, let us look more closely at the algebra of conformal transformations derived there. In particular, let us apply $\partial_{\bar{z}}$ to eq. (2.34). Using

$$2\partial_{\bar{z}}\partial_z \log|z-w| \;=\; \partial_{\bar{z}}\frac{1}{(z-w)} \;=\; \pi\delta^{(2)}(z-w) \,, \tag{3.10}$$

we find

$$\partial_{\bar{z}}T_{zz}(z)\; T_{ww}(w) \;=\; -c\cdot\frac{\pi}{12}\partial_z^3\,\delta^{(2)}(z-w) \;+\cdots \tag{3.11}$$

Thus, the existence of the central charge implies that T_{zz} is not completely analytic. By the logic of eq. (2.10), this implies that the system is not completely scale-invariant. Let us manipulate this relation further by applying $\partial_{\bar{w}}$ and using (2.9) to exchange T_{zz} for $T_{z\bar{z}}$. Then

$$T_{z\bar{z}}(z)\; T_{w\bar{w}}(w) \;=\; c\cdot\frac{\pi}{12}\cdot\partial_z\partial_{\bar{w}}\delta^{(2)}(z-w) \;+\;\cdots \tag{3.12}$$

An infinitesimal shift of ϕ is brings down from the exponent of the functional integral the term

$$\frac{1}{2\pi}\int d^2\xi\ \phi(\xi)T_{z\bar{z}}(\xi) \tag{3.13}$$

Thus, eq. (3.12) implies that a shift of ϕ produces a multiplicative c-number factor of just the form of (3.9), with coefficient proportional to the central charge c of the Virasoro algebra.[13]

In many contexts, the conformal transformation law (3.9) is perfectly acceptable. In a few cases, it is actually physically relevant; for example, in a statistical mechanical system on a strip of finite width, mapping the strip to the full plane and applying (3.9) gives the correct dependence of the free energy on the finite size of the strip.[14,15] We have already remarked, however, that the standard formulation of string theory requires that the conformal motion be a gauge symmetry of (3.1). The extra term in (3.9) is unacceptable. This higher criterion of conformal invariance imposes an additional stringent constraint on a conformal field theory: The central charge c of the Virasoro algebra must vanish identically. This criterion is made especially difficult to fulfill by two observations made in the previous section: First, the coordinate fields X^μ each contribute one unit to c, for a total of $c = D$. Second, an additional field on the world sheet can give a negative, cancelling, contribution to c only if it creates states of negative metric.

Polyakov realized that this apparent dilemma has a very natural resolution The reparametrization ghosts introduced in (3.8) create negative-metric states and must, in any case, be included in any analysis of the conformal transformation properties of the string functional integral. Let us, then, set up the conformal field theory for these ghost fields and compute their contribution to c.

The fields c^z and b_{zz} obey the classical equation of motion

$$\partial_{\bar{z}}c^z = \partial_{\bar{z}}b_{zz} = 0 . \tag{3.14}$$

Quantum mechanically, the action given in (3.8) implies the propagator

$$\langle b_{zz}(z)\, c^w(w)\rangle = \frac{1}{z-w} . \tag{3.15}$$

We may then treat c^z and b_{zz} as analytic tensor fields. Since the Faddeev-Popov procedure assigns to c^z the conformal properties of the displacement ξ^z, this field should transform as a conformal tensor of dimension (-1). The complementary antighost b_{zz} acquires dimension 2. The ghost and antighost associated with $\xi^{\bar{z}}$, $c^{\bar{z}}$ and $b_{\bar{z}\bar{z}}$, transform as anti-analytic tensors with dimensions (-1) and 2. From here on, I will drop the tensor indices and refer to these fields as $c(z)$, $b(z)$, $\bar{c}(\bar{z})$, and $\bar{b}(\bar{z})$.

The energy-momentum tensor for c and b can be reconstructed from the requirement that it reproduce the operator product (2.27) with each of c and b, assigning these field the correct dimensions. The result is

$$T(z) = -2b\partial_z c - \partial_z bc \,. \tag{3.16}$$

One may check that the single and double pole terms in the operator product of this $T(z)$ with itself are given in accordance with (2.33). More generally, we might imagine a system with anticommuting analytic tensor fields

$$\hat{b}\,, \quad \text{dimension } (j)\,; \qquad \hat{c}\,, \quad \text{dimension } (1-j)\,, \tag{3.17}$$

always with the propagator (3.15). This system is described by

$$T(z) = -j\hat{b}\partial_z\hat{c} + (1-j)\partial_z\hat{b}\hat{c}\,. \tag{3.18}$$

It is not difficult to work out the central charge of the $\hat{b}$, $\hat{c}$ system for a general value of j. We must simply compute the c-number term in the operator product of (3.18) with itself:

$$\begin{aligned} T(z)\,T(w) &= j^2\left\langle\hat{b}(z)\partial_w\hat{c}\right\rangle\left\langle\partial_z\hat{c}\hat{b}(w)\right\rangle + (j-1)^2\left\langle\partial_z\hat{b}\hat{c}(w)\right\rangle\left\langle\hat{c}(z)\partial_w\hat{b}\right\rangle \\ &\quad + j(j-1)\Big\{\left\langle\hat{b}(z)\hat{c}(w)\right\rangle\left\langle\partial_z\hat{c}(z)\partial_w\hat{b}\right\rangle + \left\langle\partial_z\hat{b}(z)\partial_w\hat{c}\right\rangle\left\langle\hat{c}(z)\hat{b}(w)\right\rangle\Big\} \\ &= j^2\left(\frac{-1}{(z-w)^4}\right) + (j-1)^2\left(\frac{-1}{(z-w)^4}\right) \\ &\quad + 2j(j-1)\left[\frac{-2}{(z-w)^2}\right]\left[\frac{1}{z-w}\right] \\ &= -\frac{(6j^2-6j+1)}{(z-w)^4}\,. \end{aligned} \tag{3.19}$$

Thus, we find for the central charge of anticommuting tensor fields:

$$c = -2C_j \equiv -2\cdot(6j(j-1)+1)\,. \tag{3.20}$$

If $\hat{b}$ and $\hat{c}$ had been commuting fields, the propagator (3.15) would have implied

$$\left\langle\hat{c}(z)\,\hat{b}(w)\right\rangle = -\frac{1}{z-w}\,. \tag{3.21}$$

and the change of sign in this equation would have induced a change of sign in the final answer. Thus, for commuting tensor fields

$$c = +2C_j , \tag{3.22}$$

where C_j is just as given in (3.20).

Setting $j = 2$ in (3.20), we find that the system of reparametrization ghosts has $c = -26$. This cancels the contributions from $X^\mu(z)$ only if the string is embedded in a space of 26 dimensions. In that case, however, the conformal field theory on the string is conformally invariant in the strong sense described at the beginning of this section: The complete functional integral is unchanged, even by overall multiplicative factors, under a conformal transformation.

3.2. Spinors and Ghosts of the Superstring

From the formalism we have developed, it is not difficult to give a parallel discussion of the embedding dimension of the superstring. The superstring is obtained from the ordinary string by extending the geometrical invariance on the world sheet from local reparametrization invariance to local supersymmetry and replacing each coordinate field X^μ by a supermultiplet (X^μ, Ψ^μ), where Ψ^μ is a Majorana fermion. Let us first discuss this new matter field, then turn to the new ghosts. A Majorana fermion in 2 dimensions has the action

$$\int \mathcal{L} = \frac{1}{4\pi} \int d^2z \, \overline{\Psi}\gamma \cdot \partial \Psi , \tag{3.23}$$

where, because of the Majorana nature of Ψ, $\overline{\Psi} = \Psi^T\gamma^0$. If we introduce the explicit representation of Euclidean Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} , \tag{3.24}$$

this takes the form

$$\mathcal{L} = \frac{1}{4\pi} \int d^2z \, \left[\psi\partial_{\bar{z}}\psi + \overline{\psi}\partial_z\overline{\psi} \right] . \tag{3.25}$$

The dynamics of the field ψ is almost exactly that of a $\hat{b}$, $\hat{c}$ system. We can make the connection explicit by combining two species of Majorana fermions to form

$$\hat{b} = \psi^1 + i\psi^2 , \qquad \hat{c} = \psi^1 - i\psi^2 . \tag{3.26}$$

The new fields are just the positive-chirality part of a Dirac fermion and its complex conjugate. (The negative-chirality part of the Dirac field is anti-analytic and may be built from $\overline{\psi}^1, \overline{\psi}^2$.) These fields have an action proportional to $\int \hat{b}\partial_{\bar{z}}\hat{c}$ and thus a propagator of the form of (3.15). Since a fermion field in 2 dimensions has dimension $\frac{1}{2}$, two Majorana fermions form a $\hat{b}$, $\hat{c}$ system with $j = \frac{1}{2}$ (plus the corresponding anti-analytic system). This system of two Majorana fermions has $c = 1$. We might have guessed this result from the well-known property that a Dirac fermion in 2 dimensions can be bosonized, that is, replaced by a equivalent system containing one real scalar field. Bosonization will come to play a major role in our analysis, beginning in the next section.

In addition to the new matter fields, we find that the superstring has new ghosts corresponding to the gauge-fixing of the new geometrical invariances of this theory. Without unravelling the whole structure of these new transformations, we can easily guess that they will correspond, in conformal coordinates, to the transformation of the world-sheet gravitino field by a spinor parameter:

$$\delta\lambda_{\bar{z}\varsigma} = \partial_{\bar{z}}\eta^{\varsigma} , \tag{3.27}$$

and the corresponding transformation with an analytic derivative. In (3.27), the (lowered) indices ς, $\bar{\varsigma}$ denote 2-dimensional spinor indices corresponding to the upper and lower components of (3.24). Since η is an anticommuting parameter, it must be replaced by a commuting ghost field γ^{ς}; this field transforms as a conformal tensor of dimension $(-\frac{1}{2})$. This field will have the action

$$\int \mathcal{L} = \frac{1}{2\pi}\int d^2z\, \beta_{z\varsigma}\partial_{\bar{z}}\gamma^{\varsigma} , \tag{3.28}$$

where I have introduced an antighost $\beta_{z\varsigma}$, also a commuting field, of dimension $\frac{3}{2}$. These two fields, which I will refer to henceforth as $\gamma(z)$ and $\beta(z)$, form a system of commuting fields with $j = \frac{3}{2}$. Thus, they contribute

$$c = +2C_{j=\frac{3}{2}} = 11 . \tag{3.29}$$

These fields have anti-analytic counterparts $\overline{\gamma}(\bar{z})$, $\overline{\beta}(\bar{z})$ which contribute to the central charge of the Virasoro operators $\overline{L}_n$.

Adding the contributions to c from all of the fields on the world sheet of the superstring—coordinates X^{μ}, fermions ψ^{μ} (taken in pairs), anticommuting ghosts b, c, and commuting ghosts β, γ—we find

$$c = D + \frac{1}{2}D - 26 + 11 = \frac{3}{2}(D - 10) . \tag{3.30}$$

The superstring is then conformally invariant in the strong sense required for string theory when it is embedded in a space of 10 dimensions.

The restriction of string theories to a particular space-time dimension is a striking requirement, unusual in the formulation of a physical theory. Much effort has been spent trying to answer the question of whether string theories are well-defined outside of this critical dimension, with results which are so far inconclusive. For the remainder of this set of lectures, I will restrict myself to working in the critical dimension. We will see that the peculiar choices of $d = 26$ and $d = 10$ actually work miraculous simplifications in many aspects of the physics of strings.

4. Vertex Operators and Tree Amplitudes

Now that we have set up the dynamics of fields on the string world surface in terms of conformal field theory, we are ready to construct the spectrum of [illegible] of the string and the interactions between these states predicted by the [illegible]. We will find that both of these features of string dynamics may be presented in a most natural way in the language of conformal field theory.

4.1. The BRST Charge

Before beginning this discussion, however, we need one more piece of technical apparatus. In the previous section, we introduced into the string world sheet the ghost fields $b(z)$, $c(z)$, noting as we did this that these fields belong to a quantum theory with negative metric. The excitations of the ghost field will naturally become a part of the spectrum of states of the string. It is important, though, that we should not find negative metric states propagating on the world sheet as a result of a physical scattering process. Actually, negative metric states can arise not only from the ghost excitations but also also from the longitudinal and timelike excitations of $X^\mu(z)$. These modes of excitation were explicitly eliminated in the light-cone gauge treatment of the string dynamics presented in Michael Green's lectures, but they are still present in the covariant formalism that I have been constructing. In order to disentangle these dangerous states, we will need some mathematical tool which will enable us to distinguish physical from unphysical modes of excitation.

The standard tool for identifying and controlling the unphysical states of a covariantly quantized field theory is the ghost charge of Becchi, Rouet, Stora,[16] and Tyupin[17] (BRST). This is a nilpotent charge—

$$Q^2 = 0 \tag{4.1}$$

—which is Hermitian, commutes with the Hamiltonian, raises the ghost number by one unit, and annihilates all ghost-free, gauge invariant states. Such a charge can be constructed in any covariant field theory quantized in a way which introduces ghosts.*

Once the BRST charge Q has been constructed, it can be applied in the following way: The relation (4.1) implies that every eigenstate of the Hamiltonian is acted on by Q as a member of one of the following types of multiplets:

$$\begin{aligned} \text{singlet}: \quad & Q\,|\psi_0\rangle \;=\; 0\;; \\ \text{doublet}: \quad & Q\,|\psi_1\rangle \;=\; |\psi_2\rangle \qquad Q\,|\psi_2\rangle \;=\; 0\;. \end{aligned} \tag{4.2}$$

Since Q is Hermitian, $|\psi_2\rangle$ has zero norm: $\langle\psi_2|\psi_2\rangle = \langle\psi_1|Q^2|\psi_1\rangle = 0$. The gauge-invariant, ghost-free states should be BRST singlets; it is necessary to check in detail for each theory that the only 1-particle states which are BRST singlets are of this form, and that all such states have positive norm. Let us assume that this is true. Then the initial state in any scattering process will satisfy $Q\,|\psi_{in}\rangle = 0$. But then, since $[Q,H] = 0$, any state obtained by time-evolving this state will also be annihilated by Q. Thus, the final state of a scattering process must be linear combination of BRST singlets and zero-norm states. If we take a matrix element of this state to compute the S-matrix, the zero-norm pieces disappear, and we find

$$\langle\psi_f|\,S\,|\psi_i\rangle \;=\; \langle\psi_{f,0}|\,S\,|\psi_{i,0}\rangle\;, \tag{4.3}$$

where the states on the right-hand side are projected onto BRST singlets. Thus, the properties of Q allow one to prove that, if S is unitary on the full Hilbert space, as is guarranteed by Hamiltonian evolution, it is also unitary when restricted to BRST singlets. No probability disappears into ghostly states which transform nontrivially under Q.

Now that I have discussed the use of Q, I would like to describe roughly how to define Q in any given theory. This discussion abstracts a general construction due to Fradkin and Vilkoviskii.[19] Consider a theory with gauge invariances generated by charges G_i. Let b_i, c^j represent ghost and antighost fields satisfying $\{b_i, c^j\} = \delta_i{}^j$. Then we can begin the construction of Q by writing $Q = c^i G_i + \ldots$; This expression annihilates gauge-invariant states and otherwise has the form of a gauge transformation with gauge parameter c^i. Now we need only complete this expression to form a nilpotent charge by adding a piece which will act nontrivially

* A beautiful and detailed review of the BRST charge and its uses in field theory has been given by Kugo and Ojima.[18]

on states with ghosts. A suitable choice is given by writing

$$Q \;=\; c^i G_i \;+\; \frac{1}{2} f_{ij}{}^k c^j c^i b_k \;, \tag{4.4}$$

where the $f_{ij}{}^k$ are the structure constants of the algebra of the charges G_i. We may compute

$$\begin{aligned} Q^2 \;&=\; c^i c^j G_i G_j \;+\; \frac{1}{2} f_{ij}{}^k c^j c^i \{b_k, c^\ell\} G_\ell \;+\; \frac{1}{8} f_{ij}{}^k f_{\ell m}{}^n \{c^j c^i b_k, c^m c^\ell b_n\} \\ &=\; \frac{1}{2} c^i c^j f_{ij}{}^k G_k \;+\; \frac{1}{2} f_{ij}{}^k c^j c^i G_k \;+\; \frac{1}{4} f_{ij}{}^k f_{\ell k}{}^m c^j c^i c^\ell b_n \\ &=\; \qquad 0 \qquad + \qquad \frac{1}{4} f_{[ij}{}^k f_{\ell]k}{}^n (c^j c^i c^\ell b_n) \;. \end{aligned} \tag{4.5}$$

The remaining term vanishes by the Jacobi identity.

To make this construction more concrete, one must, in any specific system, assign appropriate values to the charges and ghost operators given above. In string theory, we clearly would like to identify the G_i with the conformal generators L_n and the ghost operators with the Fourier components of $b(z)$ and $c(z)$. This program is complicated by two features, the fact that the gauge algebra, first, is infinite-dimensional, and, second, may contain a central charge. However, let us overlook these issues for the moment and try to construct a Q of the form $Q = \sum c_n L_n + \ldots$. A suitable completion is given by interpreting the structure constants which appear in (4.4) as arising from the action of T_{zz} on the the ghosts themselves. Thus, for the bosonic string, we write

$$Q \;=\; \oint \frac{dz}{2\pi i} \;:\; c(z)[T^{(X)}(z) \;+\; \frac{1}{2} T^{(b,c)}(z)] \;:\;, \tag{4.6}$$

where $T^{(b,c)}$ is given by (3.16). The anti-analytic fields give a second BRST charge

$$\overline{Q} \;=\; \oint \frac{d\overline{z}}{2\pi i} \;:\; \overline{c}(\overline{z})[\overline{T}^{(X)}(\overline{z}) \;+\; \frac{1}{2}\overline{T}^{(\overline{b},\overline{c})}(\overline{z})] \;: \tag{4.7}$$

built using the anti-analytic components of the ghost fields.

To check the nilpotency of this Q, we represent $Q^2 \;=\; \frac{1}{2}\{Q, Q\}$ as the difference of correlation functions in which one contour lies just outside, then just inside, the other. This reduces the computation to the analysis of singularities in the operator product of the two integrands. Using the explicit form of $T^{(b,c)}$ and the general expansion (2.34), it is not hard to show that the anticommutator vanishes if the combined system of X, b, c has total $c = 0$. (An analogous argument implies that $\overline{Q}^2 = 0$.) Here again, the choice of the critical dimensionality plays an important role in simplifying the formalism.

4.2. BOSONIZATION OF THE GHOSTS

Certain aspects of the BRST formalism for the string are made clearer if one introduces an alternative representation of the b, c system in terms of boson operators. This bosonization of fermions is a familiar feature of the physics of 2-dimensional systems.[20,21] It will soon be clear that the boson-fermion correspondences are clarified, and new generalizations are suggested, by the notation of conformal field theory. Let me now review the conventional bosonization in this language, and then generalize to the bosonization of the reparametrization ghosts.

In the previous section, I noted that a massless complex fermion (or two Majorana fermions) in 2 dimensions gave the same central charge $c = 1$ as a real boson field. I will now argue that these theories also give identical results for correlation functions, if we make the correspondence

$$\psi^1 + i\psi^2 \leftrightarrow \sqrt{2}e^{i\phi(z)}, \qquad \psi^1 - i\psi^2 \leftrightarrow \sqrt{2}e^{-i\phi(z)}. \tag{4.8}$$

If ϕ is taken to be an ordinary free boson field, (2.29) implies that both of the exponentials of this field written in (4.8) have dimension $\frac{1}{2}$. This observation and the fact that the ψ system has $c = 1$ imply that, in correlation functions of fermion operators with factors of $T(z)$, the singularities as operators $T(z)$ approach fermion operators or one another are matched if we replace operators built from ψ^1, ψ^2 with the corresponding operator built from ϕ. Using (2.30), we can compute the singularity as two of the exponentials of ϕ approach one another:

$$\sqrt{2}e^{i\phi(z)}\,\sqrt{2}e^{-i\phi(w)} \sim 2\exp\big(-\log(z-w)\big) = \frac{2}{(z-w)}; \tag{4.9}$$

this properly reproduces the singularity of the corresponding product of fermions. Thus, these correlation functions are analytic functions with the same singularities and so must be identical. The system of two fermions is therefore physically equivalent to the bosonic theory built from ϕ. This is the free-field limit of the conventional bosonization of fermions in 2 dimensions.

Describing the reparametrizat n ghost system by bosons is obviously more of a challenge, since this system has $c = -26$. I will show, however, that this system is equivalent to a theory containing one real boson field σ. If we follow the argument just given, we must find elements of the σ theory which allow us to match the central charge, the dimensions of b and c, and the singularity of the b–c propagator. Begin with the central charge. Let us choose as a trial form for $T(z)$ the somewhat more general dimension 2 operator

$$T(z) \;=\; -\frac{1}{2}(\partial_z\sigma)^2 \;+\; A\partial_z^2\sigma \;, \tag{4.10}$$

where A is a constant to be determined. This gives

$$\begin{aligned} T(w)\,T(z) \;&\sim (-\frac{1}{2})^2\big(\langle\partial_w\sigma\,\partial_z\sigma\rangle\big)\cdot 2 \;+\; A^2\;\langle\partial_w^2\sigma\,\partial_z^2\sigma\rangle \\ &\qquad\qquad + \;\text{(less singular)} \\ &\sim\; \frac{1}{2}\,\frac{1}{(w-z)^4} \;+\; A^2\,\frac{2\cdot 3}{(w-z)^4} \;+\;\ldots \end{aligned} \tag{4.11}$$

Thus, we require $12A^2 + 1 = -26$, or $A^2 = -9/4$. We can rectify the sign in this relation by changing the sign of the kinetic energy term in the σ action and in $T(z)$. This gives σ the wrong metric, but that is only to be expected of a field which describes a ghost. The σ propagator then takes the form

$$\langle\sigma(z)\sigma(w)\rangle \;=\; +\log(z-w)\;. \tag{4.12}$$

With this choice,

$$T^{(\sigma)}(z) \;=\; +\frac{1}{2}(\partial_z\sigma)^2 \;+\; \frac{3}{2}\,\partial_z^2\sigma \tag{4.13}$$

satisfies (2.34) with $c = -26$.

By analogy to (4.8), we might seek correspondents for b and c which are exponentials of σ. Let us first compute

$$\begin{aligned} T(w)\;e^{\alpha\sigma(z)} \;&\sim\; \left[\frac{1}{2}\big(\langle\partial_w\sigma(w)\;\alpha\sigma(z)\rangle\big)^2 \;+\; \frac{3}{2}\,\langle\partial_w^2\sigma(w)\;\alpha\sigma(z)\rangle\right]\,e^{\alpha\sigma(z)} \;+\;\ldots \\ &\sim\; \left[\frac{1}{2}\alpha^2 - \frac{3}{2}\alpha\right]\,\frac{1}{(w-z)^2}\;e^{\alpha\sigma(z)} \;+\;\ldots \end{aligned} \tag{4.14}$$

This implies that $e^{\alpha\sigma(z)}$ is a conformal tensor of dimension $\frac{1}{2}\alpha(\alpha-3)$. Thus, we are motivated to identify

$$b(z) \leftrightarrow e^{-\sigma(z)}, \qquad c(z) \leftrightarrow e^{\sigma(z)}, \tag{4.15}$$

since the two exponentials have dimension 2, (-1), respectively. These operators also reproduce the b–c operator product

$$e^{-\sigma(z)}\, e^{\sigma(w)} \sim \exp\big(-\log(z-w)\big) = \frac{1}{z-w}. \tag{4.16}$$

Thus, we may reproduce any correlation function of b, c, and $T^{(b,c)}$ by operators built from σ.

I would now like to rewrite the BRST charge in this bosonized language. This requires some care in the definition of operator products. Thoughout my discussion, I have been assuming that all products of operators at the same point are normal-ordered. The normal-ordering of b and c operators, however, is not the same as the normal-ordering of the corresponding σ operators, and so we must be careful when we convert complex operators from one picture to the other. Let me, then, give the explicit definition of the 3-ghost operator which appears in (4.6):

$$\begin{aligned} :c(z)\,T^{(b,c)}(z): = \lim_{z\to w} \Big[& \{:c(z)::T^{(b,c)}(w):\} \\ & + \big(\frac{1}{(z-w)^2}c(w) + \frac{2}{(z-w)}\partial_w c\big)\Big]. \end{aligned} \tag{4.17}$$

The last two terms cancel the singular terms resulting from contraction of $c(z)$ with the factors of $b(w)$ in $T^{(b,c)}$; only then is the indicated limit smooth. If we bosonize each operator on the right-hand side of (4.17), we obtain

$$
\begin{aligned}
: c(z)\,T^{b,c}(z) : \; = \; & \lim_{z \to w} \Big[e^{\sigma(z)} \cdot \big(\frac{1}{2}(\partial_w \sigma)^2 + \frac{3}{2}\partial_w^2 \sigma \big) \\
& \qquad + \big(\frac{1}{(z-w)^2} e^{\sigma(w)} + \frac{2}{(z-w)} \partial_w e^{\sigma(w)} \big) \Big] \\
= \; & \lim_{z \to w} \Big[: e^{\sigma(z)} \Big\{ - \frac{1}{(z-w)^2} - \frac{1}{(z-w)} \partial_w \sigma \\
& \qquad + \frac{1}{2}(\partial_w \sigma)^2 + \frac{3}{2}\partial_w^2 \sigma \Big\} : \\
& \qquad + \big(\frac{1}{(z-w)^2} e^{\sigma(w)} + \frac{2}{(z-w)} \partial_w e^{\sigma(w)} \big) \Big] \\
= \; & \lim_{z \to w} \Big[: \Big\{ - \frac{1}{(z-w)^2} - \frac{2}{(z-w)} \partial_w \sigma \\
& \qquad - \frac{3}{2}(\partial_w \sigma)^2 - \frac{1}{2}\partial_w^2 \sigma + \frac{1}{2}(\partial_w \sigma)^2 + \frac{3}{2}\partial_w^2 \sigma \Big\} \, e^{\sigma(w)} : \\
& \qquad + \big(\frac{1}{(z-w)^2} e^{\sigma(w)} + \frac{2}{(z-w)} \partial_w e^{\sigma(w)} \big) \Big] \\
= \; & e^{\sigma(w)} \{ -(\partial_w \sigma)^2 + \partial_w^2 \sigma \} \\
= \; & e^{\sigma(w)} \big\{ (\partial_w \sigma)^2 + \partial_w^2 \sigma \big\} - \partial_w \sigma \, \partial_w (e^{\sigma}) \, .
\end{aligned}
\tag{4.18}
$$

Since Q involves the integral of this expression, we may integrate the last term by parts; then the integral of (4.18) falls into the form $2 \cdot \oint dw \; : e^{\sigma(w)} T^{(\sigma)}(w) :$. The factor 2 is just what we need to convert (4.6) to

$$
\oint \frac{dw}{2\pi i} \; : e^{\sigma(w)} \, T^{(tot)}(w) \; : , \quad \text{with } T^{tot} = T^{(X)} + T^{(\sigma)} \, . \tag{4.19}
$$

From this expression it is even more straightforward to apply the trick described below (4.6) and show that $Q^2 = 0$ if $T^{(tot)}$ has zero central charge.

4.3. String Scattering Amplitudes

We are now ready to construct the amplitude for scattering of strings. To do this, we will pursue the following strategy: We will first identify asymptotic single-particle states of string which can represent the initial and final states of a scattering process. To avoid the propagation of unphysical modes, we will insist that these states are BRST invariant. Then we will cast these states into the form of excitations on the world sheet, and allow those excitations to propagate and eventually overlap with one another. The overlap of the time-evolved initial and final states defines the scattering amplitude.

To define asymptotic states, we apply in a powerful way the conformal invariance of the theory. Using the conformal mapping

$$z - z_0 = e^{w} , \tag{4.20}$$

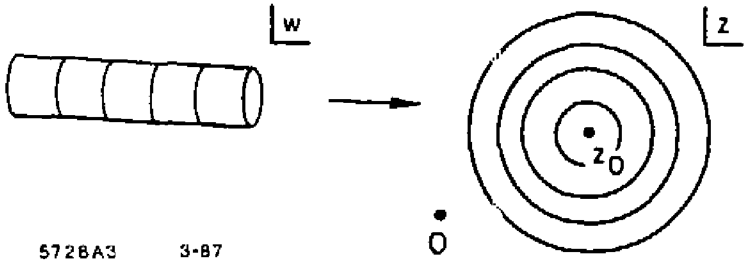

Figure 3. Identification of an asymptotic region of the closed string with the neighborhood of a point z_0 on the conformal plane.

according to the construction (2.3), we can map the asymptotic region of an infinitely extended closed-string world surface into the neighborhood of the point $z = z_0$. This transformation is shown in Fig. 3. In the discussion to follow, I will assume for simplicity that $z_0 = 0$. This entails no loss of generality; the Hamiltonian evolution described in Section 2, defined by dilatations outward from $z = 0$, could equally well have been set up about any other point.

If we put no operator at the end of the cylinder shown in Fig. 3, what propagates in from infinity is the ground state of the string in the sector with vacuum quantum numbers. Mapping this situation conformally to the z plane, the observation reads as follows: If we put *no* operator at $z = z_0$, this defines a particular state with vacuum quantum numbers as the initial state in the time evolution of radial quantization. Let us refer to this state as $|0\rangle$.

To define any other asymptotic state, we would apply some operator asymptotically on the string, or, equivalently, at $z = z_0$. The set of operators which might be used in this construction is most simply discussed in terms of the Fourier components of conformal fields, evaluated on a circle about z_0. If we translate to $z_0 = 0$, the definition of the Fourier components is that given in (2.42). Let us look more closely at the first part of this definition. Using this equation and the correspondence between operators and correlation functions, we may write

$$t_n\,|0\rangle \;=\; \left\langle \cdots \oint \frac{dz}{2\pi i}\, z^{n+d_t-1} t(z) \right\rangle \,, \tag{4.21}$$

where the contour is a small circle about $z = 0$. The operator ordering on the left-hand side gives the radial time ordering for the right-hand side; thus, if t_n is applied directly to the vacuum on the left, there should be no other operator inside the contour on the right. But then, if $n + d_t - 1 \geq 0$, we can contract this contour to $z = 0$ and find zero. This implies

$$t_n\,|0\rangle \;=\; 0 \quad \text{for} \quad n \geq 1 - d_t \,. \tag{4.22}$$

The moments of $t(z)$ which do not satisfy this condition may be evaluated by contour integration; one finds simply $t(0)$ and the successive derivatives of t evaluated at 0. We may view these as the creation operators for string excitations.

As an example, consider the moments of $\partial_z X^\mu$, which are conventionally denoted by

$$\alpha^\mu_n \;=\; +i \oint \frac{dz}{2\pi i}\, z^n \partial_z X^\mu \,. \tag{4.23}$$

For $n > 0$, α^μ_n is an L_0 lowering operator. All of these operators, and also α^μ_0, annihilate $|0\rangle$. It is natural to interpret the α^μ_n as lowering operators for the string normal-mode osciallations. Their counterparts α^μ_{-n} would be the corresponding raising operators. These operators may be equivalently represented as the derivatives of $\partial_z X^\mu$ evaluated at 0; for example, $\alpha^\mu_{-1} = \partial_z X(0)$. We can confirm this identification of raising and lowering operators by computing the commutator of two α_n^μ, using the functional method which by now should be quite familiar:

$$
\begin{aligned}
[\alpha_n^\mu, \alpha_m^\nu] &= (+i)^2 \left[\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\right] z^n \partial_z X^\mu \, w^m \partial_w X^\nu \\
&= \oint \frac{dw}{2\pi i} \, w^m \oint \frac{dz}{2\pi i} \, z^n \, \frac{1}{(z-w)^2} \, \delta^{\mu\nu} \\
&= \oint \frac{dw}{2\pi i} \, w^m \cdot (n \, w^{n-1}) \, \delta^{\mu\nu},
\end{aligned} \tag{4.24}
$$

so that

$$
[\alpha_n^\mu, \alpha_m^\nu] = n \delta(n+m) \delta^{\mu\nu} . \tag{4.25}
$$

The Fourier component α_0^μ is naturally interpreted the center-of-mass momentum operator of the string. With this interpretation, α_0^μ annihilates $|0\rangle$ because this state has zero momentum. To obtain a state of finite momentum, we can inject momentum into the string by applying the operator $e^{ip\cdot X}$ at z_0. We can check all of these identifications by computing

$$
\begin{aligned}
\alpha_n^\mu e^{ip\cdot X(0)} |0\rangle &= +i \oint \frac{dz}{2\pi i} \, z^n \partial_z X^\mu(z) \, e^{ip\cdot X(0)} \\
&= +i \oint \frac{dz}{2\pi i} \, z^n \left\{ \left(\frac{-ip^\mu}{z}\right) e^{ip\cdot X(0)} + (\text{nonsingular}) \right\} \\
&= \begin{cases} 0 & n > 0 \\ p^\mu \, |0\rangle & n = 0 \end{cases} .
\end{aligned} \tag{4.26}
$$

Indeed, the exponential shifts the center-of-mass momentum of the string while allowing all of the nonzero-frequency oscillators to remain in their ground states. We have now found that the state

$$
e^{ip\cdot X(0)} \, |0\rangle \tag{4.27}
$$

represents the asymptotic state of a ground-state string at momentum p^μ. Excited state of the string are represented by composite operators

$$
\partial_z X^\mu e^{ip\cdot X(0)} , \quad \partial_z X^\mu \partial_z X^\nu e^{ip\cdot X(0)} , \quad \partial_z^2 X^\mu e^{ip\cdot X(0)} , \quad \text{etc.} \tag{4.28}
$$

Operators such as those shown in (4.28) which are used to define asymptotic states of string are called *vertex operators*.

All of the excitations shown eq. (4.28) are created by analytic derivatives of X^μ; these represent left-moving excitations on a closed string. We could equally well have used anti-analytic derivatives of X^μ—$\partial_{\bar{z}} X^\mu(0)$ and higher derivatives. It follows from the generalization of the above argument to the anti-analytic sector that these operators create right-moving excitations in the asymptotic string states. In addition, it is straightforward to describe the analytic and anti-analytic ghost excitations as being created and destroyed by ladder operators, or, equivalently, by local vertex operators. By applying the manipulations of eq. (4.24), it is straightforward to show that the operator product

$$b(z)\ c(w)\ \sim\ \frac{1}{z-w} \tag{4.29}$$

implies that the Fourier components satisfy the anticommutation relations of ladder operators:

$$\{b_n, c_m\} \ = \ \delta(n+m)\ . \tag{4.30}$$

According to (4.22), b_n and c_m annihilate $|0\rangle$ unless $n \le 1$, $m \le -2$. Comparing with (2.42), we see that the b_n, c_m which do not annihilate $|0\rangle$ are precisely those which can be incorporated into vertex operators as the values of b and c and their successive derivatives at $z = 0$.

Once we have formulated a vertex operator $\mathcal{O}(0)$, we must decide whether the corresponding asymptotic state is BRST invariant. The state $|0\rangle$ is a BRST invariant state, since Q is defined as a contour integral, and this contour can be deformed to zero if it encloses no other operators. The operation $Q\,\mathcal{O}(z_0)\,|0\rangle$ is defined functionally by drawing the BRST contour about the operator $\mathcal{O}$ placed at z_0. Let us check whether this quantity vanishes.

$$Q\ \mathcal{O}(z_0)\ =\ \oint \frac{dw}{2\pi i}\, c^\sigma T(w)\, \mathcal{O}(z_0)\ . \tag{4.31}$$

To compute the operator product which gives the singularity of the integrand, assume that $\mathcal{O}(z_0)$ contains no ghost operators (as is true for the operators (4.28)) and that $\mathcal{O}(z_0)$ is a primary conformal field. Then (4.31) becomes

$$\begin{aligned} Q\ \mathcal{O}(z_0)\ &=\ \oint \frac{dw}{2\pi i}\, c^\sigma \left[\frac{d_{\mathcal{O}}}{(w-z_0)^2} + \frac{1}{(w-z_0)}\,\partial_z\right] \mathcal{O} \\ &=\ \left[d_{\mathcal{O}}(\partial_z c^\sigma)\cdot \mathcal{O}\ +\ c^\sigma \cdot \partial_z \mathcal{O}\right](z_0)\ . \end{aligned} \tag{4.32}$$

This result apparently can never vanish. However, if $d_{\mathcal{O}} = 1$, the final result is a total derivative, so that

$$Q \int dz\ \mathcal{O}(z) \;=\; \int dz\ \partial_z(c^\sigma \mathcal{O}) \;=\; 0\ . \tag{4.33}$$

The integral of a primary conformal field of dimension 1 thus gives a representation on the complex plane of a BRST invariant asymptotic state.

How should we properly interpret this result? Up to this point, all of the intuition we have used to motivate this construction has come from the consideration of closed strings. Closed strings have a local symmetry under the action both of the L_n and the $\overline{L}_n$; thus physical states should be invariant to both Q and $\overline{Q}$. In the closed-string theory, then, asymptotic states should be associated with operator insertions of the form

$$\int dz d\bar{z}\ \mathcal{O}(z, \bar{z})\ , \tag{4.34}$$

where $\mathcal{O}$ is primary and has dimension 1 relative to $T(z)$ and also with respect to $\overline{T}(\bar{z})$. For the simplest vertex operator

$$\mathcal{O} \;=\; e^{ip\cdot X(z,\bar{z})}\ , \tag{4.35}$$

the restriction on the dimensionality implies that

$$p^2 \;=\; 2\ , \tag{4.36}$$

that is, the asymptotic state must be an on-shell state of a particle of mass $m^2 = -2$.* This state is a tachyon, and that (unfortunately) is the correct result for the ground state of the closed string.

The construction in (4.33) has, however, another interpretation. We might recognize that (4.25), by itself, is precisely the algebra of mode creation and annihilation operators for the open string. This should tempt us to interpret the analytic sector of the system we have constructed as describing the open string. Mapping back to the cylinder according to (2.3), then slicing the cylinder into two semicircular pieces, we can see that, in this interpretation, the real axis of the z plane should be identified with the boundary of the open string. The open string boundary condition that $\partial_\sigma X^\mu = 0$ at the endpoint may be

* In Michael Green's conventions, this would read: $m^2 = -8$.

given the following interpretation: Let left-moving excitations on the open string be represented by analytic functions in the upper half-plane and right-moving excitations by analytic functions in the lower half-plane. Then the open string boundary condition is simply the requirement that these functions be continuous across the real axis.

This interpretation of the analytic sector of the world sheet dynamics suggests a new interpretation of the integral of a vertex operator: The integral $\int dz \mathcal{O}(z)$ along the real axis of the z-plane represents a BRST-invariant open string asymptotic state which couples to another open string at its boundary. The simplest such state is the one associated with the vertex operator $\mathcal{O} = e^{ip \cdot X}$. The requirement that $d_{\mathcal{O}} = 1$ again implies $p^2 = 2$, so this state is again a tachyon. This is, of course, a familiar property of the ground state of the open string.

These two constructions lead to formulae for the scattering amplitudes of open and closed strings, since the transition amplitude from a set of initial to a set of final string states can be computed as the joint correlation function of the corresponding vertex operators.

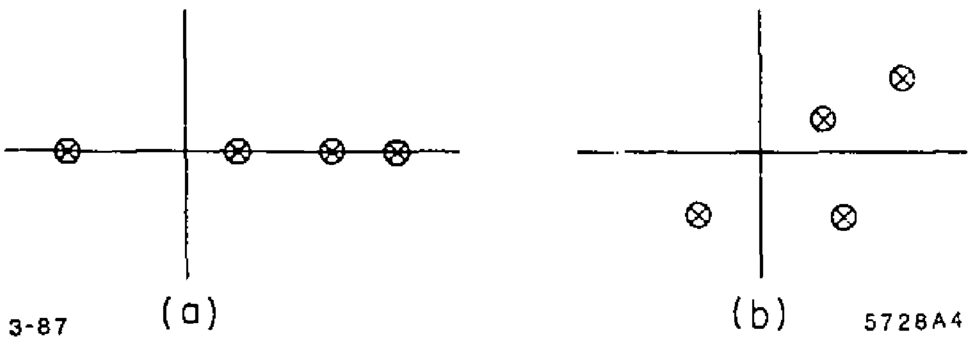

Figure 4. Correlation functions of vertex operators which give the scattering amplitudes of the (a) open and (b) closed string theories.

The open and closed string versions of this construction are illustrated in Fig. 4. We can then check our formalism by computing the scattering amplitude for four tachyons explicitly in each theory.

Begin with the scattering amplitude for open-string tachyons. This is given by a correlation function of four vertex operators integrated along the real axis:

$$A = \left\langle \int dz_1 dz_2 dz_3 dz_4 \; e^{ip_1 \cdot X(z_1)} \; e^{ip_2 \cdot X(z_2)} \; e^{ip_3 \cdot X(z_3)} \; e^{ip_4 \cdot X(z_4)} \right\rangle . \qquad (4.37)$$

As Michael Green has discussed, open strings may be given flavor quantum numbers by assigning a group theory factor (called the Chan-Paton factor) to each cyclic ordering of the operators. This factor gives the amplitude that the quantum numbers created on the string boundary by each vertex operator can be annihilated by the next operator along the line. I will ignore this factor in my discussion here, though I will assume that the cyclic order of the vertex operators is fixed. Evaluating the correlation function, we find

$$A = \int dz_1 dz_2 dz_3 dz_4 \prod_{i<j} \exp\big(p_i \cdot p_j \log(z_i - z_j)\big) . \qquad (4.38)$$

The expression (4.38) is, unfortunately, not yet well defined. The problem is one that should be familiar already from Michael Green's discussion of the string scattering amplitude: The integrand of (4.38) has a group of invariances which forces the integral to diverge. A part of this divergence comes from the fact that the integrand is translation invariant. The full symmetry is the unbounded three-parameter group of fractional linear transformations with real coefficients

$$z \to \left(\frac{az+b}{cz+d}\right) , \qquad (4.39)$$

exactly the group of eq. (2.39), specialized to real coefficients—$SL(2,R)$. This is in fact the group of conformal transformations which map the upper half plane onto itself 1-to-1. It is straightforward to check that the change of variables (4.39) leaves the integrand unchanged as long as each $p_i^2 = 2$. A natural way to cure this problem is to divide (4.38) by the group volume of $SL(2,R)$. Using the infinitesimal form of an $SL(2,R)$ transformation given in (2.41), we can find the Jacobian for the change of variables from any three of the (real) z_i to the (real) parameters α, β, γ:

$$\left| \frac{\partial(z_i, z_j, z_k)}{\partial(\alpha, \beta, \gamma)} \right| = (z_i - z_j)(z_i - z_k)(z_j - z_k) \qquad (4.40)$$

Making this change of variables using z_1, z_2, z_4, and then cancelling the integral over group parameters, we find the final result for A:

$$A = \int dz_3 \,(z_1 - z_2)(z_1 - z_4)(z_2 - z_4) \prod_{i<j} \exp\big(p_i \cdot p_j \log(z_i - z_j)\big) \,. \quad (4.41)$$

This result should be independent of the choice of z_1, z_2, z_4, as long as the four points are in the correct cyclic order. If we assign $z_1 \to \infty$, $z_2 = 1$, $z_4 = 0$, always using $p_i^2 = 2$ and momentum conservation, (4.41) simplifies to the form

$$A = \int_0^1 dz \,(1 - z)^{p_2 \cdot p_3} z^{p_3 \cdot p_4} \,. \quad (4.42)$$

This is the famous amplitude of Veneziano[22] which was in fact the first result in string theory.

The scattering amplitude of the closed string theory may be computed by following an analogous set of steps. One begins from the expression

$$A = \left\langle \int d^2z_1 d^2z_2 d^2z_3 d^2z_4 \, e^{ip_1 \cdot X(z_1)} \, e^{ip_2 \cdot X(z_2)} \, e^{ip_3 \cdot X(z_3)} \, e^{ip_4 \cdot X(z_4)} \right\rangle \,, \quad (4.43)$$

where the integrals now run over the whole complex plane. The correlation function may be evaluated as in (4.38). The resulting expression is invariant to transformations of the form (4.39) with general complex coefficients; this is the full group $SL(2,C)$. This group is in fact the group of all 1-to 1 conformal mappings of the whole z plane onto itself. The amplitude (4.43) can be made well-defined by dividing by the volume of this group. The Jacobian needed in this construction is the absolute square of (4.40). The result is

$$A = \int d^2z_3 \, |(z_1 - z_2)(z_1 - z_4)(z_2 - z_4)|^2 \prod_{i<j} \exp\big(2p_i \cdot p_j \log|z_i - z_j|\big) \,. \quad (4.44)$$

Assigning z_1, z_2, z_4 to ∞, 1, 0, we find

$$A = \int d^2z \, |1 - z|^{2p_2 \cdot p_3} \, |z|^{2p_3 \cdot p_4} \,, \quad (4.45)$$

the Virasoro-Shapiro[23,24] closed string amplitude.

To conclude this discussion of string scattering amplitudes, I would like to discuss briefly the problem of constructing BRST invariant vertex operators for higher states of the string spectrum. The open-string vector states $\zeta^\mu \alpha^\mu_{-1}|0\rangle$ are created by vertex operators

$$\mathcal{O}_\zeta(z) \;=\; \zeta^\mu \partial_z X^\mu(z) e^{ip\cdot X(z)} \,. \tag{4.46}$$

We must check not only that this operator has the correct dimension but also that it is a primary conformal field. To do this, compute its operator product expansion with $T(z)$:

$$\begin{aligned} T(w)\;\mathcal{O}_\zeta(z) \;\sim\; & -\frac{1}{2}\langle\partial_w X^\mu\; ip\cdot X(z)\rangle\langle\partial_w X^\mu\;\zeta^\nu\partial_z X^\nu\rangle\; e^{ip\cdot X(z)}\cdot 2 \\ & -\;\frac{1}{2}\Big(\langle\partial_w X^\mu\; ip\cdot X(z)\rangle\Big)^2\zeta^\nu\partial_z X^\nu\; e^{ip\cdot X(z)} \\ & -\;\frac{1}{2}\partial_w X^\mu\,\langle\partial_w X^\mu\;\zeta^\nu\partial_z X^\nu\rangle\; e^{ip\cdot X(z)}\cdot 2\;+\;\ldots \\ \sim\;& \frac{ip\cdot\zeta}{(w-z)^3}\; e^{ip\cdot X(z)} \;+\; \frac{p^2/2+1}{(w-z)^2}\;\zeta\cdot\partial_z X\, e^{ip\cdot X(z)} \;+\;\ldots \end{aligned} \tag{4.47}$$

The operator $\mathcal{O}_\zeta$ has dimension 1 if $p^2=0$, that is, if the vector particles created by this operator are massless. However, (4.47) informs us that we must impose an additional constraint in order that $\mathcal{O}_\zeta$ be primary. This is the condition $\zeta\cdot p=0$, the familiar physical requirement of transversality. In a similar way, we may find that the tensor vertex operator

$$\eta^{\mu\nu}\,\partial_z X^\mu \partial_z X^\nu e^{ip\cdot X} \tag{4.48}$$

has dimension 1 if $p^2=-2$, corresponding to $m^2=2$, but it is primary only if we satisfy as well the physical state conditions on the polarization tensor

$$p_\mu\,\eta^{\mu\nu} \;=\; 0 \;=\; \eta^\mu{}_\mu \,. \tag{4.49}$$

In general, for string modes with spin, the requirement of BRST invariance implies not only that the particle should be on its mass shell but also that the polarization should have a physically correct orientation. This is the first hint of the deep connection between BRST invariance on the world sheet and gauge invariance in space-time which we will explore in the next section.

5. Gauge Invariances of the Bosonic String

In the previous section, we formulated a set of rules for computing string scattering amplitudes at the tree level. The basic assumption of this construction was that the string scattering amplitude should involve propagation on the world sheet of physical (that is, BRST invariant) excitations. We found, however, that this condition implies that the external particles of string must also be physical propagating states on space-time. This correspondence between world-sheet and space-time properties is already remarkable in itself, but it is worth pushing the argument one step further. Our experience with local field theories with spin tells us that such theories cannot be formulated covariantly and still naturally project out unphysical polarizations unless they possess an underlying gauge invariance. We have seen also that the open string theory contains massless vector states; it would be most attractive if these were the gauge bosons of some explicit local invariance of the string theory. In this section, I will exhibit the local gauge invariances of the free string theory and clarify the relation between these invariances and the world-sheet dynamics.

5.1. More Conformal Field Theory

To carry out this analysis, we will need some additional tools from conformal field theory. In the previous section, I introduced the vacuum state $|0\rangle$, defined by putting no operator at the point $z = 0$, which forms the asymptotic past in radial quantization. To continue our discussion, it will be useful to define $\langle 0|$, and, more generally, a notion of operator adjoints and inner product appropriate to conformal field theory. All of the necessary concepts are provided in the work of BPZ.

The obvious definition of $\langle 0|$ is to put nothing at the point $z = \infty$ which forms the asymptotic future of radial quantization. Just as contour integrals which can be deformed to 0 represent operators which annihiliate $|0\rangle$, operators will annihilate this $\langle 0|$ when the corresponding contour integrals can be pushed to ∞. Conformal invariance allows us to discuss both situations at the same time, because a conformal transformation

$$z \rightarrow \hat{z} = -\frac{1}{z} \tag{5.1}$$

interchanges past and future. BPZ proposed that this conformal transformation could in fact be taken to be the definition of the adjoint of an operator or state. This definition has the virtue that it does not interchange the analytic and anti-analytic sectors. We can check the definition by applying it to a general Fourier component t_n:

$$\begin{aligned} t_n &= \oint \frac{dz}{2\pi i}\, z^{n+d_t-1} t(z) \\ &\longrightarrow \oint \frac{dz}{2\pi i}\, z^{n+d_t-1} \left(\frac{d\bar{z}}{dz}\right)^{d_t} t(\bar{z}) \\ &= \oint \frac{d\bar{z}}{2\pi i}\, \frac{1}{\bar{z}}\, (-\bar{z})^{-n-d_t}\, (\bar{z}^2)^{d_t}\, t(\bar{z}) \\ &= (-1)^{d_t+n}\, t_{-n}\, . \end{aligned} \tag{5.2}$$

(A factor of (-1) disappears in the second step because both the z and the $\bar{z}$ contours are taken to run counterclockwise around the unit circle.) The transformation (5.1) thus does interchange t_n and t_{-n}. We find as well the equivalence

$$t_n|0\rangle = 0 \quad \leftrightarrow \quad \langle 0|t_{-n} = 0\, . \tag{5.3}$$

According to (4.22), these statements apply when $n \geq 1 - d_t$.

The L_n annihilate $|0\rangle$ for $n \geq (-1)$. Reciprocally, $\langle 0|L_n = 0$ for $n \leq 1$. Thus, the three operators L_{-1}, L_0, L_1, and their anti-analytic counterparts $\overline{L}_{-1}$, $\overline{L}_0$, $\overline{L}_1$, annihilate both $|0\rangle$ and $\langle 0|$. This makes these operators true symmetries of conformal field theory: If U is a transformation generated by these operators—that is, if $U \in SL(2,C)$—

$$\langle 0|\phi(z_1)\,\phi(z_2)\cdots|0\rangle = \langle 0|(U\phi(z_1)U^{-1})(U\phi(z_2)U^{-1})\cdots|0\rangle \tag{5.4}$$

Thus, all conformal field theory matrix elements are invariant to $SL(2,C)$ transformations. This is the origin of the conformal invariances of the four-tachyon scattering amplitudes discussed at the end of the last section. Because of this property, the state $|0\rangle$ is often referred to as the $SL(2,C)$ invariant vacuum.

The antighost field $b(z)$ has the same dimension as $T(z)$, and therefore, in parallel to the above discussion, b_{-1}, b_0, and b_1 annihilate both $|0\rangle$ and $\langle 0|$. But then the identity following from the commutation relation (4.30) of the ghost ladder operators

$$\{b_n, c_{-n}\}\,|0\rangle = |0\rangle \neq 0\ ; \tag{5.5}$$

implies that the three operators c_{-1}, c_0, c_1 annihilate neither $|0\rangle$ nor $\langle 0|$. All other c_n and b_n ladder operators annihilate either $|0\rangle$ or $\langle 0|$. This set of statements implies that the basic nonzero expectation value in the theory is

$$\langle 0|\, c_{-1}\, c_0\, c_1\, |0\rangle \;\neq\; 0\,. \tag{5.6}$$

Apparently, the adjoint operation of BPZ does not preserve ghost number; indeed, it insists that ghost number is violated by 3 units in conformal field theory calculations on the plane.

To properly understand this ghost number nonconservation, it is necessary to develop in some detail the geometrically invariant formulation of the string dynamics. Since this would take us too far afield, I will simply sketch the logic of the required argument. The ghost number current $j_z = bc$ is the fermion number current of a set of chiral fermions (of unconventional spin) in 2 dimensions. Therefore, we should expect that this current has a gravitational anomaly. Its conservation law is, in fact,

$$\partial_{\bar z} j_z \;=\; -\frac{3}{8\pi}\, R\,, \tag{5.7}$$

where my conventions correspond to $\int d^2z\sqrt{g}R = 8\pi$ for a sphere. Considered for more general 2-dimensional manifolds, this integral is a topological invariant, the Euler characteristic χ; its value is

$$\frac{1}{4\pi}\int d^2z\sqrt{g}\, R \;=\; -2(g-1)\,, \tag{5.8}$$

where g is the *genus* of the surface, the number of handles. For surfaces of the topology of a sphere, $g = 0$ and we expect (-3) units of ghost number nonconservation. Since the complex plane, including the point at infinity, has this topology, we thus obtain the nonconservation law displayed in (5.6). For surfaces with handles, ghost number $3(g-1)$ should be swallowed by the conformal field theory matrix element.

When one encounters fermion number nonconservation due to the coherent effects of anomalies, the nonconservation is normally manifested in the appearance of *zero modes* of the fermion field. These are modes of the fermion field which are localized solutions to the equations of motion. If one defines the functional integral over the fermion field by decomposing in eigenmodes of the fermion action

$$\psi(x) \;=\; \sum_n \psi_n\, \varphi_n(x)\,, \tag{5.9}$$

the zero modes $\varphi_0(x)$ are annihilated by the Dirac operator, and so the components ψ_0 which multiply these solutions do not appear in the action. The fermion integral over each of these components then gives 0 unless a factor of ψ_0 is supplied by an extra fermion operator under the functional integral:

$$
\begin{aligned}
\langle 1\rangle &= \int \mathcal{D}\psi\, e^{-\int \mathcal{L}}\, 1 = 0\,, \\
\langle \psi(z)\rangle &= \int \mathcal{D}\psi\, e^{-\int \mathcal{L}}\, \psi(z) = \phi_0(z)\,.
\end{aligned}
\tag{5.10}
$$

For the anticommuting ghosts, eq. (5.8) implies that we should find 3 zero modes of $c(z)$ on the sphere, no zero modes on the torus (actually, one finds one each for $c(z)$ and $b(z)$), and $3(g-1)$ zero modes of b on surfaces of higher genus.

The zero modes of c responsible for the result (5.6) are actually easy to identify: Since the equation of motion for c is $\partial_{\bar z} c = 0$, the zero modes are analytic functions regular on the plane which satisfy the correct boundary conditions at infinity. These boundary conditions should be the ones appropriate for treating the z plane as a sphere: Choose a metric which makes the plane a compact space of constant positive curvature:

$$
g_{z\bar z} = \frac{1}{(1+A^2|z|^2)^2} \qquad g_{zz} = g_{\bar z\bar z} = 0\,. \tag{5.11}
$$

(Sending $A \to 0$ makes the neighborhood of the origin as flat as one wishes.) Then an eigenmode of c is normalizable if

$$
\|\, c\,\|^2 = \int d^2z \sqrt{g}\, \bar c^{\bar z}\, g_{\bar z z} c^z = \int d^2z\, \frac{1}{(1+A^2|z|^2)^4}\, |c^z|^2 \tag{5.12}
$$

is finite. There are three entire functions which satisfy this criterion: $c(z) \sim 1, z, z^2$. It is pleasing that these are exactly the coefficients of c_1, c_0, c_{-1} in the Fourier decomposition of $c(z)$ given by the right-hand side of (2.42).

If we normalize (5.6) to $\langle 0|\, c_{-1}c_0c_1\,|0\rangle = 1$ and apply this Fourier decomposition, we find that

$$
\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = \begin{vmatrix} 1 & 1 & 1 \\ z_3 & z_2 & z_1 \\ z_3^2 & z_2^2 & z_1^2 \end{vmatrix} = (z_1-z_2)(z_1-z_3)(z_2-z_3)\,. \tag{5.13}
$$

This formula also follows from the bosonized expression

$$
\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = \left\langle e^{\sigma(z_1)}e^{\sigma(z_2)}e^{\sigma(z_3)}\right\rangle\,. \tag{5.14}
$$

The result (5.13) is reminiscent of eq. (4.40). This is, in fact, the first sign of a general relation between the treatment of ghost zero modes and the measure for the integrations involved computing in string scattering amplitudes. I will explain the relevance of this formula later in this section; we will discuss the more general situation in Section 8.

5.2. Ghost Contributions to the String Spectrum

One of the results of the analysis just given is that the L_0 lowering operator c_1 does not annihilate the $SL(2,C)$ invariant vacuum $|0\rangle$. This means that there exists another state of the open string Hilbert space which has a lower L_0 than $|0\rangle$; this is

$$|\Omega\rangle \;=\; c_1\,|0\rangle \ , \tag{5.15}$$

Since all other lowering operators annihilate $|0\rangle$ and $(c_1)^2 = 0$, this is actually the state of minimum L_0. This new vacuum satisfies

$$\langle\Omega|c_0|\Omega\rangle \;=\; \langle 0|c_{-1}c_0c_1|0\rangle \;=\; 1 \ ; \tag{5.16}$$

then $\langle\Omega|\Omega\rangle = 0$, since this quantity has the wrong ghost number. We have found that the true vacuum of the open string theory is a peculiar ghostly state of zero norm. Nevertheless, as we will see in a moment, this observation will allow us to considerably refine our understanding of the spectrum of string states.

Since $|\Omega\rangle$ is the true open string vacuum, we should consider rebuilding the spectrum of physical string excited states by applying the creation operators for coordinate excitations α^{μ}_{-n} to $|\Omega\rangle$; this produces states of the form

$$|\psi\rangle = \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\alpha^{\lambda}_{-2}\alpha^{\sigma}_{-13}\cdots|p\rangle \ , \qquad \text{where} \quad |p\rangle \;=\; e^{ip\cdot X}\,|\Omega\rangle \ . \tag{5.17}$$

We may denote such states, equally well, by $|\psi\rangle = c(0)V(0)\,|0\rangle$, where $V(0)$ is a vertex operator built of X^{μ} and its derivatives. These states are not necessarily BRST invariant. In fact, $|\Omega\rangle$ itself is not BRST invariant: If we apply the contour integral formulae (4.6) or (4.19) to the representation

$$|\Omega\rangle \;=\; c(0)\;|0\rangle \;=\; e^{\sigma(0)}\;|0\rangle \tag{5.18}$$

and evaluate operator products, we find

$$Q\,|\Omega\rangle = \;(-1)\cdot c_0\,|\Omega\rangle \ . \tag{5.19}$$

On a state of the more general form (5.17),

$$Q\;|\psi\rangle \;=\; \oint\frac{dz}{2\pi i}\;:c(z)\Big[T^{(X)}+\frac{1}{2}T^{(b,c)}\Big]\;|\psi\rangle \;=\; \Big[\sum_{n=-\infty}^{\infty}c_n(L^{(X)}_{-n}\;-\;\delta_{n,0})\Big]\;|\psi\rangle \ , \tag{5.20}$$

where the $L^{(X)}_n$ are the Fourier components of $T^{(X)}$. Since for $n > 0$, $c_n\,|\Omega\rangle = 0$, half of the conditions for BRST invariance implied by (5.20) are trivially satisfied. We see, then, that $|\psi\rangle$ will be BRST invariant if

$$\begin{aligned} L_n^{(X)}\,|\psi\rangle &= 0\,, \qquad n \geq 1\,, \\ (L_0^{(X)} - 1)\,|\psi\rangle &= 0\,. \end{aligned} \tag{5.21}$$

These are exactly the physical state conditions presented, from a very different viewpoint, by Michael Green. The second condition states that the operator V which creates $|\psi\rangle$ from $|\Omega\rangle$ must be of dimension 1. This is exactly the mass-shell condition derived in the previous section. The first condition is equivalent to the statement that V must be a primary conformal field, since the contour integral

$$L_n^{(X)} V(0)\,|\Omega\rangle = \oint \frac{dz}{2\pi i}\, z^{n+1} T(z)\, V(0)\,|\Omega\rangle\,, \tag{5.22}$$

for $n \geq 1$, will pick up any higher terms in the operator product expansion of T with V. We have seen that this statement requires $|\psi\rangle$ to have a physical polarization.

An observation equivalent to that just presented is that operators

$$c(0)\,V(0) = e^{\sigma(0)}\,V(0)\,, \tag{5.23}$$

where $V(z)$ satisfies the conditions of the previous paragraph, are the BRST invariant vertex operators which we sought but failed to find in the previous section. Let us check this directly: If V is a primary conformal field built from X^μ,

$$\begin{aligned} Q\; c(0)V(0) &= \oint \frac{dz}{2\pi i}\; :e^{\sigma(z)}\left[T^{(X)}(z) + T^{(\sigma)}(z)\right]::\; e^{\sigma(0)}V(0): \\ &= \oint \frac{dz}{2\pi i}\; :e^{\sigma(z)+\sigma(0)}:\; \exp\left[\langle\sigma(z)\sigma(0)\rangle\right] \left[\frac{d_V - 1}{z^2} + \cdots\right]. \\ &= \oint \frac{dz}{2\pi i}\; :e^{\sigma(z)+\sigma(0)}:\; z\left[\frac{d_V - 1}{z^2} + \cdots\right]. \end{aligned} \tag{5.24}$$

This vanishes if V has dimension 1; that is, if $c \cdot V$ has dimension 0.

We can apply this observation immediately to our calculation of the 4-tachyon scattering amplitude. The amplitude constructed in eq. (4.37) remains BRST invariant if we replace factors of $\int dz V(z)$ by $c(z_0)V(z_0)$ where z_0 is any fixed point on the real axis. In fact, we are required to make 3 such replacements in order to satisfy the law that conformal field theory amplitudes annihilate 3 units of ghost number. This gives the formula:

$$A = \left\langle \left[c(z_1)e^{ip_1\cdot X(z_1)}\right] \left[c(z_2)e^{ip\cdot X(z_2)}\right] \int dz_3\, e^{ip\cdot X(z_3)} \left[c(z_4)e^{ip\cdot X(z_4)}\right] \right\rangle . \tag{5.25}$$

When we evaluate this formula by using (5.14) to compute the ghost correlation function, we find again the result (4.41). Eq. (5.25) can be made to look more symmetric by replacing

$$e^{ip\cdot X(z_3)} \quad \text{by} \quad \oint \frac{dw}{2\pi i} b(w)\; c(z_3) e^{ip\cdot X(z_3)} \tag{5.26}$$

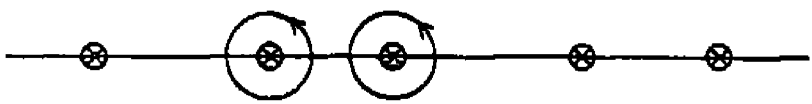

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Figure 5. Form of the 4-tachyon scattering amplitude in a formalism in which the vertex operators contain ghost factors $c(z)$.

This gives an alternative picture of the multi-string scattering amplitude which is shown in Fig. 5. Because the assignment of $c(z)$ factors serves only to fix the $SL(2,R)$ symmetry, it does not matter which of the z_i are surrounded by a $b(w)$ contour. Giddings and Martinec[25] have recently shown that one can systematically derive a formula of this last form from a set of Feynman rules for the open string dynamics.

5.3. Gauge Transformations of the String States

Let us turn now to a study of the string states which do not satisfy the first of the physical state conditions (5.21). The Virasoro commutation relation (2.36) for $[L_n, L_{-n}]$ implies that (unless the state has a particular exceptional value of of L_0) any state which is not annihilated by some $L_n^{(X)}$ can be written as a linear combination of states of the form

$$L_{-n}^{(X)}|\psi'\rangle \qquad (n > 0)\ . \tag{5.27}$$

It is instructive to write out the simplest of the states (5.27) explicitly. To do this, rewrite

$$L_{-n}^{(X)} = \int \frac{dz}{2\pi i}\, z^{n+1} T^{(X)}(z) = \frac{1}{2} \sum_{k=-\infty}^{\infty} :\alpha^{\mu}_{-n+k}\alpha^{\mu}_{-k}:\ , \tag{5.28}$$

and apply this operator to states built upon $|p\rangle = e^{ip\cdot X}|\Omega\rangle$. Because $\alpha_n^{\mu}|p\rangle = 0$ for $n > 0$, only a finite number of terms of the series (5.28) contribute; further, we may simplify by using $\alpha_0^{\mu}|p\rangle = p^{\mu}|p\rangle$. Then we find

$$L_{-1}\ |p\rangle = \alpha_{-1}\cdot\alpha_0\ |p\rangle = p\cdot\alpha_{-1}\ |p\rangle\ . \tag{5.29}$$

Similarly,

$$\begin{aligned} L_{-1}\ \lambda\cdot\alpha_{-1}|p\rangle &= p\cdot\alpha_{-1}\ \lambda\cdot\alpha_{-1}|p\rangle + \lambda\cdot\alpha_{-2}\ |p\rangle \\ L_{-2}\ |p\rangle &= \alpha_{-2}\cdot p|p\rangle + \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}|p\rangle\ . \end{aligned} \tag{5.30}$$

If we view the states which appear on the right-hand sides of (5.29) and (5.30) as components of the string states $\zeta^{\mu}\alpha^{\mu}_{-1}|p\rangle$, $\eta^{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|p\rangle$, we see that these are exactly the unphysical polarization states, the timelike and trace parts of the polarization, which cannot be associated with propagating states.

I would like to give this observation a deeper interpretation, as follows: Let me introduce a *string field*, a functional of the instantaneous position of the string in space-time:

$$\Phi[X^{\mu}(\sigma)] \tag{5.31}$$

The eigenstates of the single-string Hamiltonian, viewed as Schrödinger wavefunctions, are functionals of $X^{\mu}(\sigma)$ and, in fact provide us a basis of such functionals.

Let us expand a general functional $\Phi[X^\mu]$ in this basis. The coefficients in this expansion must carry Lorentz indices to match those of the mode creation operators for the corresponding state. It is convenient to include the dependence of $\Phi[X^\mu]$ on the center-of-mass position x^μ of the string as an x^μ-dependence of the coefficient functions. The general form of the expansion is then

$$\Phi[X^\mu(\sigma)] = [\phi(x) - iA^\mu(x)\alpha^\mu_{-1} - \frac{1}{2}h^{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\nu_{-1} - iV^\mu(x)\alpha^\mu_{-2} + \ldots]|\Omega\rangle . \tag{5.32}$$

I have assigned arbitary names to the coefficient functions; the factors of i insure that all of these functions, and $\Phi[X]$ itself, are real-valued. Notice that the coefficient functions take the form of local fields of increasing spin. Each particle in the single-string spectrum is assigned its appropriate local field.

Now let $\Phi_{(1)}[X]$, $\Phi_{(2)}[X]$ be two new string fields, with coefficient fields $\phi_{(1)}(x),\ldots$ and $\phi_{(2)}(x),\ldots$, respectively. Using the results (5.29) and (5.30), we may compute

$$\begin{aligned} L_{-1}\Phi_{(1)}[x(\sigma)] &= [-i\partial^\mu\phi_{(1)}\cdot\alpha^\mu_{-1} - \frac{1}{2}[\partial^\mu A^\nu_{(1)} + \partial^\nu A^\mu_{(1)}]\alpha^\mu_{-1}\alpha^\nu_{-1} \\ &\qquad - iA^\mu_{(1)}\alpha^\mu_{-2} + \ldots]|\Omega\rangle \qquad (5.33) \\ L_{-2}\Phi_{(2)}[x(\sigma)] &= [-\frac{1}{2}(-\delta^{\mu\nu}\phi_{(2)})\alpha^\mu_{-1}\alpha^\nu_{-1} - i\partial^\mu\phi_{(2)}\alpha^\mu_{-2} + \ldots]|\Omega\rangle . \end{aligned}$$

Now compare the two lines of (5.33) to (5.32) term by term. The coefficients of string eigenstates in (5.33), viewed as local fields, are of exactly the right form to be the variations of the fields of (5.32) under (Abelian) local gauge transformations:

$$\begin{aligned} \delta A^\mu &= \partial^\mu\phi_{(1)} , \\ \delta h^{\mu\nu} &= [\partial^\mu A^\nu_{(1)} + \partial^\nu A^\mu_{(1)}] - \delta^{\mu\nu}\phi_{(2)} , \\ \delta V^\mu &= A^\mu_{(1)} + \partial^\mu\phi_{(1)} . \end{aligned} \tag{5.34}$$

It is natural to hope that the string theory is indeed a gauge theory, with precisely these transformations as its gauge invariances.

The higher terms in the formulae (5.33), together with terms from the action of higher L_{-n} operators on new string fields, produce possible gauge variations for the higher mass fields in the expansion (5.32). At first sight, this looks like a frightening expansion of the gauge symmetry group. However, this enormous expansion of the gauge group is clearly necessary: The higher mass levels of the string theory contain increasing numbers of field with spin. These high-spin fields must obey consistent field equations which allow them to propagate and interact without producing their unphysical polarization states. It is likely that this is only possible if each of these fields possesses its own local gauge invariance.

The enormous gauge symmetry suggested by this analysis can be written in a relatively compact form[26–28]

$$\delta\Phi[X(\sigma)] = L_{-n}\Phi_{(n)}[X(\sigma)] \tag{5.35}$$

by invoking gauge parameters $\Phi_{(n)}[X]$ which are functionals of $X^\mu(\sigma)$. In ordinary physics, we speak of *global* gauge symmetries, in which the gauge parameters are constant, and *local* gauge symmetries, in which the gauge parameters are local functions of X. Here we have the next step in this hierarchy, *chordal* gauge transformations, in which the gauge parameters are functionals on the space of strings.

5.4. A Gauge-Invariant Action

In the last section, we have made considerable progress toward a gauge-invariant formulation of the open string theory. To complete this analysis, at least at the level of free strings, we need only construct an action principle with the gauge invariance (5.35). It seems, however, that something is missing. The ghosts, which played so important a role in the formulation of the string spectrum, played no role at all in the considerations which we have just completed. Before we attempt to construct a gauge-invariant action, then, I would like to discuss the ghost generalization of the string field formalism presented above.

Let us, then, extend the string field Φ to a functional of $X^\mu(\sigma)$, $b(\sigma)$, and $c(\sigma)$. The Hilbert space of string eigenmodes will then be larger, and more terms will appear in a normal-mode expansion such as (5.32). Explicitly, we find, through the second excited level:

$$\begin{aligned}\Phi[X,b,c] = \Big[&\phi(x) - iA^\mu(x)\alpha^\mu_{-1} - i\mathrm{b}(x)b_{-1} - i\mathrm{c}(x)c_{-1} \\ &- \frac{1}{2}h^{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} - iV(x)\cdot\alpha_{-2} - i\tilde{\mathrm{b}}(x)b_{-2} - i\tilde{\mathrm{c}}(x)c_{-2} \\ &- \tilde{\mathrm{b}}^\mu(x)b_{-1}\alpha^\mu_{-1} - \tilde{\mathrm{c}}^\mu(x)c_{-1}\alpha^\mu_{-1} - s(x)b_{-1}c_{-1} + \ldots\Big]|\Omega\rangle\end{aligned} \tag{5.36}$$

The new local fields appear in quite a remarkable pattern. At the first excited level, corresponding to $m^2 = 0$ states, we find the familiar vector field $A^\mu(x)$, plus two new fields $b(x)$, $c(x)$. Since Φ is a real-valued functional, while b_{-1} and c_{-1} are Grassmann-valued, the coefficient fields $b(x)$ and $c(x)$ must be anticommuting fields. They are exactly the ghost and antighost required for the standard Fadde'ev-Popov quantization of A^μ! At the next level, we have the vector and scalar gauge parameters $A^\mu_{(1)}$ and $\phi_{(2)}$ of eq. (5.34), so we expect, and find, vector and scalar ghosts $\tilde{c}^\mu$, $\tilde{c}$ and their corresponding antighosts. This amazing phen enon was discovered by Siegel,[26] and it was this discovery that set in motion t unravelling of the string gauge invariances.

I would lik to use this extended string functional as my starting point in a search for a uge-invariant action. Let us abstract the expansion (5.36) to the form $\Phi[X, b, c] = \sum \phi_i(x) |\psi_i\rangle$. If we Fourier transform with respect to x, we can rewrite the tring field as a sum over coefficient functions for string states of definite momen m:

$$\Phi = \int [dp]\ \phi_i(p)\, |\psi_i(p)\rangle\ , \tag{5.37}$$

where $|\psi_i(p)\rangle$ is a string state built by applying mode creation operators to $|p\rangle$. It is natural to ok for a free-string action of the form

$$S = (\Phi\, K|\Phi) = \int d^{26}p\ \phi_i(-p) \left\langle \psi_i(-p) \middle| K \middle| \psi_j(p) \right\rangle \phi_j(p)\ , \tag{5.38}$$

where K is an erator which acts on the single-string Hilbert space. We must find a K whos action on string eigenstates can reproduce the space-time Lagrangians of th component fields.

Since $|\phi(p)\rangle$ ncludes the ghost vacuum contribution and, possibly also, ghost excitations, it i most naturally constructed as a state in conformal field theory. Let us, in fac reinterpret the matrix element which appears in (5.38) as a conformal fiel eory expectation value involving the vertex operators which create $|\psi_i\rangle$ and $|\psi_j\rangle$ and a third operator which represents K. We can write these elements ncretely for the case in which both external states are tachyons: The state $|\psi_j(p)\rangle = |p\rangle$ is represented by the vertex operator $c(0)e^{ip\cdot X(0)}$. The state $\langle\psi_j(p)| = \langle p|$ is represented by the adjoint of this operator, that is, its conformal trans orm under $z \to -1/z$. We might regularize this operation by setting the original vertex operator at $z = \epsilon$; then the adjoint operation, carried out according to the transformation law (2.23), yields

$$c(\epsilon)\, e^{ip\cdot X(\epsilon)} \ \to\ \left(\frac{1}{\epsilon^2}\right)^{p^2/2-1} \cdot c\left(-\frac{1}{\epsilon}\right)\ e^{ip\cdot X(-1/\epsilon)}\ . \tag{5.39}$$

Each of the two vertex operators has ghost number 1. In order to satisfy the ghost number conservation law of conformal field theory matrix elements, the operator we choose for K must also have ghost number 1. The "obvious" candidate is the BRST charge Q. Indeed, using (5.20) we may compute

$$\begin{aligned}\langle -p|\,Q\,|p\rangle &= \left\langle \left(\frac{1}{\epsilon^2}\right)^{\frac{p^2}{2}-1} c(-\frac{1}{\epsilon})e^{-ip\cdot X(-1/\epsilon)} \cdot Q \cdot c(0)e^{ip\cdot X(0)} \right\rangle \\ &= \left\langle \left(\frac{1}{\epsilon^2}\right)^{\frac{p^2}{2}-1} c(-\frac{1}{\epsilon})\, e^{-ip\cdot X(-1/\epsilon)} \cdot c_0\,(\frac{p^2}{2}-1)\; c_1\, e^{ip\cdot X(0)} \right\rangle \\ &= \left(\frac{1}{\epsilon^2}\right)^{\frac{p^2}{2}-1} \left\langle c_{-1}\left(\frac{1}{\epsilon}\right)^2 \cdot c_0 c_1 \right\rangle \exp[-p^2\log(\frac{1}{\epsilon})] \cdot (\frac{p^2}{2}-1)\ ,\end{aligned} \qquad (5.40)$$

the complicated dependence on ϵ cancels completely, and what remains is

$$\langle -p|\,Q\,|p\rangle = \frac{1}{2}(p^2 - 2)\ . \qquad (5.41)$$

Since the tachyon has mass $m^2 = -2$, this gives exactly the correct free Lagrangian for the tachyon field:

$$S = \int \phi(-p)\,\frac{1}{2}\,(p^2+m^2)\;\phi(p) + \ldots\ . \qquad (5.42)$$

We shall now check the form of the Lagrangian at higher mass levels. Let us concentrate for the moment on the terms involving string states with physical polarizations. These states are created by vertex operators which are primary conformal fields. For such states, eq. (5.20) implies

$$\langle \psi_i(-p)|\; Q\; |\psi_j(p)\rangle = \langle \psi_i(-p)|\; c_0\big(L_0^{(X)}-1\big)\; |\psi_j(p)\rangle\ . \qquad (5.43)$$

If the state $|\psi_j(p)\rangle$ contains n units of oscillator excitation, then, acting on this state,

$$L_0^{(X)} - 1 = \frac{p^2}{2} + n - 1 = \frac{1}{2}(p^2 + m^2)\ . \qquad (5.44)$$

The factor of c_0 in (5.43) is needed to make the diagonal matrix element in $|\Omega\rangle$ nonzero. We have thus shown that action (5.38) with the choice $K = Q$ gives the correct free-field action for all physical components of the various string fields.

This analysis makes it very plausible that the correct action for the open string free field theory is[29–31]

$$S = (\Phi|Q|\Phi) \ . \tag{5.45}$$

To complete a demonstration that this is indeed the correct action, it is only necessary to show that this expression has a group of gauge invariances sufficiently large that we can remove all unphysical field components. Actually, it is very easy to identify the gauge invariances; since $Q^2 = 0$, any transformation of the form

$$\delta|\Phi) = Q|E) \ , \tag{5.46}$$

where $|E)$ is a new string functional, leaves (5.45) invariant. A subset of these transformations is obtained by specializing

$$|E) = b_{-n}|\Phi_{(n)}) \ . \tag{5.47}$$

Acting on this with Q, we find $\delta|\Phi) = L_n|\Phi_{(n)})$; thus our proposal above for the gauge group of the free string theory is contained as a subgroup of the transformations represented in (5.46). Since all physical states have the ghost number of $|\Omega)$, and the gauge invariances (5.35) also correspond to definite ghost number, we can simplify the theory by restricting $|\Phi)$ to states of ghost number 1. Since Q raises the ghost number by 1 unit, $|E)$ will then be restricted to ghost number 0. For this theory of definite ghost number, a careful analysis[32] shows that the number of gauge invariances is precisely correct to allow all unphysical string states to be eliminated.

It should be noted, however, that the restriction to definite ghost number 1 allows certain states to remain in the classical string theory which cannot be written as excitations of the coordinate oscillators. The first of these states appears at the second mass level, it is visible in (5.36)as the component field $s(x)$ in the term

$$\Phi = [\cdots + s(x)b_{-1}c_{-1} + \cdots]\,|\Omega) \ . \tag{5.48}$$

This field can be gauged away, but it does participate in the gauge algebra of the $m^2 = 2$ states. We will see a more important example of such a ghostly classical field in the next section.

The covariant quantization of this system is not entirely straightforward. The gauge invariances we have defined have their own gauge-invariances: $\delta|E) = Q|G)$, where $|G)$ is a new functional of ghost number (-1), leaves the transformation law of $|\Phi)$ unchanged. This means that the Faddeev-Popov action for the ghosts will have a gauge invariance, requiring ghosts of ghosts. Following this logic, one finds ghosts of ghosts of ghosts, etc.; each successive ghost involves states of ghost number 1 unit lower. The corresponding antighosts cover the set of positive ghost numbers. The final result of this procedure is a covariantly quantized theory containing all possible ghost number sectors: This theory includes the full content of the functional $\Phi[x(\sigma), b(\sigma), c(\sigma)]$. The gauge-fixed string field theory is thus exactly the field theory of the gauge-fixed string!

It is possible to supplement the free field action I have described with a gauge-invariant interaction. Many authors have offered proposals for this interaction; two of the most completely realized are those of Witten[29,33,34] and Hata, Itoh, Kugo, Kunitomo, and Ogawa.[35] Unfortunately, I do not have space here to review this work, which still seems to require a good deal of further clarification.

6. Gravity from String Theory

Now that we have seen how the gauge invariances of the open string theory arise, let us explore how these ideas generalize to the closed string. Michael Green explained that the graviton arises as a particular state of the closed string; thus the formulation of a gauge-invariant closed string action should lead us directly to general coordinate invariance and Einstein's equations. Unfortunately, many of the connections along this chain are not yet well understood. In this section, I will give three sets of arguments which give partial information about the nature of gravity in string theory; I leave it to you to formulate a more unified understanding of the connection between strings and gravity.

6.1. Scattering of Gravitons

First, let us generalize the formalism for string scattering amplitudes presented in the previous section to the closed string. To do this, we need only bring back into our discussion the anti-analytic section of the Hilbert space and construct vertex operators which create both analytic and antianalytic modes excitations.

Let us denote the ladder operators of the anti-analytic sector (corresponding to right-moving excitation on the closed string) by $\overline{\alpha}_n^\mu$; these obey the same algebra as their counterparts in the analytic sector. Denote the new ghost ladder operators by $\overline{b}_n$ and $\overline{c}_m$. The vacuum of the theory is given by

$$|\Omega\rangle = c_1 \overline{c}_1 |0\rangle . \tag{6.1}$$

This state is not BRST-invariant, but the states of nonzero momentum

$$|p\rangle = e^{ip\cdot X(0)} c_1 \overline{c}_1 |0\rangle \tag{6.2}$$

are annihilated by Q and $\overline{Q}$ if $p^2 = 2$. Michael Green identified the graviton with the massless tensor state $\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|p\rangle$. This state is created by the vertex operator

$$c(z)\overline{c}(\overline{z}) \cdot \eta^{\mu\nu}\,\partial_z X^{\mu}\partial_{\overline{z}}X^{\nu}e^{ip\cdot X(z,\overline{z})} \tag{6.3}$$

(or simply by the coordinate part of this operator, integrated over d^2z). The operator (6.3) is annihilated by both Q and $\overline{Q}$ if $p^2 = 0$ and if $p^{\mu}\eta^{\mu\nu} = \eta^{\mu\nu}p^{\nu} = 0$.

Notice that the graviton state satisfies the condition $L_0 = \overline{L}_0$; it has equal amounts of excitation in the left- and right-moving sectors. :hael Green argued that this condition must be satisfied for all physical closed-string states. Let me explain how this condition arises from the conformal field theory viewpoint. If I ignore the problem of fixing the $SL(2,C)$ invariance (which contributes, in any event, only an infinite redundancy), a scattering amplitude is computed by integrating vertex operators over all points of the z plane.

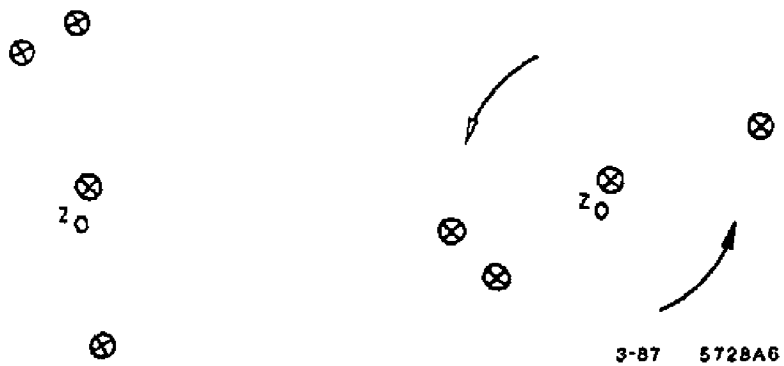

Figure 6 Typical configurations of vertex operators which contribute to a scattering amplitude for closed strings.

The contribution in which a particular vertex operator is located at z_0 involves an integral over the locations of all other vertex operators; in particular, it includes an integral over rotations of the other vertex operators about the point z_0, as shown in Fig. 6. Translate z_0 to 0. As we have seen from the discussion of $SL(2,C)$ in section 2, the two operators L_0 and $\overline{L}_0$ generate linear combinations of dilatations and rotations. Explicitly,

$$\begin{aligned} &(L_0 + \overline{L}_0) \quad \text{generates} \quad z \to z + \delta\lambda z\,, \quad \overline{z} \to \overline{z} + \delta\lambda \overline{z}\,, \\ &(L_0 - \overline{L}_0) \quad \text{generates} \quad z \to z + i\delta\alpha z\,, \quad \overline{z} \to \overline{z} - i\delta\alpha \overline{z}\,. \end{aligned} \tag{6.4}$$

Since we integrate over all possible rotated configurations of sources, only states created at z_0 which are rotationally invariant will give a nonzero contribution to the amplitude. These are the states annihilated by $(L_0 - \overline{L}_0)$. The other possible states of the closed string theory simply disappear. This phenomenon, that states which are not invariant to geometrical transformations on the world sheet disappear from all transition amplitudes, is a novel and extremely important aspect of string theory. It will play a major role in our discussion in Sections 8.

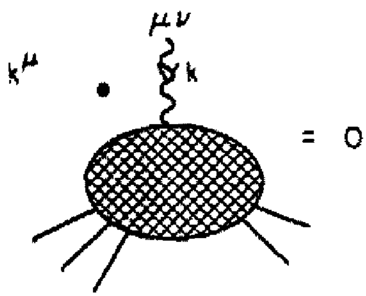

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Figure 7. The graviton Ward identity for on-shell external states.

The vertex operator formalism allows us to check the on-shell graviton Ward identity shown in Fig. 7: Contracting the graviton vertex with p^μ should cause the graviton scattering amplitude to vanish. Indeed, we can see directly

$$p^\mu \int d^2z\, \partial_z X^\mu \partial_{\overline{z}} X^\nu e^{ip\cdot X} = -i \int d^2z\, \partial_z \left[\partial_{\overline{z}} X^\nu e^{ip\cdot X}\right], \tag{6.5}$$

which integrates to 0. In principle, one might worry that nonzero contact contributions might appear due to singularities as z approaches the location z_i of some other vertex operator. However, all such contact terms are multiplied by factors $|z - z_i|^{2p \cdot p_i}$, evaluated as $z \to z_i$. This factor is properly defined by analytic continuation from the region $p \cdot p_i > 0$; hence, the contact terms are zero. This subtlety aside, the Ward identity for gravitons arises in a very natural way. This is easy to understand, because the graviton couples to the local energy-momentum density of the world sheet.

6.2. Gauge Invariances of the Closed String Theory

Let us turn now to the generalization of the construction of gauge invariances and the gauge invariant action to the closed string. My discussion of these issues will be somewhat more pedestrian than that given in the previous section; hopefully, it will clarify the physical content of the construction presented there.

Begin with a closed string field, a functional $\Phi[X^\mu(\sigma)]$ of the location of a closed string in space-time. The mode expansion of this field, restricted to components satisfying $L_0 = \overline{L}_0$, is

$$\Phi[X(\sigma)] = \left[\phi(x) + t^{\mu\nu}(x)\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1} + \ldots\right]|\Omega\rangle . \tag{6.6}$$

We have been concentrating on the graviton, a symmetric tensor, but actually the field $t^{\mu\nu}$ is a tensor of arbitrary symmetry. It is useful to decompose it into its symmetric and antisymmetric parts:

$$t^{\mu\nu}(x) = h^{\mu\nu}(x) + b^{\mu\nu}(x) . \tag{6.7}$$

The generalization of the gauge transformation law (5.35) to this system reads:

$$\delta\Phi = L_{-n}\Phi_{(n)} + \overline{L}_{-n}\Phi_{(\overline{n})} . \tag{6.8}$$

If we define the expansion coefficients

$$\Phi_{(1)} = \left[\cdots + i\xi^\nu(x)\overline{\alpha}^\nu_{-1} + \cdots\right]|\Omega\rangle , \qquad \Phi_{(\overline{1})} = \left[\cdots - i\overline{\xi}^\nu(x)\alpha^\nu_{-1}\cdots\right]|\Omega\rangle , \tag{6.9}$$

then (6.8) implies the gauge transformation laws

$$\begin{aligned} \delta h^{\mu\nu} &= \frac{1}{2}(\partial^\mu\xi^\nu + \partial^\nu\overline{\xi}^\mu) + (\mu \leftrightarrow \nu) , \\ \delta b^{\mu\nu} &= \frac{1}{2}(\partial^\mu\xi^\nu - \partial^\nu\overline{\xi}^\mu) - (\mu \leftrightarrow \nu) . \end{aligned} \tag{6.10}$$

If $h^{\mu\nu}$ is interpreted as the linearized gravitational field, the first line of (6.10) is just a linearized local coordinate transformation.

It is not difficult to write an action for $h^{\mu\nu}$ and $b^{\mu\nu}$ which is invariant to (6.10) even if one works in the restricted Hilbert space without ghost excitations. Note that, since $[L_1, L_{-1}] = 2L_0$, the operator $P = (1 - L_{-1}(2L_0)^{-1}L_1)$ satisfies

$$P\,L_{-1}\Phi_{(1)} \;=\; \left[L_{-1} - L_{-1} - L_{-1}^2\,\frac{1}{2(L_0+1)}L_1\right]\,\Phi_{(1)} \;\sim\; L_{-1}^2 L_1\Phi_{(1)}\;, \quad (6.11)$$

which has overlap only with states with two units of excitation in the analytic sector. This object thus acts on $t^{\mu\nu}\alpha_{-1}^{m}\overline{\alpha}_{-1}^{\nu}\,|\Omega\rangle$ as a projector onto gauge-invariant components. This suggests that we write the action for Φ in the form $S = (\Phi|K|\Phi)$,

$$K \;=\; \left[(L_0 - 1) + (\overline{L} - 1)\right]\,P\overline{P}\;+\;\ldots\;, \quad (6.12)$$

and the additional terms annihilate the states of the graviton mass level. Using this ansatz, we can evaluate the matrix element of K explicitly and find

$$S \;=\; \cdots + \int dx\; t^{\mu\nu}\left[-\partial^2\left[\delta^{\mu\lambda} - \frac{\partial^\mu\partial^\lambda}{\partial^2}\right]\left[\delta^{\nu\sigma} - \frac{\partial^\nu\partial^\sigma}{\partial^2}\right]\right] t^{\lambda\sigma}\;+\;\cdots\;; \quad (6.13)$$

I have used the fact that $(L_0 - 1) = \frac{1}{2}p^2 = -\frac{1}{2}\partial^2$ on this mass level.

The piece of the action (6.13) which is antisymmetric in the indices of t is

$$\begin{aligned} S \;&=\; \int dx\; b^{\mu\nu}\,(-\partial^2\delta^{\mu\lambda}\delta^{\nu\sigma} \;+\; \partial^\mu\partial^\lambda\delta^{\nu\sigma} \;+\; \partial^\nu\partial^\sigma\delta^{\mu\lambda})\;b^{\lambda\sigma} \\ &=\; \frac{1}{3}\int dx\;[\partial^\rho b^{\lambda\sigma} \;-\; \partial^\lambda b^{\rho\sigma} \;-\; \partial^\sigma b^{\lambda\rho}]^2\,. \end{aligned} \quad (6.14)$$

In this expression, $b^{\mu\nu}$ appears as an antisymmetric tensor gauge field with field strength $H^{\mu\nu\lambda} = \partial^{[\mu}b^{\nu\lambda]}$; the action takes the gauge invariant form $S \sim \int H^2$.

The part of (6.13) which is symmetric in the tensor indices is more problemmatical. This piece can be rearranged, by adding and subtracting a convenient term, to form

$$S = \frac{1}{2}\int dx\, h^{\mu\nu}\left\{ -\partial^2\left(\delta^{\mu\lambda} - \frac{\partial^\mu\partial^\lambda}{\partial^2}\right)\left(\delta^{\nu\sigma} - \frac{\partial^\nu\partial^\sigma}{\partial^2}\right) + (\lambda \leftrightarrow \sigma)\right.$$

$$+ 2\partial^2\left(\delta^{\mu\nu} - \frac{\partial^\lambda\partial^\sigma}{\partial^2}\right)\left(\delta^{\lambda\sigma} - \frac{\partial^\lambda\partial^\sigma}{\partial^2}\right) \tag{6.15}$$

$$\left. - 2\partial^2\left(\delta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right)\left(\delta^{\lambda\sigma} - \frac{\partial^\lambda\partial^\sigma}{\partial^2}\right)\right\} h^{\lambda\sigma} .$$

The first two lines of this expression are the terms quadratic in $h^{\mu\nu}$ which appear when one expands the Einstein action

$$\int dx\, \sqrt{G}\, R \tag{6.16}$$

according to $G_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. In this same expansion, the curvature scalar is given by

$$R = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h^\mu{}_\mu , \tag{6.17}$$

so the last term is the linearization of $\int dx(-2)\, R\, (-\partial^2)^{-1}\, R$. This interaction is nonlocal in space-time. To render it local, we must add to the theory a new scalar field $\phi(x)$ which couples to curvature. The action

$$S = \int \phi\partial^2\phi - 2\phi R \tag{6.18}$$

is a local expression which reproduces the last term of (6.15) when ϕ is integrated out. This new field $\phi(x)$ is called the *dilaton* field.

If we had begun in a formalism which included the ghost states of the closed string Hilbert space and which insures gauge invariance from the beginning by virtue of the identities $Q^2 = \overline{Q}^2 = 0$, we would have found the action appearing directly in the form (6.18), with the extra field ϕ identified as the coefficient of a ghostly state in the decomposition of the string field:

$$\Phi = \left[\cdots - \phi(x)\left(b_{-1}\bar{c}_{-1} + c_{-1}\bar{b}_{-1}\right) + \cdots\right]|\Omega\rangle . \tag{6.19}$$

Note that the kinetic-energy term for ϕ in (6.18) has the wrong sign; this is a signal of the field's ghostly origin. In general, the system of gravitational and dilaton fields contains one propagating scalar particle, which is a linear combination of ϕ and $h^\mu{}_\mu$.

6.3. Conformal Consistency Conditions

In the discussion just completed, we took a straightforward approach to the derivation of Einstein's equations, constructing the gauge-invariant action for the string gravitational field. We did in fact succeed in obtaining the correct equations, though only at the linearized level. Now I would like to present a second, more indirect approach to the derivation of the field equations, by considering the dynamics of strings moving in macroscopic background gravitational fields.

At the beginning of this section, I discussed the coupling of single gravitons and antisymmetric tensor particles to closed strings. This coupling was described by the insertion of vertex operators

$$\int DX\, e^{-\int \mathcal{L}} \left[\int d^2z\; (h^{\mu\nu}(p) + b^{\mu\nu}(p))\; \partial_z X^\mu \partial_{\bar z} X^\nu e^{ip\cdot X} \right] . \tag{6.20}$$

Since the insertion is of the same basic structure as the world sheet action

$$\mathcal{L} \;=\; \frac{1}{2\pi} \int d^2z\, \delta_{\mu\nu}\, \partial_z X^\mu \partial_{\bar z} X^\nu \;, \tag{6.21}$$

it is natural to view (6.20) as the first term in the linearization of the geometrically invariant expression

$$\int DX\; \exp\left[-\frac{1}{2\pi} \int d^2z\;\; \{ G_{\mu\nu}(X) \partial_z X^\mu \partial_{\bar z} X^\nu \;+\; B_{\mu\nu}(X) \partial_z X^\mu \partial_{\bar z} X^\nu \} \right] \tag{6.22}$$

about the flat background metric which appears in (6.21). The coefficients $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ in (6.22) are functions of the string coordinate $X^\mu(z)$; thus, the world-sheet dynamics described by (6.22) is nonlinear. In fact, (6.22) is precisely the action of a 2-dimensional nonlinear sigma model, in which the *target space* on which the nonlinear sigma model variables live has been identified with space-time. The $B_{\mu\nu}$ term, which is antisymmetric in world-sheet indices, may be interpreted as a Wess-Zumino term.[36]

At first sight, there seems to be no difficulty in quantizing the string in any general background geometry. However, we must recall that the quantization of the string depends crucially on conformal invariance, or, equivalently, on the existence of a BRST charge satisfying $Q^2 = 0$. We will see in a moment that this implies very stringent restrictions on the background geometry. These restrictions were first derived by Lovelace,[37] Fradkin and Tseytlin,[38] Callan, Friedan, Martinec, and Perry,[39] and Sen[40] by studying the β functions of the nonlinear sigma model. In my discussion, I will follow Banks, Nemeschansky, and Sen[41] (BNS) in approaching the problem from the viewpoint of the ghost dynamics and the BRST charge.

Before beginning this analysis, we must identify one piece which is missing from (6.22). This expression couples the string to background $G_{\mu\nu}$ and $B_{\mu\nu}$ fields but does not yet include the dilaton. Since this field has, after all, a ghostly origin, BNS propose to include it as a nonlinear coupling to the world-sheet ghosts

$$S = \frac{1}{2\pi}\int d^2z\,\Big\{G_{\mu\nu}(X)\partial_z X^\mu \partial_{\bar z} X^\nu + B_{\mu\nu}(X)\partial_{\bar z} X^\mu \partial_{\bar z} X^\nu \\ + \Big[b_{zz}\partial_{\bar z} c^z + \frac{4}{3}\, b_{zz} c^z \partial_{\bar z}\phi(X)\Big] + h.c.\Big\}\,. \tag{6.23}$$

This action is *classically* BRST invariant, with

$$Q = \oint \frac{dz}{2\pi i}\, e^{(4/3)\phi(X)}\, c(z)\big[-\frac{1}{2}G_{\mu\nu}\partial_z X^\mu \partial_{\bar z} X^\nu - b\partial_z c\big]\,. \tag{6.24}$$

One can remove the factor $e^{4/3\,\phi(X)}$ by the transformation

$$c \to e^{-4/3\,\phi}c\,, \qquad b \to e^{+4/3\,\phi}b\,. \tag{6.25}$$

Unfortunately, this transformation is generated by the ghost number current $j^z = bc$, which we already know may possess anomalies. In fact, the derivative of $\phi(X)$ couples to this chiral current like an external gauge field. We therefore expect to find an anomaly in the transformation (6.25); this generates a new term in Q proportional to $\partial_z^2\phi(X)$. The transformation (6.25) then leaves us with the BRST charge

$$Q = \oint \frac{dz}{2\pi i}\, c(z)\,\big[\tilde{T}^{(X)} + \frac{1}{2}T^{(b,c)}\big]\,, \tag{6.26}$$

where

$$\tilde{T}^{(X)} = -\frac{1}{2}G_{\mu\nu}(X)\,\partial_z X^\mu \partial_{\bar z} X^\nu + \frac{1}{2}\partial_z^2\phi(X)\,. \tag{6.27}$$

From the discussion of the BRST charge in Section 4, we know that (6.26) will satisfy $Q^2 = 0$ if the operator product of $\tilde{T}^{(X)}$ with itself takes the standard form

$$\tilde{T}^{(X)}(w)\,\tilde{T}^{(X)}(z) \;\sim\; \frac{c/2}{(z-w)^4} \;+\; \frac{2}{(w-z)^2}\,\tilde{T}^{(X)}(z) \;+\; \frac{1}{w-z}\,\partial_z \tilde{T}^{(X)} \;+\; \dots\,, \tag{6.28}$$

with $c = 26$. This result, however, is no longer so simple to obtain, because the operator product must be computed in a field theory with complicated nonlinear interactions. BNS expand the nonlinear sigma model about a flat target space metric by the standard technique of parametrizing the space in terms of Riemann normal coordinates:[42]

$$G_{\mu\nu}(X) \;=\; G_{\mu\nu}(x_0) \;-\; \frac{1}{3}\,R_{\mu\alpha\nu\beta}\,\delta X^\alpha(z)\delta X^\beta(z) \;+\; \dots \tag{6.29}$$

Integrating over the field fluctations $\delta X^a(z)$ defines a perturbation theory in powers of the background curvature. Computing to 1-loop order in this expansion, they find

$$\begin{aligned}\tilde{T}^{(X)}(w)\;\tilde{T}^{(X)}(z) \;=\; (6.28) \;+\; &\frac{\overline{w}-\overline{z}}{(w-z)^3}\cdot\frac{1}{2\pi}\,\partial_z X^\mu \partial_{\bar z} X^\nu \\ &\cdot\left\{\left[R_{\mu\nu} \;-\; \frac{1}{4}H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} \;+\; 2\nabla_\mu\nabla_\nu\phi\right]\right. \\ &\left. +\;\left[\nabla_\lambda H_{\mu\nu}{}^\lambda \;-\; 2(\nabla_\lambda\phi)H_{\mu\nu}{}^\lambda\right]\right\}.\end{aligned} \tag{6.30}$$

The new singular structure appears because the interactions couple the analytic and anti-analytic sectors. (The brackets indicate the symmetric and antisymmetric parts of this tensor.) In addition, one finds a shift of the central charge

$$c \;=\; \left[D \;+\; 6\alpha'\left\{-\nabla^2\phi \;+\; (\nabla_\mu\phi)^2 \;-\; \frac{1}{4}R \;+\; \frac{1}{48}(H_{\mu\nu\lambda})^2\right\}\right]\,, \tag{6.31}$$

where the string slope parameter α' absorbs the dimensions. To insure $Q^2 = 0$, the two terms in brackets in (6.30) must be set to zero, and c must be kept equal to 26.

The second of the two conditions generated by (6.30) is the equation

$$\nabla_\lambda H_{\mu\nu}{}^\lambda = 2(\nabla_\lambda \phi) H_{\mu\nu}{}^\lambda , \tag{6.32}$$

which is a Maxwell equation for the $B^{\mu\nu}$ field. The first condition can be rearranged into an equation of the form

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = \cdots ; \tag{6.33}$$

this is Einstein's equation for the background gravitational field. These two conditions are equivalent to the vanishing of the β functions for the nonlinear sigma model. The Bianchi identities can be used to show that, if these two conditions are satisfied, the quantity displayed in (6.31) satisfies $\nabla_\mu c = 0$; thus all that remains is to adjust the overall constant value of this quantity. The three conditions all follow consistently from the following variational principle:

$$\delta \int dx \sqrt{G}\, e^{-2\phi} \left\{ R + (\nabla_\mu \phi)^2 - \frac{1}{12} H^2 \right\} = 0 . \tag{6.34}$$

It is worth comparing this action principle with the results of our explicit anaysis of the gauge-invariant string field action. The free-field Lagrangians (6.14), (6.16), and (6.18), taken together, give exactly the linearization of (6.34). Presumably, this is no accident; the equations (6.30) and (6.31) are consistency conditions that should be automatically satisfied if the theory is properly formulated. However, the generalization of our earlier analysis to the full nonlinear theory has not yet been done, and the precise relation between these two approaches is not at all understood. I can only recommend this as a problem for your attention.

How do we solve the conformal consistency conditions? I will discuss only the simplest solutions here. The equations we have derived correspond to the results of leading-order perturbation theory in powers of the curvature; in higher orders, they receive corrections proportional to higher powers of curvature and field strength. To this order, however, they are solved by setting $\phi(x)$ to a constant value, $H = 0$, and $R_{\mu\nu} = 0$. This last condition does not necessarily imply $R_{\mu\nu\lambda\sigma} = 0$; among the additional allowed configurations are the Calabi-Yau manifolds which we will discuss briefly in Section 9. Several authors [43,44] have proposed solutions corresponding to group manifolds on which both of the symmetric tensors appearing in (6.30)and (6.31)—$R_{\mu\nu}$ and $H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma}$—take values proportional to $G_{\mu\nu}$. When the condition (6.30) is satisfied, $H_{\mu\lambda\sigma}$ acts as a parallelizing torsion. From the nonlinear sigma model viewpoint, this solution is just the fixed point found by Polyakov and Wiegman[45] and Witten[46] by the addition to the nonlinear sigma model of a Wess-Zumino term. However, the condition (6.31) cannot be satisfied at the same time unless one shifts the value of D away from its free-field value of 26. This is perfectly acceptable for the bosonic string, but in the case of the fermionic string such a shift may do violence to space-time supersymmetry. We will take up this issue, and other issues related to the compactification of space-time dimensions, again in Section 9.

7. The Covariant Superstring

In order to introduce fermions into string theory, and to formulate consistent and possibly realistic string models, one must generalize the simple bosonic string to a system with supersymmetric world-sheet dynamics. Michael Green has described to you the general outline of this method, and his own elegant light-cone formulation of the supersymmetric string. In this lecture, I will develop this theory once again from a viewpoint which allows Lorentz-covariant calculations of string scattering amplitudes. As in earlier sections, my primary tool will be the use of conformal field theory. We will see that the calculational methods of conformal field theory work together naturally with the constraints of BRST invariance to clarify the structure of this extension of the formalism of strings.

7.1. Superconformal Field Theory

As a first step in developing the theory of the supersymmetric string, I would like to introduce the supersymmetric extension of the formulation of conformally-invariant field theory given in Section 2.[47,48] This extension is surprisingly straightforward; all of the technical apparatus we require is already in place.

In principle, I should begin from a locally supersymmetric 2 dimensional action, coupled to the supermultiplets (X^μ, Ψ^μ) which we described at the end of Section 3.* However, following the logic of Section 2, I will assume directly that we have chosen the metric to be of the form (2.5), and, further, that the corresponding gravitino field vanishes. In this flat background, we can set up a superspace with bosonic coordinates z, $\bar{z}$. For $N = 1$ (or (1,1)) supersymmetry in 2 dimensions, the supersymmetry generators form a 2-component spinor which we can represent in the basis of eq. (3.24). The superspace thus should have two fermionic coordinates; we may represent these as Grassmann variables $\theta, \bar{\theta}$ which

* A clear derivation of the string dynamics from this starting point can be found ref. 49.

transform as analytic and anti-analytic objects of conformal dimension $-\frac{1}{2}$. A scalar superfield on this space has the general structure

$$X(z,\bar{z},\theta,\bar{\theta}) = X + \theta\psi + \overline{\theta\psi} + \theta\bar{\theta}F . \tag{7.1}$$

We can see that this field indeed contains the analytic supermultiplet $(X(z),\psi(z))$ and its anti-analytic counterpart, as well as an auxiliary field F. For the rest of my discussion, I will attach a Lorentz index to X and treat it as the string coordinate superfield.

The two supersymmetry generators can be represented as derivatives with respect to the anticommuting coordinates:

$$\mathrm{Q} = \partial_\theta - \theta\partial_z , \qquad \overline{\mathrm{Q}} = \partial_{\bar\theta} - \bar\theta\partial_{\bar z} , \tag{7.2}$$

corresponding to the supersymmetry algebra

$$\{\mathrm{Q},\mathrm{Q}\} = -2\partial_z , \qquad \{\overline{\mathrm{Q}},\overline{\mathrm{Q}}\} = -2\partial_{\bar z} . \tag{7.3}$$

Covariant derivatives which anticommute with Q and $\overline{\mathrm{Q}}$ are given by

$$D = \partial_\theta + \theta\partial_z , \qquad \overline{D} = \partial_{\bar\theta} + \bar\theta\partial_{\bar z} \tag{7.4}$$

A natural guess for the free-field Lagrangian of X^μ is: $\mathcal{L} = \overline{D}X^\mu DX^\mu$. Indeed, if we compute the derivatives of X^μ explicitly

$$\begin{aligned} DX^\mu &= \psi^\mu + \theta\partial_z X^\mu - \bar\theta\theta\partial_z\overline{\psi}^\mu + \bar\theta F^\mu \\ \overline{D}X^\mu &= \overline{\psi}^\mu + \bar\theta\partial_{\bar z}X^\mu + \bar\theta\theta\partial_{\bar z}\psi^\mu - \theta F^\mu \end{aligned} \tag{7.5}$$

and define $\int d^2\theta\,\bar\theta\theta = 1$, we can assemble

$$\begin{aligned} &\frac{1}{2\pi}\int d^2z\,d^2\theta\,\overline{D}X^\mu DX^\mu \\ &\qquad = \frac{1}{2\pi}\int d^2z\left\{\partial_{\bar z}X^\mu\partial_z X^\mu - \psi^\mu\partial_{\bar z}\psi^\mu - \overline{\psi}^\mu\partial_z\overline{\psi}^\mu + F^2\right\} . \end{aligned} \tag{7.6}$$

This is indeed the supersymmetric action of massless free fields. The equations of motion which follow from this action imply that ψ^μ is an analytic field while

$\overline{\psi}^\mu$ is anti-analytic and $F^\mu = 0$. This allows us to simplify (7.5) to

$$DX^\mu = \psi^\mu + \theta\partial_z X^\mu , \qquad \overline{D}X^\mu = \overline{\psi}^\mu + \overline{\theta}\partial_{\bar{z}} X^\mu . \tag{7.7}$$

Let us now explore the symmmetries of the analytic sector. The coordinate differences

$$z_{12} = z_1 - z_2 - \theta_1\theta_2 \qquad \theta_{12} = \theta_1 - \theta_2 \tag{7.8}$$

are supersymmetric, in the sense that they are annihilated by $(\mathbf{Q}_1 - \mathbf{Q}_2)$. It is useful to note that

$$D_1 z_{12} = D_2 z_{12} = \theta_{12} , \qquad \text{and} \qquad D_1^2 = \partial_{z_1} , \quad D_2^2 = \partial_{z_2} . \tag{7.9}$$

The invariance of z_{12} suggests that the propagators for the component fields of $X^\mu(z,\theta)$ can be written in a unified way as a superspace propagator

$$\langle X^\mu(z_1,\theta_1)X^\nu(z_2,\theta_2)\rangle = -\delta^{\mu\nu}\log(z_{12}) . \tag{7.10}$$

It is easy to check this by Taylor-expanding the left- and right-hand sides of (7.10) in powers of θ_1, θ_2. The two nonvanishing terms do indeed give the correct component-field propagators:

$$\begin{aligned} \langle X^\mu(z_1)X^\nu(z_2)\rangle &= -\delta^{\mu\nu}\log(z_1 - z_2) \\ -\theta_1\theta_2 \left[\langle\psi^\mu(z_1)\psi^\nu(z_2)\rangle \right. &= \left. -\delta^{\mu\nu}\frac{1}{z_1 - z_2} \right] . \end{aligned} \tag{7.11}$$

The energy-momentum tensor of the (X^μ, ψ^μ) supermultiplet can also be written in superfield form. A natural expression is

$$T = -\frac{1}{2}DX^\mu D^2 X^\mu . \tag{7.12}$$

This object is actually fermionic in character, and of dimension $\frac{3}{2}$. Its components in a expansion in θ have the form

$$T = T_F + \theta T_B . \tag{7.13}$$

T_B is a bosonic tensor of dimension 2; this should be identified with the energy-momentum tensor of the component description. Indeed, for the choice (7.12),

$$T_B = -\frac{1}{2}(\partial_z X^\mu)^2 + \frac{1}{2}\psi^\mu \partial_z \psi^\mu ; \tag{7.14}$$

this is exactly the energy-momentum tensor of the component fields X^μ, ψ^μ. The fermionic component T_F has the form

$$T_F = -\frac{1}{2}\psi^\mu \partial_z X^\mu ; \tag{7.15}$$

this is the generator of local supersymmetry transformations.

Apparently, the local conformal and supersymmetry motions come together into a unified algebra. Using our functional representation of commutators, we can work out the algebra if we know the operator products of the components of $T(z,\theta)$ with one another. Since T_B is the conformal generator introduced in Section 2, and T_F is a conformal tensor of dimension $\frac{3}{2}$, we can immediately write two of these relations:

$$\begin{aligned} T_B(w)\, T_B(z) &\sim \frac{3\hat{c}/4}{(w-z)^4} + \frac{2}{(w-z)^2} T_B(z) + \frac{1}{(w-z)} \partial_z T_B , \\ T_B(w)\, T_F(z) &\sim \frac{3/2}{(w-z)^2} T_F(z) + \frac{1}{(w-z)} \partial_z T_B . \end{aligned} \tag{7.16}$$

In the first line, I have defined $\hat{c} = \frac{2}{3}c$, so that a scalar superfield $X(z,\theta)$ will have $c = 1$. The two lines of eq. (7.16), may be recognized as components of the superfield relation

$$\begin{aligned} T(z_1,\theta_1)\, T(z_2,\theta_2) \sim \frac{\hat{c}/4}{z_{12}^3} &+ \frac{3}{2}\frac{\theta_{12}}{z_{12}} T(z_2,\theta_2) \\ &+ \frac{1}{2}\frac{1}{z_{12}} DT(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}} \partial_z T(z_2,\theta_2) . \end{aligned} \tag{7.17}$$

The leading component of eq. (7.17) gives the last of the three operator product expressions:

$$T_F(w)\,T_F(z) \sim \frac{\hat{c}/4}{(w-z)^3} + \frac{1}{2}\,\frac{1}{(w-z)}\,T_B(z)\,. \tag{7.18}$$

If we now define conformal and superconformal generators by

$$\begin{aligned} L_n &= \oint \frac{dz}{2\pi i}\, z^{n+1} T_B(z)\,, \\ G_k &= 2 \oint \frac{dz}{2\pi i}\, z^{k+\frac{1}{2}} T_F(z)\,, \end{aligned} \tag{7.19}$$

we can apply the contour integral methods of Section 2 to compute their commutation relations. The result is

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}\, n(n^2-1)\delta(n+m) \\ [L_n, G_k] &= \left(\frac{n}{2} - k\right) G_{n+k} \\ \{G_k, G_p\} &= 2L_{k+p} + \frac{\hat{c}}{2}\,(k^2 - \frac{1}{4})\delta(k+p)\,. \end{aligned} \tag{7.20}$$

This is the *superconformal*, or Neveu-Schwarz-Ramond, algebra. It is a graded extension of the Virasoro algebra incorporating local supersymmetry. This algebra will be the world-sheet gauge symmetry algebra of the supersymmetric string.

7.2. Vertex Operators for the Neveu-Schwarz Sector

The Hilbert space of states defined by the superconformal field theory of $X^\mu(z,\theta)$ should contain just the spectrum of states found in the open superstring theory, in its covariant (Neveu-Schwarz-Ramond) formulation. Let us work out that spectrum, and see what correspondence appears.

It is most straightforward to break up the superfield X^μ into its component fields $X^\mu(z)$, $\psi^\mu(z)$. $X^\mu(z)$ has exactly the action that I have already described in Section 3: Its Fourier components α^μ_{-n} act as ladder operators to create the excited oscillator modes of the string. The new information comes from $\psi^\mu(z)$.

Since this field is a conformal tensor of dimension $\frac{1}{2}$, its Fourier expansion should be written

$$\psi^\mu(z) = \sum_{k=-\infty}^{\infty} \psi_k^\mu z^{-k-\frac{1}{2}} , \qquad \psi_k^\mu = \oint \frac{dz}{2\pi i} z^{k-\frac{1}{2}} \psi^\mu(z) . \tag{7.21}$$

As we have seen in (2.44), this convention for the Fourier components implies that ψ_k^μ lowers L_0 by k units. In our previous examples, k took integeral values; however, we see from (7.21) that $\psi^\mu(z)$ can be a single-valued function on the complex plane only if, in this equation, k takes half-integer values. Since, (4.22) implies that

$$\psi_k^\mu |0\rangle = 0 \qquad \text{for} \qquad k \geq \frac{1}{2} , \tag{7.22}$$

the components with $k > 0$ act as fermion annihilation operators, and the components with $k < 0$ act as creation operators. The full Hilbert space is the space of states created from $|0\rangle$ by the action of α_{-n}^μ, with n an integer, and ψ_{-k}^μ, with k a half-integer. Comparing this result to Michael Green's development of the covariant superstring, we see that this reproduces exactly the Neveu-Schwarz sector of the theory.

At first sight, this result seems paradoxical: The Neveu-Schwarz sector of the string was defined by anti-periodic boundary conditions, while in the above discussion, I have insisted on the regularity of $\psi^\mu(z)$ and therefore on the periodicity of ψ^μ around equal-time circles. This paradox dissolves when we look back to the transformation (2.3) which gives the relation between the original string variables τ, σ and z. If we write $w = \tau + i\sigma$ and carry out this conformal mapping of a tensor of dimension $\frac{1}{2}$, we find

$$\psi(w) \rightarrow \left(\frac{dz}{dw}\right)^{\frac{1}{2}} \psi(z(w)) = e^{w/2}\psi(z(w)) . \tag{7.23}$$

Going once around the string sends $\sigma \rightarrow \sigma + 2\pi$, or $w \rightarrow w + 2\pi i$. This produces a factor $e^{i\pi} = (-1)$ on the right-hand side of (7.23). Hence, in order for ψ^μ to be single-valued on the z plane, it must have been anti-periodic ($\psi^\mu(\sigma + 2\pi) = -\psi^\mu(\sigma)$) on the original string. The result of this argument is that the states of the Neveu-Schwarz sector are very simply described on the z plane. The states of the Ramond sector require a more sophisticated construction, which will be presented later in this section.

It is not difficult to construct the vertex operators which create the asymptotic states of the Neveu-Schwarz sector. It is important to note, however, that because the reparametrization gauge group is larger for the superstring, the constraints of BRST invariance are stronger. To determine the new conditions, let us work out the BRST charge. This appears naturally in the superspace formulation just presented. If we group the reparametrization and superconformal ghosts into supermultiplets

$$C = c + \theta\gamma\,, \qquad B = \beta + \theta b\,, \tag{7.24}$$

we can obtain the actions of the (b,c) system and of the (β,γ) system from the expression

$$S = \frac{1}{2\pi}\int d^2z \int d^2\theta\; B\bar{D}C\,. \tag{7.25}$$

Using as ingredients the superfields B and C and the supersymmetric covariant derivative D, it is not difficult to construct an energy-momentum superfield whose bosonic component reproduces the energy-momentum tensors of the component ghost systems, given by (3.18) with $j = 2$ for (b,c) and $j = \frac{3}{2}$ for (β,γ). The result is:

$$T = -D^2B\cdot C + \frac{1}{2}DB\,DC - \frac{3}{2}BD^2C\,. \tag{7.26}$$

The fermionic component T_F of (7.26) may be though of as the generator of local supersymmtry on the ghost fields.

It is naturally suggested that the BRST charge for the superstring theory should be constructed as a superspace contour integral of a ghost field with this energy-momentum superfield. More concretely, this prescription gives

$$\begin{aligned} Q &= -\oint\frac{dz}{2\pi i}\int d\theta\; :C(z,\theta)\cdot\left(T^{(X)} + \frac{1}{2}T^{(B,C)}\right): \\ &= \oint\frac{dz}{2\pi i}\left\{c(z)\cdot\left(T_B^{(X,\psi)} + \frac{1}{2}T_B^{(b,c,\beta,\gamma)}\right) - \gamma(z)\cdot\left(T_F^{(X,\psi)} + \frac{1}{2}T_F^{(b,c,\beta,\gamma)}\right)\right\}. \end{aligned} \tag{7.27}$$

Indeed it is straightforward to verify that this quantity satisfies $Q^2 = 0$ as long as $T^{(X)}$ satisfies the operator product relation (7.17) with $\hat{c} = 10$. This insures that the full energy-momentum tensor $(T^{(X)} + T^{(B,C)})$ satisfies the Neveu-Schwarz-Ramond algebra with zero central charge. If the background space-time is flat, this condition is just the requirement derived in Section 3 that this background be 10-dimensional.

It is reasonable to expect that vertex operators for the Neveu-Schwarz states will have the general form:

$$\int dz \cdot V[X^\mu, \psi^\mu] . \tag{7.28}$$

Let us apply (7.27) to this structure and see what conditions result. For this, it is useful to rewrite the BRST charge in the form

$$Q = \sum_{-\infty}^{\infty} : c_n L_{-n}^{(X,\psi)} - \frac{1}{2}\gamma_k G_{-k} : + \text{(3 ghost terms)} \tag{7.29}$$

The c_n annihilate $|0\rangle$ only for $n \geq 1$, so all of the L_n for $n \geq (-1)$ must give zero when applied to (7.28). Since (7.28) is translation-invariant, it is indeed annihilated by L_{-1}. The conditions associated with L_n, $n \geq 0$, are just are those written in (5.21), with each $L_n^{(X)}$ replaced by the total Virasoro operator for the combined system of X^μ and ψ^μ. The γ_k annihilate $|0\rangle$ only for $k > \frac{1}{2}$, so G_k for $k \geq -\frac{1}{2}$ must also give zero acting on (7.28). $G_{-\frac{1}{2}}$ may be identified as the global supersymmetry generator: Using the definition (7.19) together with (7.15), and representing an infinitesimal supersymmetry parameter by ϵ, one can readily compute the commutators

$$[\epsilon G_{-\frac{1}{2}}, X^\mu(z)] = \epsilon\psi^\mu(z) , \qquad [\epsilon G_{-\frac{1}{2}}, \psi^\mu(z)] = \epsilon\partial_z X^\mu(z) . \tag{7.30}$$

This is indeed a global supersymmetry transformation

$$[\epsilon G_{-\frac{1}{2}}, X^\mu(z,\theta)] = \epsilon\, Q X^\mu(z,\theta) . \tag{7.31}$$

If (7.28) is annhilated by $G_{-\frac{1}{2}}$ and by all of the L_n for $n \geq 0$, the second relation of (7.20) implies that all of the G_k for $k > 0$ also annhilate this operator. Thus, for vertex operators of the form (7.28), the one new condition arising from the superconformal algebra is that the vertex operator be globally supersymmetric. It should be noted that (7.28) is not the most general form for a vertex operator in the Neveu-Schwarz theory. I will present some more general operators, which involve the superconformal ghosts in a nontrivial way, later in this section.

Continuing, however, with the operators of the simple form (7.28), let us write down the simplest operators satisfying the requirements of the previous paragraph. The most straightforward way to insure that $V[X^\mu, \psi^\mu]$ will be supersymmetric is to write it as a superspace integral:

$$\int dz\, V[X^\mu, \psi^\mu] = \int dz \int d\theta\, \mathcal{V}[X^\mu(z,\theta)] \,. \tag{7.32}$$

The simplest choice is

$$V_t(k) = \int d\theta\, e^{ik\cdot X(z,\theta)} = ik\cdot\psi(z)\, e^{ik\cdot X(z)} \,. \tag{7.33}$$

This operator is automatically primary. It has dimension 1 if $k^2/2 = \frac{1}{2}$. This operator thus creates a scalar particle with $m^2 = -1$; this value is negative, but half the normal quantum. The corresponding particle is the tachyon of the Neveu-Schwarz theory. The next simplest choice for V is

$$\varsigma\cdot V_v(k) = \int d\theta\, \varsigma_\mu DX^\mu e^{ik\cdot X(z,\theta)} = \varsigma_\mu\left(\partial_z X^\mu + ik\cdot\psi\psi^\mu\right) e^{ik\cdot X(z)} \,. \tag{7.34}$$

This is primary if $\varsigma\cdot k = 0$ and dimension 1 if $k^2 = 0$. This vertex thus creates the massless vector particle of the Neveu-Schwarz theory. The generalization to higher levels should be clear. One feature of this analysis seems strange, however. In the discussion of the Neveu-Schwarz spectrum given below eq. (7.22), the tachyon and vector states appeared as

$$|\Omega\rangle \,, \qquad \psi^\mu_{-\frac{1}{2}}\,|\Omega\rangle \,. \tag{7.35}$$

The vertex operators which create these states seem to contain an extra fermion field. In fact, these two arguments treat the same states in two different representations. I will reconcile these two pictures at the end of this section.

In either of the pictures, it is unambiguous that the tachyon and vector particles are created by operators with opposite Grassmann properties. This looks very dangerous for the formulations of a string field theory describing the Neveu-Schwarz states. If we insist that the Neveu-Schwarz string field is bosonic, then the expansion analogous to (5.37),

$$\Phi[X^\mu(\sigma), \psi^\mu(\sigma)] = \int d^{10}p \left\{\phi(p)V_t(p) + A_\mu(p)V_v{}^\mu(p) + \dots\right\}|0\rangle \,, \tag{7.36}$$

implies that the coefficient $A_\mu(p)$ is a c-number while $\phi(p)$ is a Grassmann number. The latter result is inconsistent with the spin-statistics theorem. To formulate a consistent theory, we must remove all terms with Grassmann coefficients.

This projection eliminates all states at half-integer mass levels. This is precisely the projection of Gliozzi, Scherk, and Olive,[50] (GSO), which was motivated from another viewpoint in Michael Green's lectures. Note that this projection turns the Neveu-Schwarz string into a theory which is free of tachyons.

7.3. Spin Operators

Let us now turn our attention to the Ramond sector. We must understand how to express the Ramond states in conformal field theory, or, equivalently, how to create these states by vertex operators. As we have already discussed, the states of the Ramond theory are states in which the fermions $\psi^\mu(z)$ are antiperiodic on circles around the origin (or, more generally, circles around the point z_0 to which the asymptotic string is mapped).

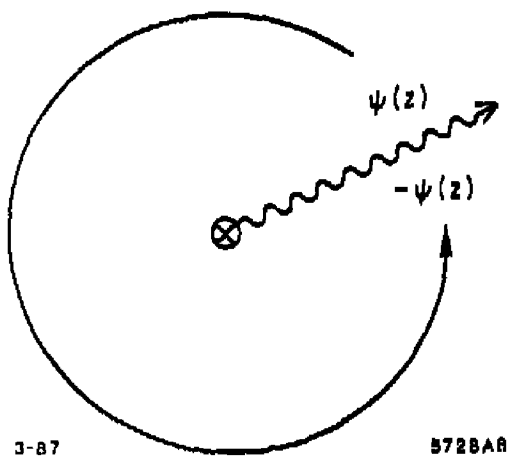

Figure 8. Analytic structure of the fermion field $\psi^\mu(z)$ for Ramond sector states.

If we view the field $\psi^\mu(z)$ as an analytic function, we would say that it has a square-root branch point at z_0. This structure is illustrated in Fig. 8. The vertex operators which create the Ramond states must, then, be operators which create this branch cut structure. I will refer to such operators as *spin operators*.

It is not difficult to build operators of this type by bosonizing the fermions $\psi^\mu(z)$. I would now like to describe that construction. This formalism is interesting from a more general point of view, because it provides a relatively simple example of the conformal field theory representation of a Kac-Moody algebra (a local current algebra). I will therefore feel free to generalize my discussion a bit from the analysis of 10-dimensional fermions to discuss a $2N$-dimensional vector field ψ^μ transforming under an $O(2N)$ current algebra.

We begin, then, with a system of $2N$ fermions with the operator products

$$\psi^\mu(z)\,\psi^\nu(w) \;=\; -\delta^{\mu\nu}\,\frac{1}{(z-w)}\,. \tag{7.37}$$

Let us relabel the fermions into pairs

$$\begin{aligned} \frac{1}{\sqrt{2}}\,(\psi^1+i\psi^2) &\to \psi^1\,, \qquad & \frac{1}{\sqrt{2}}\,(\psi^1-i\psi^2) &\to \psi^{\bar 1}\,, \\ \frac{1}{\sqrt{2}}\,(\psi^3+i\psi^4) &\to \psi^2\,, \qquad & \frac{1}{\sqrt{2}}\,(\psi^3-i\psi^4) &\to \psi^{\bar 2}\,, \end{aligned} \tag{7.38}$$

etc. Then (7.37) becomes

$$\psi^a(z)\,\psi^{\bar b}(w) \;=\; -\delta^{a\bar b}\,\frac{1}{z-w} \qquad a,b=1,2,\ldots N\,. \tag{7.39}$$

This system of fermion fields has $c = N$. It is thus natural that it can be bosonized by replacing the ψ^a, $\psi^{\bar a}$ by N boson fields according to the scheme

$$\psi^a \;\sim\; e^{i\alpha_a\cdot\phi}\,, \qquad \psi^{\bar a} \;\sim\; e^{-i\alpha_a\cdot\phi}\,, \tag{7.40}$$

where $\alpha_a{}^i = \delta_a{}^i$ is a unit vector. For each pair ψ^a, $\psi^{\bar a}$, taken separately, this construction reproduces (4.8) and thus will lead to identical correlation functions for the fermionic and bosonic theories. However, the operator assigned to ψ^a by (7.40) does not anticommute with the operator assigned to ψ^b if $a \neq b$. We may remedy this by assigning to the product of exponentials $e^{i\alpha_a\cdot\phi}\,e^{i\alpha_b\cdot\phi}$ a canonical order, or by introducing extra operators $c(a)$ (which are independent of z) to provide the correct signs when the order of these exponentials is changed. For this problem, we need operators which obey the algebra

$$c(a)\,c(b) \;=\; -c(b)\,c(a)\;, \qquad c^2(a) = 1\;. \tag{7.41}$$

These z-independent operators are called *cocycles*. A more general discussion of their significance may be found in Ref. 51. Assembling these pieces, we have

$$\psi^a(z) \;=\; ic(a)\,e^{i\alpha_a\cdot\phi(z)} \tag{7.42}$$

as the complete form of the bosonization relation.

It happens that the vector $\alpha_a{}^i$ is also the *weight vector* which characterizes an element of the vector representation of $O(2N)$.* This is not a coincidence. We can make the connection between bosonization and representation theory more explicit by constructing the generators of $O(2N)$ in their bosonized form.

The generators of $O(2N)$ are antisymmetric tensors $M^{\mu\nu}$ satisfying the algebra

$$[M^{\mu\nu}, M^{\lambda\sigma}] \;=\; M^{\mu\sigma}\delta^{\nu\lambda} \;-\; M^{\mu\lambda}\delta^{\nu\sigma} \;-\; M^{\nu\sigma}\delta^{\mu\lambda} \;+\; M^{\nu\lambda}\delta^{\mu\sigma}\;. \tag{7.43}$$

We can represent this algebra in terms of fermions by

$$M^{\mu\nu} \;=\; \oint \frac{dz}{2\pi i}\, j^{\mu\nu}(z)\;, \qquad j^{\mu\nu}(z) \;=\; -:\psi^\mu\psi^\nu:\;. \tag{7.44}$$

The operator product (7.37) implies

$$\begin{aligned} j^{\mu\nu}(z)j^{\lambda\sigma}(w) \;&\sim\; \psi^\mu(z)\,\frac{-\delta^{\nu\lambda}}{(z-w)}\,\psi^\sigma(w) \;-\; (3\text{ perms.}) \;+\; \frac{\delta^{\mu\sigma}\delta^{\nu\lambda}-\delta^{\mu\lambda}\delta^{\nu\sigma}}{(z-w)^2} \\ &\sim\; \frac{+\delta^{\nu\lambda}}{(z-w)}\,j^{\mu\sigma}(w) \;-\; \frac{\delta^{\nu\sigma}}{(z-w)}\,j^{\mu\lambda} \;-\; (\mu\leftrightarrow\nu) \\ &\qquad +\; \frac{\delta^{\mu\sigma}\delta^{\nu\lambda}-\delta^{\mu\lambda}\delta^{\nu\sigma}}{(z-w)^2}\;, \end{aligned} \tag{7.45}$$

and it is easily checked from this relation that $M^{\mu\nu}$ defined by (7.44) indeed satisfies (7.43). The double poles in (7.45) drop out of the calculation of the

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* A brief but very clear explanation of the representation theory of Lie groups may be found in the book of Cahn.[52]

commutators of charges $M^{\mu\nu}$. They have an effect, though, on the current algebra: In equal-time commutators $[j^{\mu\nu}(\sigma), j^{\rho\lambda}(\sigma')]$ of operators local in the string coordinate σ, they contribute Schwinger terms, c-number terms proportional to $\delta'(\sigma - \sigma')$.

It is now straightforward to convert the indices μ, ν of (7.44) to $a, \bar{a}$ and to bosonize this operator. For $a \neq b$, this procedure gives

$$
\begin{aligned}
j^{ab} &= c(a)\, c(b)\; e^{i\alpha_{ab}\cdot\phi}\,, \qquad \alpha_{ab}{}^{i} = \delta_a{}^{i} + \delta_b{}^{i}\,, \\
j^{a\bar{b}} &= c(a)\, c(b)\; e^{i\alpha_{a\bar{b}}\cdot\phi}\,, \qquad \alpha_{a\bar{b}}{}^{i} = \delta_a{}^{i} - \delta_b{}^{i}\,.
\end{aligned}
\tag{7.46}
$$

For $j^{a\bar{a}}$, we must be somewhat more careful:

$$
\begin{aligned}
j^{a\bar{a}} &= \lim_{z\to w} \left\{ e^{i\alpha_a\cdot\phi(z)}\, e^{-i\alpha_a\phi(w)} \;-\; (\text{singular terms}) \right\} \\
&= \lim_{z\to w} \left\{ :e^{i\alpha_a[\phi(z)-\phi(w)]}:\; \frac{1}{(z-w)} \;-\; (\text{singular terms}) \right\} \\
&= i\partial_z \phi^a\,.
\end{aligned}
\tag{7.47}
$$

The N generators $M^{a\bar{a}}$, which generate rotations in N orthogonal planes, provide a *Cartan subalgebra* of $O(2N)$, a maximal set of mutually commuting generators. The representation theory of $O(2N)$, and, more generally, of any Lie algebra, involves in an essential way the eigenvalues of these generators acting on an element Φ of an irreducible representation:

$$[M^{a\bar{a}}, \Phi] = w^a\, \Phi\,. \tag{7.48}$$

The set of eigenvalues w^i is called the *weight vector* of Φ. The representation (7.47) allows us to compute the weight vector of any operator which is the exponential of a boson field: The operator product

$$j^{a\bar{a}}(z)\; e^{i\alpha_I\cdot\phi(w)} \sim \frac{1}{(z-w)}\, \alpha_I{}^a\, e^{i\alpha_I\phi(w)}\,, \tag{7.49}$$

which follows immediately from the form of $j^{a\bar{a}}(z)$, integrates to the commutator

$$[M^{a\bar{a}},\; e^{i\alpha_I\phi(w)}] = \alpha_I{}^a\; e^{i\alpha_I\phi(w)}\,. \tag{7.50}$$

Thus, the weight vector for this operator is exactly $\alpha_I{}^i$. The remaining generators of $O(2N)$ may be seen from (7.46) to raise and lower the weights of fields of the exponential form by translating the weight vector $\alpha_I{}^i$ by $\alpha_{ab}{}^i$ or $\alpha_{a\bar{b}}{}^i$, the weights of the adjoint representation. This is exactly in accord with the representation theory of $O(2N)$.

Let us now return to the question that motivated this whole analysis: How do we construct spin operators which create branch cuts for the $\psi^a(z)$? Notice that if we assign to an exponential of boson fields the weight vector

$$\alpha_A{}^i \ = \ (\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\ldots)\ , \tag{7.51}$$

we have the operator product

$$\begin{aligned} \psi^a(z)\ e^{i\alpha_A\cdot\phi(w)} \ &\sim\ ic(a)\, e^{i\phi^a(z)}\ e^{i\alpha_A\cdot\phi(w)} \\ &\sim\ (z-w)^{\alpha_A^a}\ :ic(a)\, e^{i(\alpha_a+\alpha_A)\cdot\phi(w)}:\ . \end{aligned} \tag{7.52}$$

The singular term apparent in the second line is always $(z-w)^{\pm\frac{1}{2}}$, so the operator with weight $\alpha_A{}^i$ induces exactly the desired singularity in $\psi^a(z)$. The group-theoretic apparatus that we have developed allows us to see the remarkable interpretation of this construction: $\alpha_A{}^i$ is the weight vector for a representation on which the angular momentum generators take half-integer eigenvalues—a spinor representation. The spin operators on the world sheet, and the Ramond theory states which these operators create, transform as spinors in 10-dimensional space-time.

It is instructive to pursue the operator product relation between $\psi^a(z)$ and the spin operator a bit further. To do this, it will be convenient to choose a convenient representation of the $O(2N)$ Dirac algebra $\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}$. The $O(2N)$ Dirac matrices are $2^N \times 2^N$ matrices, which may be thought of as acting on products of N 2-component spinors. In this basis, we may represent their algebra by writing

$$\begin{aligned} \gamma^{1,2} \ &= \ i\sigma^{1,2}\otimes 1_2\otimes 1_2\otimes\ldots\ , \\ \gamma^{3,4} \ &= \ \sigma^3\otimes i\sigma^{1,2}\otimes 1_2\otimes\ldots\ , \end{aligned} \tag{7.53}$$

etc., where $\sigma^1,\sigma^2,\sigma^3$ are the Pauli sigma matrices and 1_2 is the 2×2 unit matrix. In the γ^a, $\gamma^{\bar{a}}$ basis, this representation takes the form

$$\begin{aligned} \gamma^1 \ &= \ \sqrt{2}\, i\sigma^+\otimes 1\otimes 1\otimes\ldots \qquad & \gamma^{\bar{1}} \ &= \ \sqrt{2}\, i\sigma^-\otimes 1\otimes 1\otimes\ldots \\ \gamma^2 \ &= \ \sigma^3\otimes\sqrt{2}\, i\sigma^+\otimes 1\otimes\ldots \qquad & \gamma^{\bar{2}} \ &= \ \sigma^3\otimes\sqrt{2}\, i\sigma^-\otimes 1\otimes\ldots \end{aligned} \tag{7.54}$$

The generalization of γ^5 to this algebra is

$$\Gamma = \gamma^1\gamma^2\cdots\gamma^{2N} = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \cdots \tag{7.55}$$

A spinor state has $\Gamma = \pm 1$ according to whether the number of entries $(-\frac{1}{2})$ in its weight vector $\alpha_A{}^i$ is even or odd.

Let us now define the spin operator corresponding to the spin-$\frac{1}{2}$ representation of $O(2N)$ a bit more carefully. Write

$$S_A(z) = c(A)\, e^{i\alpha_A\cdot\phi(z)} , \tag{7.56}$$

$c(A)$ is a cocycle defined by applying factors $c(a)$ to the cocycle $c(\hat{A})$ associated with the highest-weight state $\alpha_{\hat{A}}{}^i = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$. If $\alpha_A{}^i$ has components $(-\frac{1}{2})$ as its $b, c, d, \ldots$ entries, for $b < c < d < \ldots$, I will define

$$c(A) = c(b)c(c)c(d)\cdots c(\hat{A}) . \tag{7.57}$$

With these definitions, one can show by explicit computation using the bosonized form (7.42) that the singular term in the operator product of ψ^a with S_A is given by

$$\psi^a(z)\; S_A(w) \sim \frac{1}{\sqrt{2}}\,\frac{1}{(z-w)^{\frac{1}{2}}}\,(\gamma^a)_{AB} S_B(w) , \tag{7.58}$$

with the Dirac matrices in the representation (7.54). From this relation, we can build up

$$j^{ab}(z) S_A(w) \sim \frac{1}{(z-w)}\,\left(\frac{1}{4}[\gamma^a, \gamma^b]\right)_{AB}\, S_B(w) . \tag{7.59}$$

This equation implies that the action of M^{ab} on S_B is exactly that of $\frac{1}{2}\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$, just as the spinor transformation properties of S_B require.

7.4. Vertex Operators for the Ramond Sector

It is tempting to say that the spin operators $S_A(z)$ are all that we require to create states of the Ramond sector. However, there is still something wrong with this choice. The operator $S_A(z)$ has dimension $|\alpha_A|^2/2 = N/8$; for a 10-dimensional string theory. this equals 5/8. Thus, $S_A(z)$ alone cannot satisfy the requirement on a BRST-invariant vertex operator that its dimension should be 1. The resolution of this problem was discovered by Friedan, Martinec, and Shenker[53] and Knizhnik[54] in the dynamics of the superconformal ghost system. It is inconsistent with local supersymmetry to choose antiperiodic boundary conditions for $\psi^\mu(z)$ unless we also choose antiperiodic boundary conditions for $\beta(z)$ and $\gamma(z)$. Thus, the operators which create the Ramond states must also include spin operators which create branch cuts for these ghost fields. This operator provides the last ingredient needed to assemble BRST-invariant operators for the Ramond sector.

Just as we did for the coordinate fermions, we can construct the spin operators for the superconformal ghosts by bosonization. Let me, then, present the bosonization formulae for the superconformal ghost system. Since the logic of this development follows exactly that of the bosonization of the reparametrization ghosts in Section 4, and the techniques necessary to follow this analysis should also be familiar to you, my discussion here will be brief.

One comes ver. close to bosonizing the (β,γ) system by defining a new scalar field $\phi(z)$ (not to be confused with the fields $\phi^i(z)$ which enter the bosonization of ψ^μ) with energy-momentum tensor

$$T^{(\phi)} = \frac{1}{2}(\partial_z\phi)^2 - \partial_z^2\phi \,. \tag{7.60}$$

This produces a system with $c = 13$; exponentials of the field $\phi(z)$ have dimensions given by:

$$d_O = -\frac{1}{2}a(a+2)\,, \qquad \text{for} \qquad O = e^{a\phi(z)}\,. \tag{7.61}$$

This system differs from the (β,γ) system, however, in three important ways: First, the value of the central charge is wrong, since the (β,γ) system has $c = 11$. Second, the operators $e^{-\phi}$, e^{ϕ} which we would like to associate with β and γ have the wrong dimensions. Third, these operators anticommute, whereas we would like to find a representation for β and γ as commuting fields. This last problem may be expressed as the statement that, while we have much experience bosonizing fermions, we need here a bosonization of bosons.

All of these problems are solved by adding a system of *fermions* (ξ, η) with dimensions $(0, 1)$ and energy-momentum tensor

$$T^{(\xi,\eta)} = \partial_z \xi \cdot \eta \,. \tag{7.62}$$

This system has $c = -2$, so it is exactly what we need to combine with the ϕ system to give $c = 11$. In addition, we might note that evaluating (7.61) for $a = \pm 1$ gives values differing by 1 unit from the dimensions we require for β and γ. This difference can be made up by adding dimension 1 operators from the (ξ, η) system. The combinations

$$\beta(z) \rightarrow \partial_z \xi(z)\, e^{-\phi(z)} \,, \qquad \gamma(z) \rightarrow \eta(z)\, e^{\phi(z)} \tag{7.63}$$

have the correct dimensions and the correct operator product with one another to reproduce the correlation functions of $\beta(z)$ and $\gamma(z)$ with themselves and with $T(z)$.

This bosonization of the superconformal ghosts can be introduced into the BRST charge in the same way that we introduced the bosonized form of the reparametrization ghosts in Section 4. Let us, then, modify the formula for Q given in (7.27) by making the replacements (7.63) as well as (4.15). The energy-momentum tensor of the ghosts is replaced according to

$$T_B^{(b,c,\beta,\gamma)} \rightarrow T^{(\sigma)} + T^{(\phi)} + T^{(\xi,\eta)} \,, \tag{7.64}$$

and the T_F of the ghosts undergoes a similar (and somewhat more transparent) rearrangement. Inserting these new structures into (7.27) and taking care, as we did in (4.18), to correct the definition of normal-ordering appropriately, we find at last

$$Q = \oint \frac{dz}{2\pi i} \left(q_0 + q_1 + q_2\right) , \tag{7.65}$$

where

$$\begin{aligned} q_0 &= : e^{\sigma}(T^{(X)} + T^{(\psi)} + T^{(\sigma)} + T^{(\phi)} + T^{(\xi,\eta)}) : \\ q_1 &= : \frac{1}{2}\eta e^{\phi} \psi \cdot \partial_z X : \\ q_2 &= : -\frac{1}{4} e^{-\sigma} \eta \partial_z \eta\, e^{2\phi} : \end{aligned} \tag{7.66}$$

Now at last we have all of the equipment we need to construct the vertex operator for states of the Ramond theory. The simplest candidate for a spin operator for the superconformal ghosts is the exponential

$$e^{-\frac{1}{2}\phi(z)} \,. \tag{7.67}$$

From eq. (7.61), we see that this operator has dimension 3/8, exactly what is needed to bring the coordinate spin operator $S_A(z)$ up to dimension 1. Adding also a reparametrization ghost factor $c(z) = e^{\sigma(z)}$, we can assemble a complete vertex operator with spinor quantum numbers:

$$V_{-\frac{1}{2}}(z) \;=\; \overline{u}^A(k)\, S_A(z)\; e^{ik\cdot X(z)}\; e^{\sigma(z)}\; e^{-\frac{1}{2}\phi(z)} \,, \tag{7.68}$$

where $\overline{u}$ is a c-number polarization spinor. This operator has total dimension $k^2/2$. This must equal 0 for BRST invariance. In addition, BRST invariance requires that the field be primary. This implies that that $\not{k}u(k) = 0$. $V_{-\frac{1}{2}}(z)$ given by (7.68) thus creates a massless fermion with proper on-shell spin orientation. This is exactly the lowest-lying state in the Ramond sector.

7.5. PICTURES

We now have a start on the formalism for covariant calculation of fermion emission amplitudes in superstring theories. However, we have left many unanswered questions along the way. Among these is the question of the relation of the Neveu-Schwarz vertex operators (7.33) and (7.34) to the corresponding states of the spectrum. Further puzzles come from the new ghost sector: The replacement of the (β, γ) system by a boson plus a fermion pair has apparently led to some multiplication of the number of states. We would like to know whether (7.68) is the unique choice for a massless fermion vertex operator, or, if not, what other choices we are allowed.

A peculiar property of the bosonization of the superconformal ghosts which I have just described is that the final system contains two ghost number currents. The first of these is obtained by generalizing the bosonization relation $j^{(b,c)} = \partial_z\sigma(z)$ to the ϕ system; this gives the current $j^{(\phi)} = \partial_z\phi$ which assigns the charge n to exponentials $e^{n\phi(z)}$. In addition, there is a fermion number for ξ and η. The ghost number of the fields β, γ is a particular linear combination of these two charges. The orthogonal linear combination corresponds to a degree of freedom which was not at all obvious in the notation of β and γ.

Friedan, Martinec, Shenker[53] (FMS) interpret this new degree of freedom by relating it to a pathology of the original superconformal ghost action (3.28). This action is first order in derivatives while involving boson rather than fermion fields. Any such action leads to a spectrum which is not bounded below. In fact, it is not difficult to construct states of the (β, γ) theory of arbitrarily negative L_0. From (4.22), we see that $\gamma_{\frac{1}{2}} |0\rangle \neq 0$. Because $\gamma_{\frac{1}{2}}$ is a bosonic operator, we can apply it arbitrarily many times to $|0\rangle$, lowering L_0 by $\frac{1}{2}$ unit at each step.

A more sensible vacuum state would be annihilated by $\gamma_{\frac{1}{2}}$. To see how to construct this state, compute

$$\begin{aligned} \gamma_k\, e^{n\phi(0)} |0\rangle &= \oint \frac{dz}{2\pi i}\, z^{k-\frac{3}{2}}\, \eta(z)\, e^{\phi(z)}\, e^{n\phi(0)} \\ &= \oint \frac{dz}{2\pi i}\, z^{k-\frac{3}{2}} z^{-n}\; : \eta(z)\, e^{\phi(z)}\, e^{n\phi(0)} : \ . \end{aligned} \tag{7.69}$$

This vanishes only if $(k - \frac{3}{2} - n) \geq 0$. The unique state of the class $e^{n\phi(0)}|0\rangle$ which is annihilated by all γ_k and β_k for $k > 0$ is the state given by $n = -1$. The state annihilated by all L_0-lowering ghost operators is, then,

$$|\hat{\Omega}\rangle \ = \ c(0)\, e^{-\phi(0)}\, |0\rangle \ . \tag{7.70}$$

According to (7.61), the exponential has dimension $\frac{1}{2}$, so that the state $|\hat{\Omega}\rangle$ has $L_0 = -\frac{1}{2}$. This is just the position of the tachyon in the Neveu-Schwarz sector. We may identify (7.70) as the vacuum of the Neveu-Schwarz theory which properly includes all ghost contributions.

The transformation from $|0\rangle$ to $|\hat{\Omega}\rangle$ cannot be achieved by applying any finite number of γ_k and β_k operators to the $SL(2,C)$-invariant vacuum. The two states live in disjoint Hilbert spaces, within each of which the γ_k and β_k operators act. FMS visualize this by imagining that the states of (3.28) contain a condensate of bosonic ghosts with indefinite β, γ number—the *Bose sea*. They interpret the ϕ charge as the filling level of this Bose sea.

There is no conceptual problem in working within a given Bose sea level. We can, in fact, describe processes with an arbitrary number of external Neveu-Schwarz particles without changing the Bose sea level by using vertex operators of the form (7.33), (7.34), etc. However, two features of the formalism force us to study the relation of the various Bose sea levels. The first of these is the fact that the fermion vertex operator (7.68), and, more generally, any spin field for the superconformal ghosts, necessarily changes the Bose sea level. The second is the fact that for the superconformal ghosts, as for the reparametrization ghosts, conformal field theory matrix elements on the plane violate ghost number by a fixed amount. We derived in Section 5 the result that the ghost number of b and c is violated by 3 units in string tree diagram calculations, and that this observation plays an important role in determining the structure of scattering amplitudes. The basic statement of this nonconservation was the expectation value $\langle 0| c_{-1}c_0c_1 |0\rangle = 1$, or, in bosonized form,

$$\langle 0| e^{+3\sigma(0)} |0\rangle \;=\; 1 \,. \tag{7.71}$$

This statement has a twofold generalization to the bosonized superconformal ghosts: Both ghost number currents are anomalous. FMS argued that the deficits are (-2) and 1 unit for the ϕ and (η,ξ) systems, respectively, corresponding to 2 and 1 normalizable zero modes. Explicitly,

$$\langle 0| e^{-2\phi(0)} |0\rangle \;=\; 1 \,, \quad \text{and} \quad \langle 0| \;\xi(0)\; |0\rangle \;=\; 1 \,. \tag{7.72}$$

These statements couple different Bose sea levels. Further, they rule out the use of vertex operators with predetermined exponentials of ϕ to describe arbitrary scattering processes. For example, except for processes involving exactly 4 fermions, the changes in Bose sea level by $(-\frac{1}{2})$ unit produced by the insertion of successive vertex operators (7.68) will not add up properly to fulfill the first relation of (7.72).

To solve this problem, we need to find alternative forms of the fermion vertex operator, equivalent to (7.68), which change the Bose sea charge by a different number of units. This can be done, in an effective but very counterintuitive way, as follows: Because of the second relation of (7.72), the z-independent Fourier component ξ_0 corresponds to a zero mode; thus, every nonvanishing matrix element of vertex operators must contain a factor ξ_0 to saturate this zero mode. This zero mode does not appear anywhere else in our formalism; in particular, the bosonization formulae (7.63) depend only on $\partial_z\xi$. Taking this into account, let us consider the transforming the fermion vertex operator (7.68) according to

$$V_{\frac{1}{2}}(z) \;=\; [Q,\; \xi(z)\, V_{-\frac{1}{2}}(z)] \,. \tag{7.73}$$

The commutator is defined, as usual, by taking the contour in the definition of Q to encircle the point z at which the vertex operator is inserted. If $V_{-\frac{1}{2}}$ is BRST-invariant, as is guaranteed by the on-shell conditions, the BRST contour passes through $V_{-\frac{1}{2}}$ and acts on $\xi(z)$, eliminating the redundant factor ξ_0 that could potentially appear. The result is a new vertex operator which is BRST-invariant by virtue of the relation $Q^2 = 0$. Technically, (7.73) is the second member of a BRST doublet. However, this is somewhat obscure, because the first member of the doublet contains ξ_0, which is, in some sense, outside our formalism. Evaluating (7.73) explicitly, we find

$$V_{\frac{1}{2}}(z) \;=\; \bar{u}^A(k)\left\{e^{\phi/2}(\partial_z X^\mu + \frac{i}{4}k\cdot\psi\psi^\mu)\gamma^\mu_{AB}S_B \;+\; e^{3\phi/2}\eta b S_A\right\}\cdot c\cdot e^{ik\cdot X}\,. \tag{7.74}$$

This is a sensible-looking vertex operator which creates a spin-$\frac{1}{2}$ fermion state. If $V_{-\frac{1}{2}}$ is BRST invariant, this object is also. (It is not hard to check directly, using (7.61), that (7.74) has dimension 0 if $k^2 = 0$.) Thus, (7.74) is a second vertex operator for the massless fermion state of the Ramond sector, differing from (7.68) in that it *raises* the Bose sea charge by $\frac{1}{2}$ unit.

FMS refer to the transformation on vertex operators defined in the previous paragraph as the *picture-changing* operation:

$$X\left\{V(z)\right\} \;=\; [Q,\, \xi(z)V(z)]\,. \tag{7.75}$$

They emphasize that X works quite generally as a method of transforming BRST-invariant vertex operators to new operators carrying the same space-time quantum numbers but different Bose sea charge. Let me present two more examples of this relation:

$$\begin{aligned}\left[Q,\,\xi\cdot\{ik^\mu\psi_\mu e^{ik\cdot X}e^{\sigma}e^{-2\phi}\}\right] \;&=\; e^{ik\cdot X}\cdot e^{\sigma}e^{-\phi}\\ \left[Q,\,\xi\cdot\{\varsigma^\mu(\partial_z X^\mu + ik\cdot\psi\psi^\mu)e^{ik\cdot X}e^{\sigma}e^{-2\phi}\}\right] \;&=\; \varsigma^\mu\psi^\mu e^{ik\cdot X}\cdot e^{\sigma}e^{-\phi}\,.\end{aligned} \tag{7.76}$$

The operators (7.33) and (7.34) may thus be recognized as picture-changed versions of the vertex operators which create the most natural forms of the low-lying states of the Neveu-Schwarz theory:

$$e^{ik\cdot X}\,|\hat{\Omega})\;,\quad \psi^\mu_{-\frac{1}{2}}e^{ik\cdot X}\,|\hat{\Omega})\;. \tag{7.77}$$

To complete our discussion of the picture-changing operator, I would like to argue that picture-changed versions of the same vertex operator are equivalent for the purpose of computing scattering amplitudes. To understand how to make this argument, let us recall the transformation

$$c(z)V(z) \;\rightarrow\; \oint\frac{dw}{2\pi i}\,b(w)\;c(z)V(z) \tag{7.78}$$

which we introduced in eq. (5.26) to define the bosonic string scattering amplitudes. This transformation, involving the contour integral of a ghost operator, is

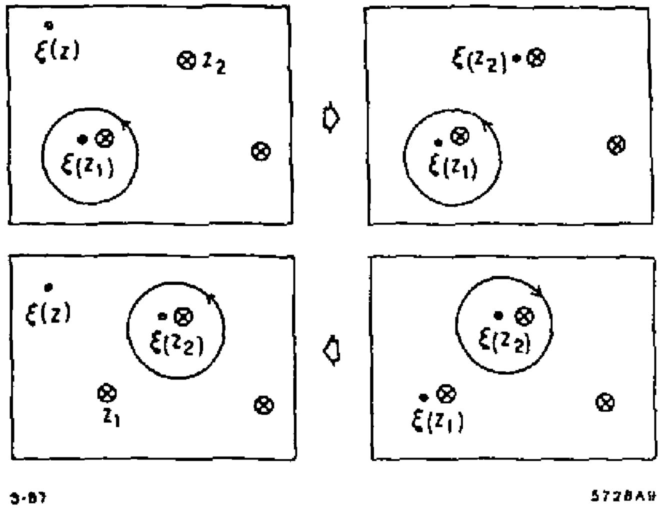

Figure 9. How to move the picture-changing operator from one on-shell vertex operator to another.

similar in structure to the picture-changing operation. In our discussion of (5.26), we saw that the b contours could be freely moved from one vertex operator to any other vertex operator in the same correlation function. If the same is true of the picture-changing contour, we can move this contour onto or off of any given operator without changing the value of the scattering amplitude.

The argument that we can move the picture-changing operator from a given vertex operator $V(z_1)$ to a second operator $V(z_2)$ is illustrated in Fig. 9. To begin, note that the amplitude, to be nonzero, must contain a factor ξ_0. Since ξ_0 is z-independent, we can consider this operator to come from the Fourier expansion of $\xi(z)$ placed at any desired point, say, z_2. Now deform the BRST contour so that it winds around $\xi(z_2)V(z_2)$. This contour passes through all BRST-invariant vertex operators, but it sticks on the factor $\xi(z_2)$. Finally, replace the newly isolated $\xi(z_1)$ by ξ_0. This argument proves that picture-changed versions of the same vertex operator are equivalent for the computation of on-shell scattering amplitudes.

In our discussion of the reparametrization ghosts, we saw that the structure of the BRST charge and the ghost Hilbert space fixed the structure of the theory even off-shell, since these elements gave directly the form of the gauge-invariant action. This construction generalizes directly to allow one to contruct a gauge-invariant action for the Neveu-Schwarz sector in the picture with Bose sea charge (-1) which contains the state $|\bar{\Omega}\rangle$. For the Ramond theory, Witten[65] has shown how to combine these elements with the picture-changing operation to construct a gauge-invariant action for the Ramond theory in the picture with Bose sea charge $(-\frac{1}{2})$. The role of states with other values of the Bose sea charge in the off-shell formalism remains obscure, and, more generally, the full structure of the gauge-invariant interacting theory is much in need of further investigation.

8. One-Loop Amplitudes for Strings

At several points in our argument, we have found that the string theory is naturally projected onto a subspace of the full Hilbert space of states on the world sheet. The projection onto the ghost-free subspace is expected in any gauge-invariant theory, but two other projections which we made—the projection onto states with $L_0 = \overline{L}_0$ in the closed string theory and the GSO projection in the superstring theory—have no natural analogue in conventional field theory. It would be useful to explore these operations further.

The origin of these physical state projections, and their relation to other intrinisically stringy aspects of the formalism, is made most clear through their role in the formulation of loop corrections to the string scattering amplitudes. In this section, I would like to illustrate this by computing the one-loop amplitudes for bosonic and fermionic strings. This computation is interesting in its own right because it reveals that the ultraviolet divergences of the string theory, even in a space-time of very high dimension, are much less severe than the divergences of a local field theory. But it will be most illuminating because of the role played in this analysis by the invariances of the world-sheet geometry. For various reasons, the analysis of loop amplitudes is simpler for *closed* strings, so I will consider only that case. I will also restrict my discussion to the 0-point amplitude, the vacuum energy shift or cosmological constant renormalization.

8.1. Moduli

Let us begin with the bosonic closed string theory. Tree-level amplitudes in this theory correspond to integrals over conformally invariant fields on a plane, or, equivalently, on a sphere. At the one-loop level, we must include a virtual closed string breaking off from the sphere and then reattaching. This gives the world sheet the topology of a torus. Using local conformal invariance, we may consider the world sheet to be precisely a torus. Our problem, then, is to functionally integrate over the coordinate fields $X^\mu(z)$ on a base space which is a torus, taking proper account of the geometrical invariances of the string theory. This calculation was first done, in the formalism of dual models, by Shapiro;[56] a very clear and complete modern treatment has been given by Polchinski.[57] My discussion of this problem will clarify most of the issues of physics, but I will refer you to ref. 57 for a proper treatment of the reparametrization ghosts.

For the bosonic string theory at tree level, the 0-point amplitude is trivial. At the one-loop level, however, a complication arises which gives even this amplitude an interesting structure. Although any surface with the topology of a torus can be converted to a flat torus by making a conformal transformation, it is not true that any flat torus can be conformally tranformed into any other.

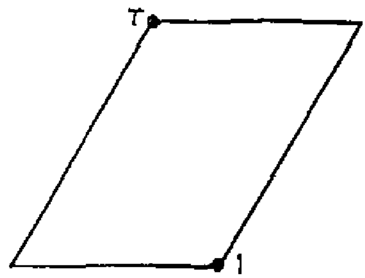

Figure 10. A torus, viewed as a parallelogram with opposite sides identified.

We may visualize a 2-dimensional torus as a parallelogram with opposite sides identified (Fig. 10). By conformal transformations, we can convert the metric on the space to the form $g_{\alpha\beta} = \delta_{\alpha\beta}$ and scale the length of the bottom edge to 1. But this construction leaves the length and orientation of the left-hand edge undetermined. These two parameters may be summarized as a complex number

$$\tau = \tau_1 + i\tau_2 . \tag{8.1}$$

τ parametrizes classes of tori which are conformally inequivalent. The conformally-invariant functional integral over $X^\mu(z)$ should then depend on τ, and the full one-loop correction should contain an integral over all inequivalent values of this parameter. In higher loops, where the base space is a surface of higher genus, one finds more degrees of freedom in the world-sheet geometry which cannot be removed by conformal transformations. The parameters of conformal equivalence classes of 2-dimensional surfaces are called *moduli*.

The existence of moduli forces us to reconsider the procedure described in Section 3 for replacing the integral over all metrics on the world-sheet by a Fadde'ev-Popov integral over reparametrization ghosts. The result of that discussion must be changed to allow for the fact that those degrees of freedom in $g_{\alpha\beta}$ which do not correspond to gauge transformations of the theory—reparametrizations and conformal transformations—should not be eliminated by the Fadde'ev-Popov procedure but should remain in the final answer. This means that eq. (3.8) should be replaced by

$$\frac{1}{V_{rep}V_{conf}}\int Dg \;=\; \int Db\,Dc\; e^{-\int(b\partial_{\bar z}c+\bar b\partial_z\bar c)}\;\cdot\int\prod_k d^2\tau^k\cdot\det(J[\tau^k])\;, \quad (8.2)$$

where the τ^k are the moduli, J is an appropriate Jacobian, and, to be precise, the ghosts $b(z)$ should be integrated only over their nonzero modes. It can be shown that the zero modes of b are in one-to-one correspondence with the moduli. This relation, combined with our discussion of the zero modes of b from eqs. (5.7) and (5.8) in Section 5, tells us that $3(g-1)$ (complex) moduli will appear in the expression for the g-loop amplitudes.*

For the case of a torus, it is not hard to derive the Jacobian J explicitly. It is most convenient to begin by mapping the general torus shown in Fig. 10 into a fixed square: $0 \le \xi_1 \le 1$, $0 \le \xi_2 \le 1$. In these coordinates, the line element becomes

$$ds^2 \;=\; |d\xi_1 \;+\; \tau d\xi_2|^2 \;=\; g_{\alpha\beta}d\xi^\alpha d\xi^\beta\;, \quad (8.3)$$

where

$$g_{ab} \;=\; \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2+\tau_2^2 \end{pmatrix}\;. \quad (8.4)$$

The Jacobian J is obtained by differentiating with respect to τ_1 and τ_2 the modes of $g_{\alpha\beta}$ orthogonal to reparametrizations and conformal transformations. A plau-

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* The formula (8.2) has been derived and analysed with exemplary clarity by Alvarez.[58]

sible, properly invariant, formula for $J^T J$ is

$$(J^T J)_{ij} = g^{\alpha\beta} \cdot \left(\frac{\partial}{\partial \tau_i} g_{\beta\gamma}\right) \cdot g^{\gamma\delta} \cdot \left(\frac{\partial}{\partial \tau_j} g_{\delta\alpha}\right) \tag{8.5}$$

Evaluating this expression, and taking the square root of its determinant, gives

$$\det J = 2\sqrt{2}/\tau_2^2 . \tag{8.6}$$

The result (8.6) can actually be obtained from more general considerations, which also illuminate somewhat more the nature of the moduli. In our discussion of the origin of τ, we considered only the possible equivalence of tori under infinitesimal reparametrizations and conformal transformations. However, the torus shown in Fig. 10 can be transformed into tori with different values of τ by making discrete reparametrizations. For example, by rotating and scaling the right-hand edge of Fig. 10 onto the interval (0, 1), we transform $\tau \to -1/\tau$. By taking the upper right-hand corner instead of the upper left-hand corner to define τ, we transform $\tau \to \tau+1$. These two transformations generate the *modular group*, the group of fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} , \tag{8.7}$$

for which a, b, c, d are *integers* satisfying $ad - bc = 1$.

The integrand of (8.2) should depend only on the intrinsic geometry of the world sheet, so it should be invariant to modular transformations. This is automatically true for the functional integral over the X^μ, as we will see in a moment. The Jacobian J must convert the integral $d^2\tau$ into a modular-invariant measure. For the value of J we have obtained in (8.6), this works out just right; it is easy to check that

$$\int |d\tau|^2 \cdot \frac{1}{\tau_2^2} \tag{8.8}$$

is explicitly invariant to (8.7).

Since the integrand of (8.2) is modular-invariant, the integral over τ which is indicated in this equation overcounts unless this integral includes only values τ which are not equivalent by modular transformations. A suitable integration

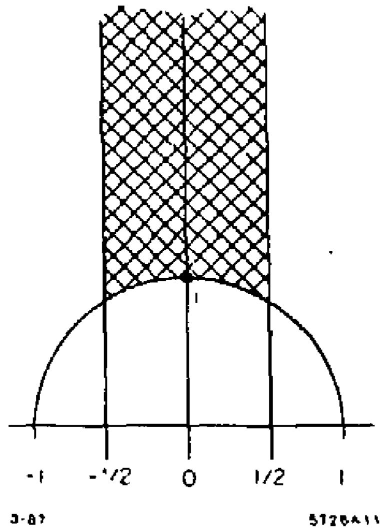

Figure 11. The fundamental domain for integration of modular-invariant quantities.

region can be found as follows: Of the two transformations which generate the modular group

$$\tau \to \tau + 1 , \qquad \tau \to -\frac{1}{\tau} , \tag{8.9}$$

the first carries the strip of the complex τ plane $-\frac{1}{2} \leq \operatorname{Re}\tau < \frac{1}{2}$ into an adjacent strip, and the second carries the exterior of the unit circle in the τ plane into the interior.

The overlap of these two regions, shown in Fig. 11, is the largest region which contains no pairs of points carried one into another by either of these transformations. It is not hard to see that this conclusion still holds when more general modular transformations are considered. This region thus gives the correct domain for the $d^2\tau$ integration in the 1-loop amplitude.

This whole discussion generalizes to the treatment of the moduli of higher-genus surfaces. In our discussion earlier in this section, we stated that (8.2) must be integrated over a parameter space of $3(g-1)$ complex dimensions. If we consider all values of the parameters corresponding to surfaces inequivalent with respect to infinitesimal reparametrizations and conformal transformations, we find an unbounded parameter space called *Teichmüller space*. In general, though, this space must be divided up according to the action of the group of discrete reparametrizations, the *mapping class group*. The maximal subspace of Teichmüller space which contains only points inequivalent under this group represents the space of moduli. Like the region of Fig. 11 for the case of tori, taken with the measure (8.8), the moduli space of any higher genus is compact in its natural invariant measure.*

8.2. CLOSED BOSONIC STRING

Having now clarified the role of the moduli, let us return to the calculation of the one-loop bosonic string amplitude. The function of τ which must be integrated over (8.8) is given by the functional integral over coordinate and ghost fields on the torus. I will compute the integral over coordinate fields X^μ explicitly; however, I will treat the ghosts only by assuming that they precisely cancel the contribution of two coordinate degrees of freedom, which one might imagine to be the longitudinal and timelike modes of oscillation. This result is justified in the paper of Polchinski, ref. 57. With this replacement, the one-loop amplitude of the bosonic string takes the form

$$\mathcal{A} = \int \frac{d^2\tau}{\tau_2^2} \left[\mathcal{A}_X\right]^{24} , \tag{8.10}$$

where

$$\mathcal{A}_X = \int DX \, e^{-\frac{1}{2}\int d^2z \; X(-\partial^2)X} . \tag{8.11}$$

I have rescaled the field X^μ from my previous convention for convenience in this context.

We can evaluate the integral $\mathcal{A}_X$ by making use of the connection between Euclidean functional integrals and Hamiltonian evolution. Write the variable on the plane of Fig. 10 as $z = x_1 + ix_2$. Then x_2 is a Euclidean time with periodicity τ_2. If $\tau_1 = 0$, this situation of a periodic Euclidean time gives precisely the functional representation of $\mathrm{tr}\left[\exp(-\tau_2 H)\right]$, where H is a Hamiltonian defined on a ring. To reintroduce τ_1, we define an operator $T(\tau_1)$ which twists the ring

* The theory of higher-loop string amplitudes and higher-genus surfaces has recently been reviewed by Alvarez-Gaumé and Nelson.[69]

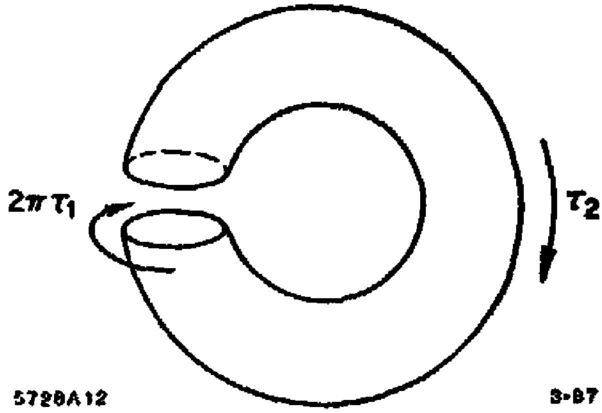

Figure 12. Evaluation of the one-loop string amplitude by relating it to a quantum partition function.

through an angle $2\pi\tau_1$. Then

$$A_X = \mathrm{tr}\left[T(\tau_1)\, e^{-\tau_2 H}\right] , \tag{8.12}$$

The relation between the geometry of the torus and the Hamiltonian interpretation of A_X is illustrated in Fig. 12.

To see what Hamiltonian to use in evaluating (8.12), let us Fourier decompose $X^\mu(x_1)$ at fixed x_2:

$$X(x_1) = X_0 + \sum_{n\neq 0} X_n e^{2\pi i n x_1} . \tag{8.13}$$

Introducing (8.13) into the exponent of (8.11), we find

$$S = \frac{1}{2}\int X(-\partial^2)X = \int d\xi_2 \left\{\frac{1}{2}\dot{X}_0^2 + \sum_n [\dot{X}_n\dot{X}_{-n} + (2\pi n)^2 X_n X_{-n}]\right\} . \tag{8.14}$$

This is just a set of harmonic oscillators, one for each normal mode on the ring.

Let us separate out the zero mode part of the dynamics by writing

$$H = H_1 + H_2 , \tag{8.15}$$

where $H_1 = \frac{1}{2}P_0^2$ and H_2 is a sum of Hamiltonians for harmonic oscillators of frequency $\omega_n = 2\pi n$.

The portion of the expression (8.12) involving the zero modes is readily evaluated by using simple quantum mechanics:

$$\mathrm{tr}\; e^{-\tau_2 P_0^2/2} = \int dX_0\; \langle X_0 | e^{-\tau_2 P_0^2/2} | X_0 \rangle = \int dX_0 \frac{1}{\sqrt{2\pi\tau_2}} . \tag{8.16}$$

The contribution of the nonzero modes is the product of terms assoicated with the left- and right-moving normal modes. At $\tau_1 = 0$, this factor is

$$\begin{aligned} \mathrm{tr}\left[e^{-\tau_2 H_2}\right] &= \left|\mathrm{tr}\; \exp\Big[-\tau_2 \cdot \sum_{n>0}(2\pi n a_n^\dagger a_n + \pi n)\Big]\right|^2 \\ &= \left|e^{-2\pi\tau_2 Z} \cdot \prod_{n>0} \sum_{k=0}^{\infty} e^{-\tau_2 \cdot 2\pi n k}\right|^2 \\ &= \left|e^{-2\pi\tau_2 Z} \cdot \prod_{n>0} \left(1 - e^{-2\pi n \tau_2}\right)^{-1}\right|^2 . \end{aligned} \tag{8.17}$$

In this equation, Z is the sum of the zero-point energies of the oscillators. We may regularize this sum using the Riemann zeta function:

$$Z = \sum_{n>0} \frac{1}{2} n = \zeta(-1) = -\frac{1}{24} . \tag{8.18}$$

Reintroducing τ_1 adds to each term a factor

$$T(\tau_1) = e^{2\pi i K} , \tag{8.19}$$

where K is the momentum around the ring, equal to $(+n)$ for each quantum of excitation of a left-moving mode and $(-n)$ for each quantum of excitation of a right-moving mode. This changes the second line of (8.17) according to

$$e^{-\tau_2\, 2\pi n k} \to e^{2\pi i \tau_1 \cdot nk} e^{-\tau_2 \cdot 2\pi nk} = e^{2\pi i \tau \cdot nk} ; \qquad (8.20)$$

this modification, in turn, changes $(i\tau_2)$ to τ throughout the rest of the analysis. Assembling all the pieces (and ignoring the overall normalization), we find

$$A_X = \int dX_0 \cdot \frac{1}{\sqrt{\tau_2}} \cdot \left| e^{\pi\tau_2/12} \prod_{n>0} \left(1 - e^{2\pi i n\tau}\right)^{-1} \right|^2 . \qquad (8.21)$$

Finally, inserting this answer into (8.10), we find

$$A = \int d^{26}X_0 \int \frac{d^2\tau}{\tau_2^2} \left(\frac{1}{\tau_2}\right)^{12} e^{+4\pi\tau_2} \left| \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right) \right|^{-48} . \qquad (8.22)$$

If we have computed correctly, this expression should be modular-invariant. To check this, we need only check the invariance of the integrand to the two transformations (8.9) which generate the modular group. The invariance under $\tau \to \tau + 1$ is obvious. The invariance under $\tau \to -1/\tau$ is not clear to the unaided eye. However, someone familiar with the theory of Jacobi theta functions will recognize immediately that (8.22) contains the Dedekind η-function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{1}^{\infty} \left(1 - e^{2\pi i\tau n}\right) . \qquad (8.23)$$

This function has a very polite transformation law under modular transformations:

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau) , \qquad \eta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\eta(\tau) \qquad (8.24)$$

These relations are, in fact, just what we need to prove the invariance of (8.22). The factor $|\sqrt{\tau}|^{-48}$ generated by the second transformation above is cancelled exactly by the modular transformation of the prefactor $(\tau_2)^{-12}$.

As this example makes clear, a student of string theory will find it indispensible to have some acqaintance with the Jacobi theta functions. I would therefore like to digress and present some of the main properties of these functions. These properties and other are discussed very clearly in the textbook of Whittaker and Watson.[60] (Please note, though, that my conventions differ slightly from those of Whittaker and Watson in order to follow the modern string literature.)

The Jacobi theta functions are a canonical set of natural analytic functions on the torus. Whittaker and Watson define four basic functions, of which the simplest is

$$\vartheta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau} e^{2\pi i n z} . \tag{8.25}$$

This function is analytic on the whole z plane. If z is translated by a period of the torus, $\vartheta_3(z|\tau)$ transforms according to

$$\vartheta_3(z+1|\tau) = \vartheta_3(z|\tau) , \qquad \vartheta_3(z+\tau|\tau) = e^{-i\pi\tau} e^{-2\pi i z} \vartheta_3(z|\tau) . \tag{8.26}$$

Thus, ϑ_3 is periodic in z with period 1, but it is only quasi-periodic with period τ. If this function had been periodic under $z \to z + \tau$, it would have been an analytic function without singularities on the torus and therefore a constant; thus, ϑ_3 is as close as anything can come to being a nontrivial analytic function on the torus. The periodicity relations (8.26) can be used to count the number of zeros N_Z of ϑ_3: Taking the contour around the boundary of the parallelogram in Fig. 10, we may compute

$$N_Z = \frac{1}{2\pi i} \oint dz \, \frac{\vartheta_3'(z|\tau)}{\vartheta_3(z|\tau)} = 1 . \tag{8.27}$$

It is not hard to see that the zero is located at the center of the parallelogram, $z = \frac{1}{2} + \frac{\tau}{2}$.

The other three theta functions are essentially translations of ϑ_3 with zeros at the other three half-periods. They may be characterized by

$$\begin{aligned} \vartheta_1(0|\tau) &= 0 , \qquad & \vartheta_2(\tfrac{1}{2}|\tau) &= 0 , \\ \vartheta_3(\tfrac{1}{2} + \tfrac{\tau}{2}|\tau) &= 0 , \qquad & \vartheta_4(\tfrac{\tau}{2}|\tau) &= 0 . \end{aligned} \tag{8.28}$$

Series representation for the four $\vartheta_i(z|\tau)$ are given in the appendix to this section.

Modular transformations such as those shown in (8.9) relabel τ and thus interchange the labels of the four half-periods. Thus, modular transformations interchange the various ϑ_i. The transformations at $z = 0$ are especially useful;

let me, then, record them as follows:

$$
\begin{aligned}
\vartheta_1'(0|\tau+1) &= e^{i\pi/4}\vartheta_1'(0|\tau)\,, & \vartheta_1'(0|-\tfrac{1}{\tau}) &= (-i\tau)^{\frac{3}{2}}\vartheta_1'(0|\tau)\,.\\
\vartheta_2(0|\tau+1) &= e^{i\pi/4}\vartheta_2(0|\tau)\,, & \vartheta_2(0|-\tfrac{1}{\tau}) &= (-i\tau)^{\frac{1}{2}}\vartheta_4(0|\tau)\,.\\
\vartheta_3(0|\tau+1) &= \vartheta_4(0|\tau)\,, & \vartheta_3(0|-\tfrac{1}{\tau}) &= (-i\tau)^{\frac{1}{2}}\vartheta_3(0|\tau)\,.\\
\vartheta_4(0|\tau+1) &= \vartheta_3(0|\tau)\,, & \vartheta_4(0|-\tfrac{1}{\tau}) &= (-i\tau)^{\frac{1}{2}}\vartheta_2(0|\tau)\,.
\end{aligned}
\tag{8.29}
$$

Since ϑ_1 vanishes at $z = 0$, I have quoted the result for its first derivative.

In addition to series representations of the form of (8.25), all four theta functions possess infinite product representations, which are derived in Whittaker and Watson. For example,

$$
\vartheta_1(z|\tau) = 2e^{i\pi\tau/4}\sin\pi z \cdot \prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)\left(1-e^{2\pi in\tau}e^{2\pi iz}\right)\left(1-e^{2\pi in\tau}e^{-2\pi iz}\right)\,. \tag{8.30}
$$

The product representations for all four $\vartheta_i(z|\tau)$ are given in the appendix to this section. Eq. (8.30) implies immediately that

$$
\vartheta_1'(0|\tau) = 2\pi\,\eta^3(\tau)\,. \tag{8.31}
$$

The modular transformation formulae for $\eta(\tau)$ quoted in (8.24) follow from (8.29) through this connection.

Let us now return to the bosonic string one-loop amplitude (8.22) and note three important properties of this expression. First, once we have regulated the zero-point energy by the prescription of eq. (8.18), this expression has no further ultraviolet divergences. In the case of the bosonic string, this subtraction is made on an ad hoc basis (though it is in fact necessary to maintain world-sheet conformal invariance), but for the superstring the corresponding zero-point energy term is automatically finite as the result of a cancellation between the bosonic and fermionic degrees of freedom on the world sheet. The expression (8.22) does contain a divergent integral, however, because the integrand blows up exponentially as $\tau_2 \to \infty$. To understand this divergence, let us look more closely at the form of the integrand in that limit. Setting $\tau = 2\pi\tau_2$, we find

$$
A \sim \int^{\infty}\frac{d\tau}{\tau}\left(\frac{e^{-m^2\tau}}{\tau^{26/2}}\right)\,, \tag{8.32}
$$

with $m^2 = -2$. The integrand of (8.22) is just the asymptotic form of the propagator of a Klein-Gordon particle in 26 dimensions. The mass which appears is tachyonic precisely because the state of lowest mass in the spectrum of the bosonic string is a tachyon. We might have suspected from the beginning that the presence of a tachyon in the spectrum would lead to inconsistencies in the theory. It is now clear that this leads to an infrared divergence in the one-loop amplitude. In fact, this term and a similar, weaker singularity due to the dilaton are the only divergences in this quantity.

A second feature of (8.22) is found in its general structure. The complicated part of the expression is the absolute square of an analytic function of the modulus τ. This form arises for a simple reason: The string dynamics treats the left- and right-moving modes separately, and the functional integrals over these modes are respectively analytic and anti-analytic in τ, except for the factors explicitly associated with the zero modes X_0. Belavin and Knizhnik[61] have shown that this decomposition continues to higher loops; the bosonic string amplitude at all higher loops has the form

$$\mathcal{A} = \int d^2\tau^k \; \mu(\tau^k,\bar{\tau}^k) \; |F(\tau^k)|^2 \, , \tag{8.33}$$

where the τ^k are the moduli and $\mu(\tau^k,\bar{\tau}^k)$ is a trivial nonanalytic factor which can be determined directly. This observation has become the basis of a calculational method for higher-loop amplitudes. This set of developments is reviewed in ref. 59.

Finally, let us note the significance of the dependence of the integrand of (8.22) on τ_1. This dependence comes only from exponentials of τ or $\bar{\tau}$. If we expand the integrand of (8.22) in powers of $e^{2\pi i\tau}$ (an expansion clearly valid for τ_2 large), a typical term has the form

$$\int d\tau_1\, d\tau_2 \left(\prod_{n>0} C_{k_n} e^{2\pi i n k_n \tau} \right) \left(\prod_{\bar{n}>0} \overline{C}_{\bar{k}_n} e^{-2\pi i \bar{n}\bar{k}_n \bar{\tau}} \right) \, , \tag{8.34}$$

where $k_n, \bar{k}_n$ are integers and $C_{k_n}, \overline{C}_{k_n}$ are numerical constants. Setting $r = 2\pi\tau_2$ as above, this takes the form

$$\int dr \; e^{-r(\sum nk_n + \sum \bar{n}\bar{k}_n)} \cdot \int d\tau_1 \; e^{2\pi i \tau_1 \cdot (\sum nk_n - \sum \bar{n}\bar{k}_n)} \, . \tag{8.35}$$

The first exponent is the contribution of mode excitations to $L_0 + \overline{L}_0$, and thereby to the mass of the string state. The second exponent contains $L_0 - \overline{L}_0$. The integral over τ_1 is exactly the projector into $L_0 - \overline{L}_0 = 0$. The projection which we found in Section 6 from the geometry of operator insertions in the world sheet is thus also imposed on all virtual particles that can appear in loop amplitudes. It appears precisely because we must sum over all possible world-sheet geometries.

8.3. Closed Superstring

Let us now generalize this calculation to the superstring. The computation of the 0-point one-loop amplitude can be found as the product of two terms. The first of these is the integral over coordinate fields X^μ and anticommuting ghosts on the torus. Except for a change in the number of total space-time dimensions, this factor is calculated exactly as in the previous section. The new ingredient is the functional integral over fermions and commuting ghosts. It is useful to group the fermions ψ^μ in pairs; the contribution of the commuting ghosts cancels that of one of these pairs. The superstring one-loop amplitude then takes the form

$$A = \int \frac{d^2\tau}{\tau_2^2} \left[A_X\right]^8 \left|\left[A_\psi\right]^4\right|^2 , \tag{8.36}$$

where A_X is given by (8.11) and

$$A_\psi = \int D\psi_1 D\psi_2 \, e^{-\int d^2x \, \psi_1(\partial/\partial x_1 + i\partial/\partial x_2)\psi_2} . \tag{8.37}$$

Notice that the exponent of (8.37) is the action of an analytic fermion field. We must square this quantity, as is indicated in (8.36), to account for both left- and right-moving fermion modes.

Just as we did for A_X, we can calculate A_ψ by rewriting it in Hamiltonian form:

$$A_\psi = \operatorname{tr}\left[T(\tau_1)\, e^{-\tau_2 H}\right] . \tag{8.38}$$

Alvarez-Gaumé, Moore, and Vafa[62] have noted that it involves no extra trouble to assign to the fermions arbitrarily twisted periodic boundary conditions:

$$\begin{aligned} \psi_1(z+1) &= -e^{-2\pi i\theta}\psi_1(z) , & \psi_1(z+\tau) &= -e^{-2\pi i\phi}\psi_1(z) , \\ \psi_2(z+1) &= -e^{+2\pi i\theta}\psi_2(z) , & \psi_2(z+\tau) &= -e^{-2\pi i\phi}\psi_2(z) . \end{aligned} \tag{8.39}$$

These boundary conditions are the most general consistent with the requirement that the action in (8.37) be periodic. The factor (-1) in the boundary conditions in time appears automatically from the operator expression $\operatorname{tr}[e^{-\beta H}]$, as in finite-temperature perturbation theory. The corresponding minus sign in the θ boundary condition is included for convenience.

The Hamiltonian corresponding to (8.37) is a sum of fermionic oscillators (that is, two-level systems) corresponding to the Fourier components of $\psi_1(x_1)$, $\psi_2(x_1)$. The Fourier components of ψ_1 have wavenumber $2\pi(n - \theta + \frac{1}{2})$; they are creation or annihilation operators for the fermionic oscillators according to whether this quantity is negative or positive. The corresponding annihilation or creation operators are the Fourier components of ψ_2. The excitation energy for each oscillator is $|n - \theta + \frac{1}{2}|$. With this information, we may easily evaluate $\mathcal{A}_\psi$ for $\tau_1 = 0$, $\phi = 0$:

$$\mathcal{A}_\psi(\tau_1 = 0, \phi = 0) \;=\; \mathrm{tr}\left[e^{-\tau_2 H}\right]$$

$$= e^{-2\pi\tau_2 Z(\theta)} \prod_{n=1}^{\infty} \left(1 + e^{-2\pi\tau_2(n+\theta-\frac{1}{2})}\right)\left(1 + e^{2\pi\tau_2(n-\theta+\frac{1}{2})}\right) . \tag{8.40}$$

The sums run over the ground state and the one excited state of each fermion oscillator. The quantity $Z(\theta)$ is the θ-dependent zero-point energy, which is given by

$$Z(\theta) \;=\; \sum_{n=0}^{\infty} \frac{1}{2}(n - \theta + \frac{1}{2}) . \tag{8.41}$$

With a suitable regularization, this should satisfy the functional relation $Z(\theta) = Z(\theta + 1) + \frac{1}{2}(\frac{1}{2} + \theta)$ and should reduce to the result Z of eq. (8.18) for $\theta = \frac{1}{2}$. The unique function satisfying these properties is

$$Z(\theta) \;=\; \frac{1}{2}\zeta(-1) \;+\; \frac{1}{4}(\frac{1}{4} - \theta^2) \;=\; \frac{1}{2}(\frac{1}{24} - \frac{1}{2}\theta^2) . \tag{8.42}$$

Now τ_1 may be reintroduced by the prescription of eq. (8.19). The phase ϕ associated with a timelike circuit of the torus has a similar effect: It requires each state created by a component of ψ_1 to be rotated by $e^{-2\pi i\phi}$, and each state created by a component of ψ_2 to suffer the opposite rotation. Assembling the pieces, we find at last

$$\mathcal{A}_\psi(\theta,\phi) \;=$$

$$e^{2\pi i\tau[\theta^2/2 - 1/24]} \prod_{n=1}^{\infty} \left(1 + e^{2\pi i\tau(n+\theta-\frac{1}{2})}e^{2\pi i\phi}\right)\left(1 + e^{2\pi i\tau(n-\theta+\frac{1}{2})}e^{-2\pi i\phi}\right) . \tag{8.43}$$

Let us now apply this calculation to the superstring one-loop amplitude. To treat the Neveu-Schwarz and Ramond sections, we must take $\theta = 0$ and $\theta = \frac{1}{2}$, respectively. It is clearly inconsistent with the geometrical invariances of the problem, however, to allow periodic or antiperiodic boundary conditions around one cycle of the torus without allowing the same choice of boundary conditions around the other cycle. The operation of summing over periodic and antiperiodic boundary conditions in time, however, has a very direct physical interpretation:

$$\mathcal{A}_\psi = \operatorname{tr}\left[(1+(-1)^F)\,T(\tau_1)\,e^{-\tau_2 H}\right] , \tag{8.44}$$

where F is the total number of fermions in the state. This is exactly the GSO projection. Once we have insisted that our string theory contain both bosons and fermions (that is, both Neveu-Schwarz and Ramond particles), the GSO projection follows from the sum over all possible world-sheet geometries.

To define the superstring amplitude, then, we must sum over $\theta = 0, \frac{1}{2}$ and independently over $\phi = 0, \frac{1}{2}$. It is possible, and, as we will see, enlightening, to sum coherently over these sectors separately for the left- and right-moving degrees of freedom. To write the amplitude explicitly, we need compact expressions for $\mathcal{A}_\psi(\theta,\phi)$ at the required values of its arguments. Fortunately, evaluating (8.43) at these four points produces exactly the product representations of the four theta functions:

$$\begin{aligned} \mathcal{A}_\psi(\tfrac{1}{2},\tfrac{1}{2}) &= \frac{\vartheta_1(0|\tau)}{\eta(\tau)} = 0 , & \mathcal{A}_\psi(\tfrac{1}{2},0) &= \frac{\vartheta_2(0|\tau)}{\eta(\tau)} , \\ \mathcal{A}_\psi(0,0) &= \frac{\vartheta_3(0|\tau)}{\eta(\tau)} , & \mathcal{A}_\psi(0,\tfrac{1}{2}) &= \frac{\vartheta_4(0|\tau)}{\eta(\tau)} . \end{aligned} \tag{8.45}$$

The analytic part of the fermion contribution to (8.36) then takes the form

$$\left[\left[\frac{\vartheta_2(0|\tau)}{\eta(\tau)}\right]^4 + \left[\frac{\vartheta_4(0|\tau)}{\eta(\tau)}\right]^4 - \left[\frac{\vartheta_3(0|\tau)}{\eta(\tau)}\right]^4 \right] , \tag{8.46}$$

where I have supplied the relative signs so that the form of the expression is preserved as the theta functions transform under modular transformations according to (8.29). Multiplying in the remaining pieces of (8.36), we find as our final expression:

$$\mathcal{A} = \int d^{10}X_0 \int \frac{d^2\tau}{\tau_2^2}\,\frac{1}{(\tau_2)^4} \left| \left(\frac{1}{\eta(\tau)}\right)^{12} \left(\vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau) - \vartheta_3^4(0|\tau)\right) \right|^2 . \tag{8.47}$$

Using (8.24) and (8.29), it is straightforward to show that (8.47) is modular invariant. Notice that an expression of this form could not possibly have been modular invariant if we had not included both sectors with $\phi = 0$ and sectors with $\phi = \frac{1}{2}$.

Two properties of this expression are worth noting. First, let us consider the behavior of the integrand as $\tau_2 \to \infty$. Just as in the bosonic string, the factors of the η function in the denominator lead to a divergence in this limit:

$$\left|\eta(\tau)\right|^{-24} \sim e^{+2\pi\tau} . \tag{8.48}$$

To compare to eq. (8.32), we again set $r = 2\pi\tau$, then this factor is associated with the appearance of a particle of mass $m^2 = -1$, the Neveu-Schwarz tachyon. However, ϑ_2 vanishes as $\tau_2 \to \infty$ and ϑ_3 and ϑ_4 tend to the same value. Thus, the tachyon contribution to (8.36), and its associated divergence, cancels. This is of course the result of the GSO projection; alternatively, this cancellation follows directly from modular invariance.

Actually, the formula (8.47) contains stronger cancellations. Among the theta function identities proved in Whittaker and Watson is the following remarkable result of Jacobi (*"aequatio identicc satis abstrusa"*):

$$\vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau) - \vartheta_3^4(0|\tau) = 0 ! \tag{8.49}$$

The entire expression vanishes. The vacuum energy of the superstring is not renormalized. This is clearly a sign of an underlying space-time supersymmetry in the theory, which requires all of the possible periodic and antiperiodic sectors for its implementation. I will display this supersymmetry more explicitly in the next section.

8.4. Heterotic String

Since we have now derived all of the technology for computing one-loop amplitudes, I would like to complete my discussion by extending this analysis to the case of the heterotic string. This string is obtained by combining the right-moving degrees of freedom of the superstring with the left-moving degrees of freedom of the bosonic string. The spectrum of this theory has been discussed with some care by Michael Green. The massless states are those of 10-dimensional supergravity, together with the gauge bosons and gauginos of an $E_8 \times E_8$ or $O(32)$ gauge theory.

The heterotic string theory contains 10 right-moving coordinate degrees of freedom $X^\mu(z)$ but 26 left-moving coordinates $X^\mu(\bar{z})$. It is easiest to understand how to deal with the 16 left-moving coordinates with no right-moving partners by fermionizing them. This gives a string with 10 ordinary coordinate fields plus 32 left-moving fermions (in addition to the 10 right-moving fermions of the superstring and their associated commuting ghosts). The $E_8 \times E_8$ version of the heterotic string theory is obtained by GSO projecting these 32 fermions in groups of 16. Using the expressions (8.45) for the fermion functional integrals, we can immediately evaluate the required integrals over each set of 8 pairs of fermions and sum these coherently. Multiplying together these two contributions and including the contributions of the remaining coordinates and the right-moving fermions, we find for the value of the one-loop amplitude of the heterotic string

$$\mathcal{A} = \int d^{10}X_0 \int \frac{d^2\tau}{\tau_2^2} \frac{1}{(\tau_2)^4} \left[\frac{1}{\eta^8}\left(\frac{\vartheta_2^8(0|\tau) + \vartheta_4^8(0|\tau) + \vartheta_3^8(0|\tau)}{(\eta(\tau))^8}\right)^2\right] \times \overline{\left[\frac{1}{\eta^8}\left(\frac{\vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau) - \vartheta_3^4(0|\tau)}{\eta^4}\right)\right]}. \tag{8.50}$$

This expression is indeed modular-invariant. Another modular-invariant expression can be obtained from the same ingredients by GSO-projecting all 16 pairs of fermions together; this gives the one-loop amplitude of the $O(32)$ heterotic string. In either case, the amplitude vanishes by virtue of (8.49), that is, by virtue of the space-time supersymmetry of the theory.

It is instructive to obtain the result (8.50) in a different way, by treating the 16 purely left-moving coordinates as bosonic coordinates compactified on a self-dual lattice. It is easy to functionally integrate over such bosons by extending the results of our previous analysis. All that we need to do is to extend the Fourier decomposition (8.13) to allow for configurations of X^μ which are periodic in x_1 only up to translation by a lattice vector $\vec{\ell}$:

$$\vec{X}(x_1) = \vec{X}_0 + \sqrt{\pi}\,\vec{\ell}\cdot(x_1 - x_2) + \sum_{n\neq 0} X_n\, e^{2\pi i n(x_1 - x_2)}. \tag{8.51}$$

The nonzero modes give exactly the same result as before. The zero modes contribution, however, is replaced by a factor which sums the contributions to e^{-S} from each possible value of $\vec{\ell}$. This gives for the full one-loop amplitude

$$\mathcal{A} = \int d^{10}X_0 \int \frac{d^2\tau}{\tau_2^2} \frac{1}{(\tau_2)^4} \left(\frac{\mathbf{F}}{\eta(\tau)^{24}}\right)\cdot \overline{\left[\frac{\vartheta_2^4(0|\tau) + \vartheta_4^4(0|\tau) - \vartheta_3^4(0|\tau)}{\eta^{12}(\tau)}\right]}, \tag{8.52}$$

where

$$\mathbf{F} = \sum_{\ell} e^{i\pi|\vec{\ell}|^2\tau}. \tag{8.53}$$

The symmetry group $E_8 \times E_8$ is obtained by taking each group of 8 left-moving coordinates to be compactified on the lattice generated by the root vectors of E_8. This lattice, which gives the possible values for $\vec{\ell}$, is:

$$\vec{\ell} = \begin{cases} (n_1, n_2, \ldots, n_8)\ , & \sum_i n_i = \text{even integer} \\ (n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2})\ , & \sum_i n_i = \text{even integer} \end{cases} \tag{8.54}$$

The condition that the components n_i sum to an even integer can be implemented by summing over all values of the n_i with the weight $(1 + e^{i\pi \sum n_i})$. If we use this trick, we can write the contribution to $\mathbf{F}$ from the vectors in the first line of (8.54) as

$$\prod_i \sum_{n_i} e^{i\pi n_i^2 \tau} \cdot \left(1 + \prod_i e^{i\pi n_i}\right) = \vartheta_3^8(0|\tau) + \vartheta_4^8(0|\tau); \tag{8.55}$$

I have recognized each term as the series representation of a theta function. Similarly, the vectors in the second line of (8.54) give the contribution

$$\prod_i \sum_{n_i} e^{i\pi (n_i + \frac{1}{2})\tau} \cdot \left(1 + \prod_i e^{i\pi(n_i + \frac{1}{2})}\right) = \vartheta_2^8(0|\tau) + \vartheta_1^8(0|\tau)\ . \tag{8.56}$$

Inserting (8.55) and (8.56) into (8.52), we find again the result (8.50). This nicely checks the equivalence of the fermionic and bosonic forms of the theory.

8.5. Appendix: Representations of Jacobi Theta Functions

In this appendix, I list for your reference the series and product representations of the four Jacobi theta functions which appear in the analysis of this section. These relations are all derived in the textbook of Whittaker and Watson.[60] The series representations are:

$$\begin{aligned} \vartheta_1(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi(n+\frac{1}{2})^2\tau} e^{2\pi i(n+\frac{1}{2})(z-\frac{1}{2})} \\ \vartheta_2(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi(n+\frac{1}{2})^2\tau} e^{2\pi i(n+\frac{1}{2})z} \\ \vartheta_3(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau} e^{2\pi i n z} \\ \vartheta_4(z|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau} e^{2\pi i n(z+\frac{1}{2})}\ , \end{aligned} \tag{8.57}$$

The product representations are:

$$
\begin{aligned}
\vartheta_1(z|\tau) &= 2e^{i\pi\tau/4}\sin\pi z\cdot\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)\left(1-e^{2\pi in\tau}e^{2\pi iz}\right)\left(1-e^{2\pi in\tau}e^{-2\pi iz}\right)\\
\vartheta_2(z|\tau) &= 2e^{i\pi\tau/4}\cos\pi z\cdot\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)\left(1+e^{2\pi in\tau}e^{2\pi iz}\right)\left(1+e^{2\pi in\tau}e^{-2\pi iz}\right)\\
\vartheta_3(z|\tau) &= \prod_{n=1}^{\infty}\left(1-e^{2\pi i\tau}\right)\left(1+e^{2\pi i(n-\frac{1}{2})\tau}e^{2\pi iz}\right)\left(1+e^{2\pi i(n-\frac{1}{2})\tau}e^{-2\pi iz}\right)\\
\vartheta_4(z|\tau) &= \prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)\left(1-e^{2\pi i(n-\frac{1}{2})\tau}e^{2\pi iz}\right)\left(1-e^{2\pi i(n-\frac{1}{2})\tau}e^{-2\pi iz}\right)\,.
\end{aligned}
\tag{8.58}
$$

Many properties of the theta functions can be checked directly from these relations. For example, it is easy to see that $\vartheta_1(z|\tau)$ is odd in z, while the other three functions are even. One can also check the relations

$$
\begin{aligned}
\vartheta_2(z|\tau) &= \vartheta_1(z+\frac{1}{2}|\tau)\\
\vartheta_3(z|\tau) &= e^{i\pi\tau/4}e^{i\pi z}\vartheta_1(z+\frac{1}{2}+\frac{\tau}{2}|\tau)\\
\vartheta_4(z|\tau) &= -ie^{i\pi\tau/4}e^{i\pi z}\vartheta_1(z+\frac{\tau}{2}|\tau)
\end{aligned}
\tag{8.59}
$$

which interconnect the four theta functions and imply the locations for their zeros given in (8.28).

9. Three Grand Questions

I would now like to pause from my development of the formalism of superstring theories to discuss three deep issues which are important to the physical interpretation of the theory. The first two of these concern the properties of the theory in the idealized setting of a flat 10-dimensional background space. We would like to know whether the space-time supersymmetry of which we saw a hint in the previous section is manifested in scattering amplitudes and whether this supersymmetry implies the finiteness of the theory to all orders of loops. The third issue concerns the transition from this idealized setting to a more realistic one, via the compactification of six of these spatial dimensions.

The first of these three questions has a well-defined answer, but one which still leaves many puzzles. What we know about the second and third issues are mostly guesses. The problem of compactification is a very deep one, however, so even our present very tentative knowledge has fueled the development, by Witten and others, of a very beautiful mathematical theory. To present this theory in detail would require another full course, so I will content myself here with a brief description of the simplest scenario. The reader seeking an introduction to the mathematics needed to analyze compactification schemes will find an accessible and quite elaborate presentation in the second volume of the book of Green, Schwarz, and Witten.[6]

9.1. Supersymmetry and Finiteness

Let us first discuss the supersymmetry of the superstring theory, in the covariant formulation that we have studied in Section 7. To verify supersymmetry, we must identify a conserved charge which interchanges space-time bosons and fermions and satisfies the supersymmetry algebra, at least on shell. Once we have identified this object, I will use it to make some intuitive statements about the finiteness of the superstring theory.

Since the supersymmetry charge changes bosons into fermions and vice versa, it must carry the space-time spinor character and the branch cut on the world sheet characteristic of a vertex operator for a state in the Ramond sector. Like a vertex operator, the supersymmetry charge must be BRST invariant. The simplest way to construct a supersymmetry charge is, then, to start with a Ramond vertex operator at zero momentum. Taking (7.68) as a starting point, we would guess:

$$\mathbf{Q}_A = \oint \frac{dz}{2\pi i} V_{-\frac{1}{2}A}(k,z)\Big|_{k=0} = \oint \frac{dz}{2\pi i} S_A e^{-\phi/2} \tag{9.1}$$

It is not difficult to check directly (taking ϵ to be a constant spinor supersymmetry parameter) that

$$[\bar{\epsilon}Q, V_{\frac{1}{2}}(k,z)] = \bar{\epsilon}\gamma^\mu u(k)\,(\partial_z X^\mu + ik\cdot\psi\psi^\mu)\,e^{ik\cdot X}\,, \tag{9.2}$$

and that commutation of $\bar{\epsilon}Q$ with $V_{-\frac{1}{2}}$ gives a picture-changed version of this relation. The right-hand side of (9.2) is just the vertex operator $V_v(k)$ of eq. (7.34). Commuting one step further,

$$[\bar{\epsilon}Q, \varsigma\cdot V_v(k,z)] = \frac{i}{2}(\bar{\epsilon}[\varsigma\cdot\gamma, k\cdot\gamma])^B\,S_B\,e^{ik\cdot X}\,. \tag{9.3}$$

The result is just $V_{-\frac{1}{2}}(k)$. The commutation relations (9.2), (9.3) have exactly the structure of the supersymmetry which links the massless gaugino and vector states of 10-dimensional Yang-Mills theory, except that the double commutation returns a picture-changed version of the original fermion vertex operator rather than that operator itself.

The charge Q_A defined in the previous paragraph apparently is not a true supersymmetry except on shell, where we can identify picture-changed representations of the same state. The algebra of Q_A with itself has the same unsatisfactory feature. To display this algebra most clearly, relabel Q_A as $Q_{-\frac{1}{2}A}$, and define its picture-changed counterpart $Q_{\frac{1}{2}A}$ from $V_{\frac{1}{2}}(k)$. Then, we can compute

$$\begin{aligned}[\bar{\epsilon}Q_{-\frac{1}{2}}, \bar{\epsilon}'Q_{\frac{1}{2}}] &= \left[\bar{\epsilon}\cdot\oint\frac{dz}{2\pi i}\,V_{-\frac{1}{2}}(k=0,z)\,,\quad \bar{\epsilon}'\cdot\oint\frac{dz'}{2\pi i}\,V_{+\frac{1}{2}}(k=0,z')\right]\\ &= \bar{\epsilon}\gamma^\mu\epsilon'\cdot\oint\frac{dz}{2\pi i}\,\partial_z X^\mu\,.\end{aligned} \tag{9.4}$$

On shell, where we can identify $Q_{-\frac{1}{2}}$ and $Q_{\frac{1}{2}}$, this is precisely the supersymmetry algebra

$$[\bar{\epsilon}Q, \bar{\epsilon}'Q] = \bar{\epsilon}\gamma^\mu\epsilon'\cdot P_\mu\,. \tag{9.5}$$

The restriction to mass shell, and thus to physical scattering amplitudes, is an awkward feature of this formalism. In scattering amplitudes, however, either $Q_{\frac{1}{2}}$ or any of its picture-changed variants do implement correctly the constraints of supersymmetry.

The fact that the supersymmetry charge of the string theory appears as a contour integral on the world sheet suggests a general method for proving nonrenormalization theorems following from supersymmetry.

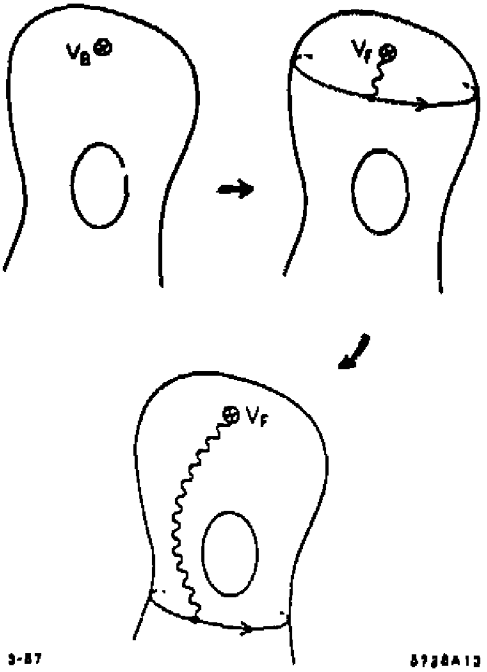

Figure 13. Method for proving supersymmetry nonrenomalisation theorems.

The technique is illustrated in Fig. 13. Let $V_B(z)$ be the vertex operator for some particular boson in the theory. If $V_B(z)$ can be written as a supersymmetry variation of a fermion vertex operator, we can write

$$V_B(z) = [Q_A, V_F(z)] = \oint \frac{dw}{2\pi i}\, q_A(w)\, V_F(z) . \qquad (9.6)$$

If we are computing a loop amplitude, the contour of w is a closed path on a compact 2-dimensional surface. If we can push this contour to the opposite side of the surface and contract it to zero, the expectation value of V_B will vanish. Thus, we can potentially prove to all orders in string perturbation theory the vanishing of the tadpole diagram for the particle created by V_B, as well as the vertex of this particle with any other particles whose vertex operators commute with Q_A. Martinec[63] has argued in this way that the tadpole of the dilaton, and, thus, the cosmological constant, vanishes to all orders in any background that preserves a space-time supersymmetry.

The subtlety in this argument is made clear in the figure: As we pull the operator $q_A(w)$ to the back side of the surface, this operator carries its branch cut with it. Thus, the boundary conditions are changed from periodic to antiperiodic (or vice versa) around any closed loop through which $q_A(w)$ is moved. In the discussion of the one-loop superstring amplitude given in the previous section, we found supersymmetry and vanishing cosmological constant only if we summed coherently over all possible sets of boundary conditions, independently for the analytic and anti-analytic degrees of freedom. Speaking loosely, the geometry of Fig. 13 makes clear that the same prescription is necessary to guarantee that supersymmetry of higher-loop amplitudes. This is already quite an interesting conclusion. However, the exact prescription for the relative phases and normalizations of the various sectors, and the invariance of the resulting sum of amplitudes under the higher-loop generalization of the modular group, has not yet been checked in detail.*

It is not difficult to imagine that the vanishing of the 0-point function might be checked directly by a similar analysis, diagrammed in Fig. 14. Represent one propagator in the diagram as a sum over states in the theory. We know that—on shell—these states form boson-fermion pairs. If the same relation held off-shell, one could represent the fermion states as supersymmetry commutators with bosons, distort the contours as shown, relabel the boundary conditions, and show the explicit cancellation of bosonic and fermionic contributions. I would very much like to know whether this argument is merely a heuristic, or whether it can be made into a rigorous proof of the nonrenormalization of the vacuum energy.

9.2. A Philosophical Digression

After so much formal analysis, it may seem overdue that I begin at last to discuss the relevance of string theories to the observed world of particle phenomena. If you think that this unseemly delay betrays my personal position on string theory, you are right. I ask you, the reader, for your indulgence as I digress to state that position in some detail.

Over the past two years, string theory has been hailed as the solution to all fundamental problems in physics and damned as "recreational mathematics". These extreme positions highlight a situation of great uncertainty about

* A part of this analysis has been presented recently by Atick and Sen.[64]

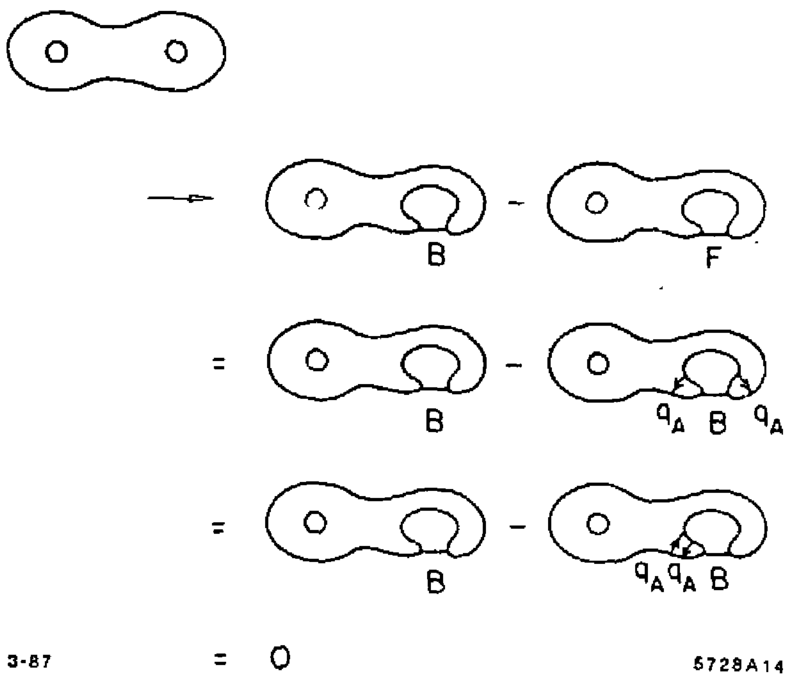

Figure 14. Method for directly proving the vanishing of the 0-point function at higher loops in a supersymmetric string theory, following Ref. 63.

the eventual position of strings in fundamental theory. I believe that the situation in the recent history of particle physics which is most precisely analogous is that of Yang-Mills theory in the early 1960's. At that time, Yang-Mills theory was known to be a profound generalization of Quantum Electrodynamics. Some mathematical methods were known for computing in this theory, but they were quite incomplete. At the same time, the beauty of the theory was already unmistakable, and connections to the observed forces were strongly suggested. Glashow used the Yang-Mills Lagrangian to build a weak-interaction theory which was correct except for the embarrassment of having massless vector bosons. Sakurai used the Yang-Mills Lagrangian as a model for the strong interactions, and postulated the vector mesons ρ, ω, ϕ. In both cases, the phenomenological application of the theory led to significant physical insight. But, on the other hand, the precise and eventually correct application of the principles of non-Abelian gauge symmetry required another ten years and the development of further crucial mathematical methods—the Higgs mechanism, the Fadde'ev-Popov procedure, and the renormalization group.

String theory has now been developing for almost twenty years, but we still seem to be quite far from having a complete mathematical grasp on its formalism. Because our calculational methods are incomplete, it is difficult to assess how precisely strings will finally appear in our understanding of the fundamental forces. Considerably more work must be done to develop the foundations of the theory and to improve its calculational methods. In particular, we desperately need some methods for performing nonperturbative string computations, since many interesting properties of string theories are apparently determined only at the nonperturbative level. At the same time, it will certainly be useful to try to apply our present, incomplete, understanding of string theory to the major phenomenological questions of the day, the problem of the vanishing of the cosmological constant and the origin of quark and lepton generations and their mass spectra. In exploring the relevance of string theory for these problems, however, we should be aware that we might learn more about the mathematics of string theory than (directly) about how these problems are solved. I would therefore favor approaches which involve string computations which are as explicit as possible, even if the phenomenological scenario does not seem sufficiently plausible in itself to justify such detailed analyses.

At the moment, the cosmological constant problem looks as difficult to solve in string theory as it has in any supergravity model. However, the most direct physical interpretations of the string theory have provided some interesting new approaches to the problem of quark and lepton generations. In the remainder of these lectures, I would like to concentrate on approaches to this question which use the string theory in an essential way. I do not know whether the analysis I will present is more important for what it teaches us about quarks and leptons or for what it teaches us about strings; in any event, I hope that these material somehow leads us nearer the goal of finding the role of strings in a unified understanding of Nature.

9.3. Calabi-Yau Ideology

The case that the compatification of 10-dimensional string theory might produce the quarks and leptons directly as string eigenstates was made forcibly just after the discovery of consistent $E_8 \times E_8$ string theories, in a remarkable paper of Candelas, Horowitz, Strominger, and Witten[4] (CHSW). These authors proposed an interpretation of string theory of great mathematical beauty which gives nontrivial answers to many of the basic questions of phenomenology. No discussion of the physical interpretation of strings could be complete without a discussion of this program, so I will review it briefly here. However, because this program allows relatively few explicit computations, and because it is reviewed in great detail in the book of Green, Schwarz, and Witten,[6] I will limit myself to an outline of its contents.

In Section 6, we derived conformal consistency equations for background gravitational fields which might interact with a bosonic string. To leading order in a perturbation theory in the curvature of the background geometry, we found as a criterion for this consistency that the Ricci tensor of the background space should vanish. If one applies this method of analysis to the heterotic string, one finds the same result at first order, but, in the next order of perturbation theory, there appears also a contribution from background Yang-Mills fields.[38–40] For example, one finds

$$c = (d-26) + 6\alpha'\{-\frac{1}{4}R + \ldots\} + (\alpha')^2\left\{\frac{1}{30}\operatorname{tr}(F_{\mu\nu})^2 - (R_{\mu\nu\lambda\sigma})^2\right\} . \quad (9.7)$$

CHSW suggested that one look for a solution to this condition and the other consistency equations with $H_{\mu\nu\lambda} = 0$, $\phi = 0$, and vanishing Ricci tensor. For obvious reasons, they wanted to find a solution in which 6 dimensions were compactified and 4 dimensions were left extended. They also wanted to insure that the compactified theory maintained a 4-dimensional supersymmetry which might survive to energies well below the compactification scale. This requirement is satisfied if the compactified 6-dimensional space possesses a covariantly constant spinor $\eta_A(y)$; then the subgroup of 10-dimensional local supersymmetry generated by the 10-dimensional spinor built up as a product of this object with a 4-dimensional constant spinor

$$\epsilon_A \otimes \eta_A(y) \quad (9.8)$$

remains a symmetry of the compactified theory.

All of these requirements fit together neatly. The equation for the covariant constancy of $\eta_A(y)$ can be manipulated as follows: Begin with

$$\nabla_\mu \eta = 0 \quad \Rightarrow \quad [\nabla_\mu, \nabla_\nu]\eta = R_{\mu\nu\rho\lambda}\Sigma^{\rho\lambda}\eta = 0, \quad (9.9)$$

where $\Sigma^{\rho\lambda} = \frac{1}{4}[\gamma^\rho, \gamma^\lambda]$ is the generator of rotations of spinors. Now contract this equation with γ^ν and break up the resulting object into pieces symmetric and antisymmetric between ν and ρ or λ. The completely antisymmetric piece vanishes because of the Bianchi identity $R_{\mu[\nu\rho\lambda]} = 0$. The remaining pieces involve contractions of indices. Thus (9.9) implies

$$R_{\mu\rho}\gamma^{\rho}\eta = 0 . \tag{9.10}$$

This equation has a nontrivial solution only if $R_{\mu\rho} = 0$. Once we have a covariantly constant spinor, this spinor may be viewed as a preferred direction in the tangent space of the 6-dimensional manifold. On any curved manifold, tangent vectors parallel-transported around closed loops suffer rotations from their original orientations. The group of such rotations is called the *tangent-space group* or the *holonomy group*. For a general 6-dimensional manifold, we would expect that these rotations would fill out the whole of $SO(6)$. Let us recall, though, that $SO(6)$ is isomorphic to $SU(4)$, with the spinor of $SO(6)$ identified with the **4** of $SU(4)$ and the vector of $SO(6)$ with the antisymmetric tensor (**6**) of $SU(4)$. Thus, a covariantly constant spinor may be though of as a preferred orientation in this $SU(4)$ which is not changed by parallel translation. The holonomy group of the manifolds we seek should then be SU(3). Under the decomposition $\mathbf{4} \to \mathbf{1}+\mathbf{3}$, the antisymmetic combination of two **4**'s decomposes as $\mathbf{6} \to \mathbf{3} + \bar{\mathbf{3}}$. This implies that the 6-dimensional coordinate y may be represented as complex conjugate triplets $(y^i, \bar{y}^i)$, $i = 1, 2, 3$; that is, the manifold has a natural complex structure. This geometry of the curvature may be linked naturally to the geometry of gauge fields by identifying the $SU(3)$ holonomy group with an $SU(3)$ subgroup of the gauge group and setting up gauge fields equal to the spin connection $\omega_{\mu\lambda\sigma}$ of the manifold:

$$A_\mu^a T^a = \frac{1}{2}\omega_{\mu\lambda\sigma}\Sigma^{\lambda\sigma} , \qquad \text{or} \qquad F^{\mu\nu a}T^a = \frac{1}{2}R_{\mu\nu\lambda\sigma}\Sigma^{\lambda\sigma} . \tag{9.11}$$

This identification causes the last term in (9.7) to vanish and also solves the remaining constraints on the appearance of a low-energy supersymmetry. Actually, it is now known that a manifold of the form chosen by CHSW does not satisfy the conformal consistency conditions at the fourth order in perturbation theory;[65] however, a small deformation of such a manifold is known to give a solution to any finite order.[66,67]

The problem of finding compactification spaces for the heterotic string is thus reduced to that of constructing 6-dimensional complex manifolds with holonomy group $SU(3)$. The existence of such manifolds is guarranteed by a general theorem conjectured by Calabi[68] and proved by Yau.[69] Let M be a 6-dimensional complex manifold satisfying the additional condition that its metric is (locally) the derivative of a potential

$$g_{i\bar{j}} = \frac{\partial^2}{\partial y^i \partial y^{\bar{j}}} K[y^k, y^{\bar{k}}] . \tag{9.12}$$

Such a manifold is called a *Kähler manifold.* A 6-dimensional complex manifold naturally has $U(3)$ holonomy. We may view the spin connection on $\mathcal{M}$ as a gauge field of $U(3) = SU(3) \times U(1)$. Let the $U(1)$ part of this gauge field have trivial topology, in the sense that the associated field strength, the projection onto this $U(1)$ of $\frac{1}{2} R_{\mu\nu\lambda\sigma} \Sigma^{\lambda\sigma}$, has zero flux through closed surfaces. The theorem then states that the metric on $\mathcal{M}$ may be continuously deformed into a metric with $SU(3)$ holonomy. It should be noted that this result falls far short of an actual construction of the metric for a space of $SU(3)$ holonomy, even when the geometry of the original manifold $\mathcal{M}$ is understood in detail. Manifolds of $SU(3)$ holonomy generally have no isometries, and, in fact, no metrics for such manifolds are known explicitly.

However, the geometry of the original manifold $\mathcal{M}$ in the argument of Calabi and Yau can be used to compute the topological invariants of the manifold with $SU(3)$ holonomy, and these give a certain amount of definite information about the effective theory resulting from compactification. If the fundamental theory contains chiral fermions in a complex representation of the gauge $SU(3)$, an index theorem will relate the number of fermion zero modes to the topology of this gauge field, and so, after the identification of eq. (9.11), to the topology of the manifold itself. Let us take the fundamental theory to be the $E_8 \times E_8$ heterotic string theory and identify the $SU(3)$ gauge field as that of a factor of the maximal subgroup $E_6 \times SU(3)$ of one E_8. Decomposing the adjoint representation of E_8 onto $E_6 \times SU(3)$, we find

$$\mathbf{248} \rightarrow (\mathbf{78},\mathbf{1}) + (\mathbf{27},\mathbf{3}) + (\overline{\mathbf{27}},\overline{\mathbf{3}}) + (\mathbf{1},\mathbf{8}) . \tag{9.13}$$

Then for each zero mode of a $\mathbf{3}$ of $SU(3)$, we find a $\mathbf{27}$ of E_6 surviving to low energies. Evaluating the indices, one finds that the number of $\mathbf{27}$'s and the number of $\overline{\mathbf{27}}$'s are determined topologically;[4] the net number of generations is given by the remarkable formula

$$\mathcal{N}(\mathbf{27}) - \mathcal{N}(\overline{\mathbf{27}}) = \frac{1}{2}\chi , \tag{9.14}$$

where χ is the Euler characteristic of the 6-dimensional compact space.

As an example, CHSW considered the submanifold of CP^4 specified as the solution to

$$\sum_{i=1}^{5} z_i^5 = 0 . \tag{9.15}$$

This space satisfies Calabi's condition and so can be deformed into a manifold with $SU(3)$ holonomy. This manifold has $\chi = -200$. However, the manifold has an isometry group $Z^5 \times Z^5$; identifying points carried into one another by this group gives a multiply-connected manifold with $\chi = -8$. Compactification on this manifold produces 4 generations of quarks and leptons at low energy. One could envision breaking the grand unification symmetry E_6 to a smaller group by adding to the multiply-connected manifold E_6 gauge fields with zero field strength but nonzero values for the Bohm-Aharonov loop integrals $\int dx \cdot A$. Note that we can decrease the rank of the original E_6 only by identifying the generators of the two Z_5's with two mutually noncommuting values of the loop integral. Thus, this class of models generally leads to low-energy gauge groups of rank 5 or 6, that is, with at least an extra $U(1)$ gauge boson in addition to the standard model $SU(3) \times SU(2) \times U(1)$.

A noteworthy, and disturbing, property of the CHSW scheme, is that the solution to the consistency equations exists independently of the value of the radius of the compactified space. This radius is one of a number of parameters of the compactification scheme which are apparently not determined at all by the consistency equations. A second such parameter is the expectation value of the dilaton field $\phi(x)$. This apparently obscure field has great physical importance: Because the dilaton field appears in an overall prefactor in the effective action (eg., eq. (6.34))

$$S_{\rm eff} = \int dx\sqrt{G}\, e^{-2\phi}\{R + \ldots\} , \tag{9.16}$$

it can be absorbed into the effective gauge and gravitational couplings. Thus, it is the dilaton vacuum expectation value which determines dynamically the values of these couplings. Unfortunately, Dine and Seiberg[70] have argued that, if these two parameters are not determined at the string tree level, they are not determined to any finite order in string perturbation theory. Even more unfortunately, the failure of our present calculational methods to determine these parameters seems to apply not only to this particular scheme of compactification but to more general schemes as well. Apparently, though some consequences of string compactification are readily computed, others will be understood only with a major improvement of our mathematical methods.

10. Orbifolds

In the previous section, we discussed the scheme of Candelas, Horowitz, Strominger, and Witten for compactifying string theory from 10 to 4 dimensions. I sketched the elegant mathematical arguments offered by these authors, which led to the suggestion that the compact spatial dimensions form a Calabi-Yau manifold, a complex manifold with $SU(3)$ holonomy. However, this conclusion was in some sense disappointing. The geometries of the Calabi-Yau manifolds are not known explicitly, and the problem of solving a nonlinear sigma model with one of these manifolds as the target space seems extremely difficult, even if the exact conformal invariance of the system allows some simplifications. How could we improve this situation?

One response is to start from the other extreme position by exploring compactification schemes which are known to be tractable and adding complication until they become as realistic as possible. The simplest compactification which is not completely trivial is given by taking the compact space to be a torus. As we saw in the discussion of the heterotic string, the problem of a string moving on a torus is not appreciably more difficult than that of a string in extended space. The nonzero frequency modes of the string have the same spectrum in these two cases. The center-of-mass coordinate is modified in two ways: First, the momentum is quantized in the compactified directions. Second, the string may have additional *soliton* states which wind around the torus, corresponding to the boundary conditions

$$x(\sigma = 0) = x_0 , \qquad x(\sigma = 1) = x_0 + \vec{\ell} , \tag{10.1}$$

where $\vec{\ell}$ is a displacement by periods of the torus which is identified with 0. The set of displacements $\vec{\ell}$ identified with 0 in this compactification forms a lattice

$$\mathcal{L} = \{ \sum_i n_i \vec{\ell}_i , \quad n_i \in \mathbf{Z} \} , \tag{10.2}$$

and we may alternatively describe the torus as the coset space of Euclidean space divided by this lattice: $T^d = \mathbf{R}^d/\mathcal{L}$. This scheme of compactification is excessively simple; in particular, it cannot break the original gauge symmetry, since all of the zero-mass bosons of the uncompactified theory are still present as states in the $\vec{\ell} = 0$ sector. Worse, the solition sectors can sometimes contribute additional zero-mass particles. This is, after all, the way most of the $E_8 \times E_8$ bosons arise in the heterotic string theory.

In order to break $E_8 \times E_8$, and in order to break 10-dimensional supersymmetry down to 4-dimensional supersymmetry, we must deform the torus in a way that will eliminate some of these massless states. Dixon, Harvey, Vafa, and Witten[6] (DHVW) introduced an elegant method for accomplishing this: They proposed that one identify points on the torus which are taken into one another by a group Γ of discrete symmetries of the lattice $\mathcal{L}$. The resulting space may be though of as the coset space T^d/Γ, or, equivalently, the quotient of $\mathbf{R}^d$ by the crystallographic group of translations by elements of $\mathcal{L}$ plus rotations and reflections in Γ. In general, some points of the torus will be fixed under the action of Γ. The identifications will then tie up the neighborhoods of these points into complicated singularities. Thus, this space is not a manifold; DHVW refer to it as an *orbifold*.

DHVW show, quite remarkably, that the singularities of the orbifold can simulate many aspects of string propagation in curved background spaces while retaining most of the simplifying features of string dynamics on a torus. In this sense, compactification on an orbifold represents an ideal compromise between calculability and realism. These compactification schemes are therefore worthy of our close attention. In this section, I would like to present a simple example of an orbifold and to compute the low-energy modes of local fields and of strings arising from this compactification.

10.1. Orbifold Geometry

To compactify the heterotic string, we need a 6-dimensional orbifold.* Let me begin, however, by discussing the geometry of a simple 2-dimensional orbifold. By taking 3 copies of this space, we will find a 6-dimensional space which will serve as my main illustrative example.

Begin, then, with the torus obtained by identifying points in the plane connected by the translations of a 2-dimensional triangular lattice. This torus may be pictured as the space obtained by identifying opposite edges of the equilateral parallelogram, with internal angles 60° and 120°, as shown in Fig. 15. Now reduce this space by making a further identification of points related by the group Γ of 120° rotations about the origin. This prescription identifies, for example, the left-hand edge and the bottom edge of the parallelogram, as shown. The origin is a fixed point under the rotation. As a consequence of this, the origin becomes a conical singularity of the orbifold, since lines through the origin at 120° to one another are glued together by the identification.

* More general schemes of which share the advantages of orbifold compactification have recently been presented in refs. 71–74.

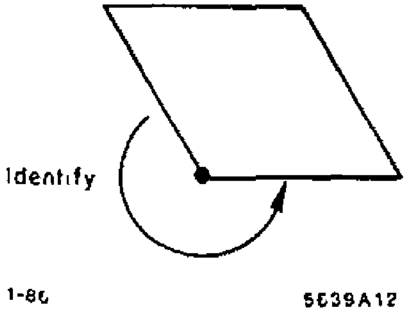

Figure 15. A simple 2-dimensional orbifold.

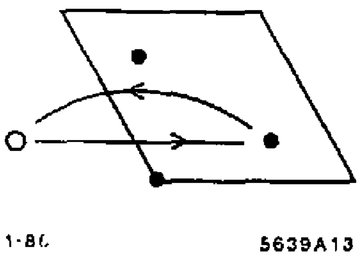

Figure 16. Three fixed points of the orbifold of Fig. 15.

Fig. 16 shows that the points

$$x^1 + ix^2 = \frac{1}{\sqrt{3}} \cdot e^{\frac{i\pi}{6}} , \qquad x^1 + ix^2 = \frac{1}{\sqrt{3}} \cdot i \tag{10.3}$$

on the long diagonal are also fixed under the combined action of rotations and lattice translations. These two points also become conical singularities of the orbifold.

It is not difficult to solve the field equations of scalar fields on the torus subject to the identification of points connected by the group Γ. The treatment of local fields with spin, however, is more subtle, since the identification subjects such fields to nontrivial boundary conditions. When we identify the left and bottom edges of the parallelogram in Fig. 15, we also identify the vectors tangent to these edges. Thus, at a point P on the bottom edge identified with a point P' on the left edge, a vector field $\vec{V}(x)$ must obey the boundary condition

$$\vec{V}(P') = \Re(120^\circ) \cdot \vec{V}(P) , \tag{10.4}$$

where $\Re(120^\circ)$ denotes a rotation through 120°. Spinor fields, and field of more general spin, obey a similar boundary condition with the rotation matrix in the appropriate representation of the rotation group.

In string theory, the situation becomes even more involved.

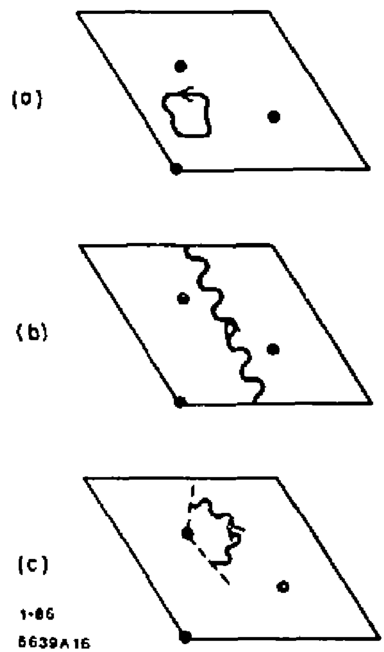

Figure 17. Three possible configurations for closed strings on an orbifold: (a) trivial; (b) winding; (c) twisted.

A closed string may describe a configuration of any of the three types shown in Fig. 17. The first two cases appear already in the dynamics of strings on a torus. The third case is new: A closed string may run from one point on the torus to a second point identified with the first under Γ. Strings of this last type are called *twisted states*. In general, the boundary conditions of twisted strings may be represented as a rotation about one of the fixed points. For the configuration shown in Fig. 17(c), if we represent the 2-dimensional variable as $x = x^1 + ix^2$ and call the fixed point x_0, the boundary condition is

$$\left(x(\sigma = 1) - x_0\right) = e^{2\pi i/3}\left(x(\sigma = 0) - x_0\right) . \qquad (10.5)$$

This condition implies the Fourier decomposition

$$x(\sigma) = x_0 + \sum_{q=n+\frac{1}{3}} x_q e^{2\pi i q \sigma} , \qquad (10.6)$$

where n is an integer. The twisted states thus differs from states of the other two types in having fractional-integer quantization of their nonzero modes.

DHVW describe an orbifold as the singular limit of a manifold. However, it is worth noting some circumstances where is correspondence does not literally hold. First of all, a manifold with highly concentrated curvature would have very large curvature tensor, and (in any reasonable coordinates) a singularity in its metric, at the point of concentration. This would produce a large nonlinear term in the sigma model constructed with the manifold as its target space. The orbifold string theory does not include this nonlinear term, and cannot, because this term would ruin the exact solubility of the model. This prescription seems to be a perfectly consistent one.

The second subtlety comes in describing the topology of the orbifold. It is often straightforward to compute the Euler characteristic χ of the smooth manifold which the orbifold appproximates. For the example of Fig. 15, that computation goes as follows: One must recall from (5.8) that a torus has Euler characteristic 0 and that (for an appropriate definition of the boundary contribution) a disc, being half a sphere, has Euler characteristic 1. Beginning from the torus of Fig. 15, remove disks about each of the fixed points, indentify triples of points under Γ, and then restore disks at the fixed points which smooth the conical singularities. The Euler characteristic of the final surface is

$$\chi = \frac{1}{3}\left(0 - 3\right) + 3 = 2 ; \qquad (10.7)$$

that is, the smoothed orbifold is topologically a sphere. Given this value of the Euler characteristic, one might be tempted to use an index theorem to predict the number of zero modes of a massless fermion field on the orbifold. However, it is possible that some of these zero modes might shrink to zero size and disappear in the singular orbifold limit. We will see an example of the disappearance and reappearance of zero modes later in this section.

10.2. FIELDS ON THE Z-ORBIFOLD

To understand more concretely the mechanics of fields on orbifolds, let us solve explicitly, in a specific example, for the modes of local fields which correspond to massless particles after compactification. A simple example to choose is a 6-dimensional space called by DHVW the *Z-orbifold*. It is built from the direct product of three of the equilateral tori shown in Fig. 15. Call the complex coordinates on these three spaces (x_1, x_2, x_3). Identify points connected by the simultaneous rotation of all three tori:

$$(x_1, x_2, x_3) \rightarrow (e^{2\pi i/3}x_1, e^{2\pi i/3}x_2, e^{2\pi i/3}x_3) . \tag{10.8}$$

Consider first the dynamics of a local field ϕ with no gauge charges. Massless particles after compactification will correspond to solutions of the field equations which can consistently have zero momentum in the extended directions; these will be zero-mode solutions of the wave equation for that field on the 6-dimensional surface. Since the 6-dimensional space is a product of flat spaces, a zero mode solution will, quite generally for any spin, be a solution to

$$\partial_x \phi = 0 , \quad \text{or} \quad \partial_{\bar{x}} \phi = 0 \tag{10.9}$$

which is normalizable on the compact space. The only solution to these equations is given by taking ϕ to be constant.

If ϕ represents a scalar field, the constant mode of ϕ is a perfectly acceptable zero mode. However, if ϕ has spin, we must impose the nontrivial boundary condition which arises from identifying tangent vectors as be identify points. For the case at hand, the boundary condition on ϕ may be represented quite generally as

$$\phi(P') = \Re_1(120°) \cdot \Re_2(120°) \cdot \Re_3(120°) \cdot \phi(P) , \tag{10.10}$$

where the three rotations are made in the planes of the three tori. This boundary condition is compatible with ϕ being constant only if $\Re_1 \cdot \Re_2 \cdot \Re_3 = 1$. This condition has very few nontrivial solutions. Consider, for example, the case in which ϕ is a 10-dimensional vector field. Decompose ϕ^M as follows, chosing complex components for the compactified dimensions:

$$\phi^M = (\phi^\mu, \phi^{5+i6}, \phi^{5-i6}, \phi^{7+i8}, \phi^{7-i8}, \phi^{9+i10}, \phi^{9-i10}) . \qquad (10.11)$$

The components ϕ^μ, $\mu = 1, \ldots, 4$, coorresponding to vector components which point into the uncompactified directions, are scalars with respect to the rotations $\Re_1, \Re_2, \Re_3$. Thus each of these components will have a zero mode, and these four zero modes form a Lorentz 4-vector after compactification. On the other hand, the component ϕ^{5+i6} picks up a nontrivial phase $\Re_1 = e^{2\pi i/3}$, while $\Re_2 = \Re_3 = 1$ on this component. Thus, this field has no zero mode. A similar conclusion holds for the remaining components of (10.11).

A similar analysis may be carried through for a 10-dimensional spinor field. To do that, it will be useful to digress briefly and set up a bit of formalism for spinors in 10 dimensions. Let us choose a representation of the 10-dimensional (Minkowski) Dirac matrices similar to that displayed in (7.54):

$$\begin{aligned}
\gamma^\mu_{(10)} &= \gamma^\mu_{(4)} \otimes 1 \otimes 1 \otimes 1 \\
\gamma^{5\pm i6}_{(10)} &= \gamma^5 \otimes \sqrt{2} i\sigma^\pm \otimes 1 \otimes 1 \\
\gamma^{7\pm i8}_{(10)} &= \gamma^5 \otimes \sigma^3 \otimes \sqrt{2} i\sigma^\pm \otimes 1 \\
\gamma^{9\pm i10}_{(10)} &= \gamma^5 \otimes \sigma^3 \otimes \sigma^3 \otimes \sqrt{2} i\sigma^\pm ,
\end{aligned} \qquad (10.12)$$

where $\gamma^\mu_{(4)}$ are 4-dimensional Dirac matrices and γ^5 is the usual 4-dimensional chirality. The 10-dimensional chirality is given by

$$\Gamma = \gamma^1\gamma^2 \ldots \gamma^{10} = \gamma^5 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \equiv \Gamma_4\Gamma_6 . \qquad (10.13)$$

In this basis, the 10-dimensional charge conjugation operator, which implements $C(\gamma^M)C^{-1} = -(\gamma^M)^T$, is

$$C = \gamma^0\gamma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 . \qquad (10.14)$$

Charge conjugation of a spinor is the operation

$$\Psi \to (\overline{\Psi} C)^T ; \qquad (10.15)$$

this preserves Γ, so in 10 dimensions it is possible to define spinors which are simultaneously Majorana (self-charge-conjugate) and Weyl ($\Gamma = +1$). Note, however, that (10.15) flips the parity of both Γ_4 and Γ_6. Thus, a 10-dimensional Majorana-Weyl spinor has the structure:

$$\Psi = \begin{pmatrix} \psi \\ (\overline{\psi}C)^T \end{pmatrix} \qquad \begin{matrix} \leftarrow \ \Gamma_4 = +1 \quad \Gamma_6 = +1 \\ \leftarrow \ \Gamma_4 = -1 \quad \Gamma_6 = -1 \ , \end{matrix} \tag{10.16}$$

where the components ψ are chirality-projected. This whole object is determined by its $\Gamma_4 = \Gamma_6 = +1$ part. Thus, the modes of Ψ which correspond to massless fermions in the extended 4-dimensional space may be built by attaching a constant 4-dimension positive-chirality spinor to a Dirac zero mode with $\Gamma_6 = +1$ on the compact 6-dimensional space to form the object ψ in (10.16).

Let us now search for these Dirac zero modes on the orbifold specified by (10.8). In the basis we have chosen for the Dirac matrices, the three rotation operators in (10.10) take the form

$$\begin{aligned} \Re_1 &= 1 \otimes e^{i\pi\sigma^3/3} \otimes 1 \otimes 1 \\ \Re_2 &= 1 \otimes 1 \otimes e^{i\pi\sigma^3/3} \otimes 1 \\ \Re_3 &= 1 \otimes 1 \otimes 1 \otimes e^{i\pi\sigma^3/3} \ . \end{aligned} \tag{10.17}$$

The boundary condition which must be satisfied is

$$\psi = (\pm 1)\Re_1 \cdot \Re_2 \cdot \Re_3 \cdot \psi \ , \tag{10.18}$$

where the factor (± 1) reflects the usual ambiguity that we may choose periodic or antiperiodic boundary conditions for fermions. I would like to choose the sign (-1), for reasons that will be made clear in a moment. There are four possible choices for $\Gamma_6 = +1$ spinors:

$$\begin{aligned} ((\sigma^3)_1, (\sigma^3)_2, (\sigma^3)_3) &= (+1,+1,+1) & \Re_1\Re_2\Re_3 &= -1 \\ &= (+1,-1,-1) & &= e^{-i\pi/3} \\ &= (-1,-1,+1) & &= e^{-i\pi/3} \\ &= (-1,+1,-1) & &= e^{-i\pi/3} \ . \end{aligned} \tag{10.19}$$

Only the first of these satisfies the boundary condition (10.18).

The zero mode that we have just identified is a covariantly constant spinor on the orbifold. Thus we should expect that, if we started with a supersymmetric theory in 10 dimensions, compactification on this space should yield a 4-dimension effective theory with supersymmetry. To check this in the present example, begin with a 10-dimensional matter supermultiplet consisting of a vector and a Majorana-Weyl spinor (A^M, Ψ). We have seen that compactification on this orbifold produces, as massless modes after compacitification, one 4-dimensional vector and one 4-dimensional Majorana spinor. This is exactly the content of a 4-dimensional supermultiplet consisting of a gauge boson and a gaugino.

Let us now alter this construction to mimic more features of the Calabi-Yau compactification discussed in the previous section. In particular, I would like to add to the space an $SU(3)$ gauge field equal to the spin connection. The only aspect of the spin connection on the orbifold which is actually visible is its integral around a closed curve, the nontrivial rotation (10.10) which connects identified points. We may introduce a nontrivial gauge field by introducing a gauge rotation around the same closed curve. Let us assume that a **3** of $SU(3)$ carried between identified points picks up a the Bohm-Aharonov phase $e^{i\int dx\cdot A} = e^{-2\pi i/3}$. Then a field in a general representation of $SU(3)$ will acquire the phase

$$\mathcal{G} = (e^{-2\pi i/3})^t , \tag{10.20}$$

where t is the *triality* of the representation ($t = 1$ for **3**, $t = -1$ for $\overline{\mathbf{3}}$, $t = 0$ for **1**, **8**). With this addition, the boundary conditions (10.10), (10.18) should be generalized to

$$\phi = (-1)^F \Re_1 \cdot \Re_2 \cdot \Re_3 \cdot \mathcal{G} \cdot \phi . \tag{10.21}$$

In this equation, the rotation matrices depend on the spin of the particle, the factor $\mathcal{G}$ depends on the $SU(3)$ properties, and $F = 1$ for a spinor field. It is actually this enhanced space that DHVW define to be the Z-orbifold.

For fields in the **1** or **8** of $SU(3)$, the analysis of zero modes we carried out earlier goes through unchanged. However, for fields in the **3** or $\overline{\mathbf{3}}$, there are interesting modifications. Consider first the case of spinor fields. Adding the phase $\mathcal{G}$ to the results of (10.19), we see that the constant spinor solutions of the types $(+--)$, $(-+-)$, and $(--+)$, for spinor fields in the **3**—but not the $\overline{\mathbf{3}}$—satisfy (10.21) and thus give massless particles of the compactified theory. In addition, the components of a 10-dimensional vector ϕ^{5+i6}, ϕ^{7+i8}, ϕ^{9+i10} corresponding to fields in the **3** now satisfy (10.21). These give massless scalar fields in 4 dimensions. Zero modes for the $\overline{\mathbf{3}}$ can be built from the remaining vector components and from negative-chirality spinors; these form the antiparticles of the scalars and spinors from the **3**'s. Thus, a 10-dimensional supermultiplet (A^M, Ψ) in the

3 of $SU(3)$ produces 3 4-dimensional supermultiplets, each of which contains a chiral spinor and a complex scalar (and their antiparticles).

If we use the Z-orbifold to compactify a 10-dimensional local field theory with $E_8 \times E_8$ gauge symmetry, according to the program outlined in the previous section, we find chiral fermion generations and accompanying E_6 gauge fields. According to the decomposition (9.13) of the adjoint representation of E_8, fermions in the **3** of the embedded $SU(3)$ are **27**'s of the accompanying E_6 group. The zero modes we have identified correspond to a set of massless particles arising from compactification which contain one supermultiplet each of gauge bosons and gauginos for the **78** of E_6 and the **8** of $SU(3)$, plus 3 supermultiplets of spinors and scalars in the representation (**27**, **3**) of $E_6 \times SU(3)$. These latter multiplets give 9 generations of quarks and leptons.

It is worth inquiring how this number compares to the Euler characteristic of the orbifold. Candelas, Horowitz, Strominger, and Witten [4] computed the Euler characteristic χ of the corresponding smooth manifold in the following way: The original product of tori has $\chi = 0$. The Z-orbifold has $3 \times 3 \times 3 = 27$ fixed points. Remove a small sphere about each of these fixed points; this gives an object of $\chi = -27$. Divide by Z_3, and then repair each hole by inserting an object with $\chi = 3$. The final surface has $\chi = 72$, by the analysis of CHSW, it should produce 36 generations. Unfortunately, we have only found 9 of these. The others must disappear from local field theory in the limit in which the smooth manifold becomes a singular orbifold.

10.3. Strings on the Z-Orbifold

The analysis of zero modes which I have presented for local fields on the Z-orbifold can be repeated for strings on the Z-orbifold in the topologically trivial sector of Fig. 17. The same set of zero-mass states is reproduced. However, in string theory, we have available also the nontrivial sectors shown in Fig. 17. These can give rise to additional zero modes which have no simple interpretation in local field theory. Particularly interesting are the twisted sectors, whose existence is unique to orbifold compactifications. As a final exercise for this course, then, I would like to explore the spectrum of the twisted sectors of string theory on the Z-orbifold, in order to identify any further massless states which arise there.

For simplicity, I will perform this analysis in the light-cone gauge. The position of the ground state will be determined by the zero-point energy of the system of oscillators. The excited states will be raised above this ground state by ladder operators; as we have seen in eq. (10.6), these operators should have a fractional offset in their quantization. A crucial ingredient in this analysis will be the formula for the zero point energy of a set of such shifted oscillators given by (8.42). We may rewrite this result as:

$$
\begin{aligned}
Z(\alpha) &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(n+\alpha) \\
&= \left(\frac{1}{48} - \frac{1}{4}\left(\alpha - \frac{1}{2}\right)^2\right) = \left(-\frac{1}{24} + \frac{1}{4}\alpha(1-\alpha)\right) .
\end{aligned}
\tag{10.22}
$$

The second term in the second line displays the shift of the zero-point energy from the situation of periodic boundary conditions on the string. The first term in this line displays the shift of the zero-point energy from the situation of antiperiodic boundary conditions.

As a first step, let us recall the way in which the $E_8 \times E_8$ quantum numbers of the 10-dimensional heterotic string arise from a fermionic representation of the left-moving degrees of freedom. This analysis will give the massless states represented in the partition function of eq. (8.50). In a light-cone quantization, the left-moving sector of this string contains 8 bosonic and 32 fermionic fields. The fermions should be divided into two groups of 16 fields, each of which will produce the quantum numbers of one E_8. Let us refer to the two E_8 groups as E_8^b and E_8^u, to indicate the groups which will be *broken* and *unbroken* after compactification. I will also label the corresponding fermions as $(\psi^b)^i$, $(\psi^u)^i$.

In the uncompactified string theory, these fermions may have simple periodic or antiperiodic boundary conditions around the string. Taking each choice of boundary condition for each set of fermions, we find 4 sectors. The zero-point energy of each sector can be computed by summing values of $Z(\alpha)$ given in (10.22). (We must recall that fermionic oscillators give a negative contribution.) For example, in the sector in which the ψ^b fermions have Neveu-Schwarz (here, antiperiodic) boundary conditions and the ψ^u fermions have Ramond (periodic) boundary conditions, the total zero-point energy of 8 bosons and two sets of 16 fermions is:

$$
8 \cdot \left(-\frac{1}{24}\right) + 16 \cdot \left(+\frac{1}{24}\right) + 16 \cdot \left(-\frac{1}{48}\right) = 0 . \tag{10.23}
$$

The zero-point energies of the four sectors are:

$$
\begin{aligned}
(NS)^b &\otimes (N-S)^u &\rightarrow\quad Z &= -1 \\
(R)^b &\otimes (NS)^u &\rightarrow\quad Z &= 0 \\
(NS)^b &\otimes (R)^u &\rightarrow\quad Z &= 0 \\
(R)^b &\otimes (R)^u &\rightarrow\quad Z &= +1
\end{aligned}
\tag{10.24}
$$

The last sector produces only massive states. The middle two sectors produce spectra which begin at the massless level. The ground states are spinors of the $O(16)$ associated with the spin operator construction using the 16 pairs of fermions with Ramond boundary conditions. After the GSO projection, these become chiral spinors. These two sectors then contribute massless states in the representation $(\mathbf{128},\mathbf{1})+(\mathbf{1},\mathbf{128})$ of $O(16)\times O(16)$. The first sector of (10.24)produces a spectrum which begins at a tachyonic level. (There are no tachyons in the spectrum of the complete string because there are no tachyonic states in the right-moving sector to satisfy the condition $L_0 = \overline{L}_0$.) The massless states in this sector are of the states

$$(\psi^b)^i_{-\frac{1}{2}}(\psi^b)^j_{-\frac{1}{2}}|\Omega\rangle \ , \qquad (\psi^u)^i_{-\frac{1}{2}}(\psi^u)^j_{-\frac{1}{2}}|\Omega\rangle \ , \tag{10.25}$$

The states $(\psi^b)^i_{-\frac{1}{2}}(\psi^u)^j_{-\frac{1}{2}}|\Omega\rangle$ are removed by the separate GSO projections for b and u. The states of (10.25) form a $(\mathbf{120},\mathbf{1})+(\mathbf{1},\mathbf{120})$ of $O(16)\times O(16)$. Each $\mathbf{120}$ assembles with the corresponding $\mathbf{128}$ to form the $\mathbf{248}$, the adjoint represetation, of an E_8. The right-moving sector endows these states with the space-time quantum numbers of a 10-dimensional matter supermultiplet (A^M,Ψ).

Now we can easily describe the generalization of this construction to a twisted sector of the heterotic string compactified on the Z-orbifold. We have already noted in eq. (10.6) that, in a twisted state on this orbifold, the bosonic oscillators corresponding to compactified dimensions have their quantization shifted by $\frac{1}{3}$ or $\frac{2}{3}$ unit. We might describe this by saying that the 8 transverse coordinates X^i, which form an 8 of the transverse rotational symmetry $O(8)$, break up into the representation $(\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\overline{\mathbf{3}})$ of the new transverse tangent-space group $O(2)\times SU(3)$. We can represent the identification of the gauge connection with the spin connection by assigning 6 of the fermions $(\psi^b)^i$ also to transform as a $(\mathbf{3}+\overline{\mathbf{3}})$ under $SU(3)$, and to be similarly offset in their quantization by $\frac{1}{3}$ unit. This assignment breaks the fermionic interchange symmetry $O(16)$ to $O(10)\times SU(3)$.

Let us now recompute the zero point energies of the four sectors. The result will differ from (10.24) only because of the shifts of the contributions from oscillators with fractional offset. These can be read from eq. (10.22). For example, in the sector with Neveu-Schwarz boundary conditions for both b and u, the shift due to replacing 6 $\alpha = 0$ bosons by $\alpha = \frac{1}{3},\frac{2}{3}$ bosons and replacing 6 $\alpha = \frac{1}{2}$ fermions by $\alpha = \frac{1}{6},\frac{5}{6}$ fermions is given by

$$\Delta Z \ = \ 6\cdot\frac{1}{18} \ + \ 6\cdot\frac{1}{36} \ , \tag{10.26}$$

so that the total zero-point energy is now $(-\frac{1}{2})$. For the four possible sectors, we find

$$
\begin{array}{llllll}
(NS)^b & \otimes & (N-S)^u & \rightarrow & Z = -\frac{1}{2} & \\
(R)^b & \otimes & (NS)^u & \rightarrow & Z = 0 & \\
(NS)^b & \otimes & (R)^u & \rightarrow & Z = \frac{1}{2} & \\
(R)^b & \otimes & (R)^u & \rightarrow & Z = +1 &
\end{array}
\tag{10.27}
$$

The two sectors with $Z > 0$ contain no massless states. The second sector listed has a spectrum which begins at zero mass; the massless states form a spinor of $O(10)$, the **16**. The first sector has states at the zero-mass level of the form

$$
\begin{array}{ll}
(\psi^b)^k_{-\frac{1}{2}} \, |\Omega\rangle \ , & k = 1, \ldots, 10 \\
(\psi^b)^a_{-\frac{1}{6}} \, (\psi^b)^b_{-\frac{1}{6}} (\psi^b)^c_{-\frac{1}{6}} \, |\Omega\rangle \ , & a, b, c = 1, 2, 3 \ ;
\end{array}
\tag{10.28}
$$

these form a $(\mathbf{10} + \mathbf{1})$ of O(10). These three representations assemble into a **27** of E_6. The left-moving sector supplies for this multiplet the space-time quantum numbers of a 4-dimensional supermultiplet consisting of a spinor plus a scalar. Each twisted sector of the Z-orbifold compactification contains another E_6 generation of quarks and leptons.

Since there are 27 fixed points on the Z-orbifold, we find 27 new generations. It is worth noting that this accounts precisely for the difference noted earlier between the topological estimate of the number of quark and lepton generations and the number of such generations found as zero modes in the local field theory limit. Apparently, the elements of topology which shrink to points and become invisible as the manifold collapses to an orbifold are still visible to strings propagating on the surface. DHVW argue that this correlation between the Euler characteristic of the smoothed manifold and the counting of generations in the string theory on the orbifold holds under quite general conditions.

Let me conclude with a few comments about the 3-point couplings of the massless particles we have constructed by compactifying strings on orbifolds. These couplings have a direct importance for phenomenology, since, if we can identify two of the massless scalars as the Higgs bosons and a set of the massless fermions as the known quarks and leptons, these couplings are exactly the Yukawa couplings of the low-energy theory which are responsible for generating the fermion masses. In orbifold compactifications, it is possible to compute these couplings explicitly, at least as a perturbation expansion in string loops.

The analysis of these Yukawa couplings has been worked out in some detail for the case of the Z-orbifold.[75-77] The couplings of the 9 multiplets in the trivial sector can be found directly from the corresponding local field theory, by inserting the explicit form of the zero-mode solutions into the standard gauge coupling

$$\int \delta\mathcal{L}_{(10)} = \int d^{10}x\, \overline{\psi}\, A\!\!\!/\psi\,. \tag{10.29}$$

The couplings of multiplets in the twisted sector have no direct interpretation in terms of local fields, but they can be computed as the amplitude for a world-sheet process in which three twisted strings join and disappear into the vacuum. Dixon, Friedan, Martinec, and Shenker[77] have developed methods for evaluating these amplitudes explicitly by generalizing the technology used for computing expectation values of spin operators.

Fig. 18 shows two qualitatively different processes which give rise to these 3-point couplings. In Fig. 18(a), three twisted strings at the same fixed point join and contract in a classically allowed process which obviously has a large amplitude. In Fig. 18(b), three twisted strings at different fixed points join in a process which involves virtual string propagation across the orbifold. This process involves intermediate string configurations with $\partial_\sigma X \sim R$, where R is the physical size of the orbifold. Thus, the coupling arising from this process should be suppressed by a semiclassical factor[76,77]

$$e^{-S} \sim e^{-cR^2/\alpha'}\,. \tag{10.30}$$

This result is noteworthy for two reasons. First, we have seen in Section 6 that α'/R^2 is effectively the coupling constant of the 2-dimensional nonlinear sigma model which represents the string world-sheet dynamics. Apparently, orbifold calculations can represent effects nonperturbative in this coupling constant, and such effect may be qualitatively important to the string physics. Second, and much more importantly, this effect offers a plausible mechanical interpretation for the magnitudes of Yukawa couplings. If R is at all large compared to the natural string scale, we find that these couplings form a natural hierarchy.

What an unusual and bizarre, but also rich and wonderful, picture of the quark and lepton generations emerges from string compactification! Surely we have much to learn from our further exploration of this theory.

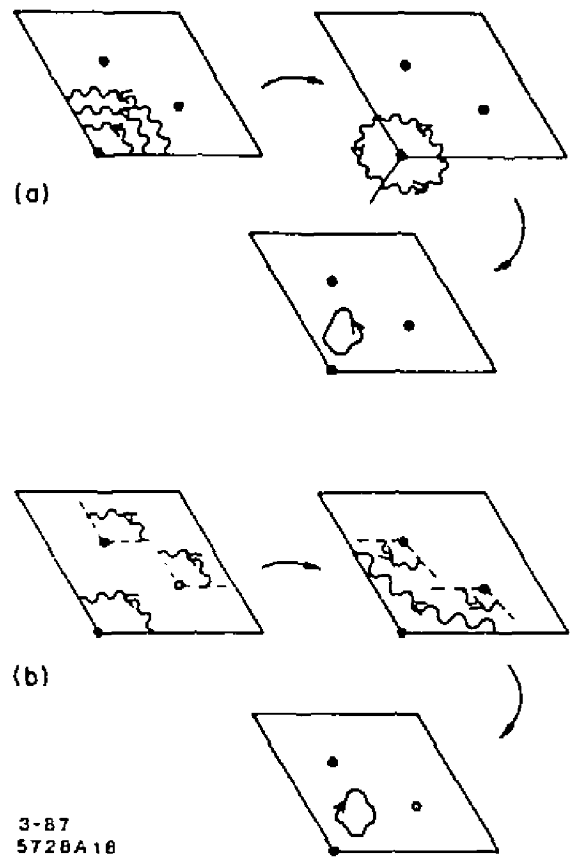

Figure 18. Two world-sheet processes which give rise to 3-point couplings of twisted states in an orbifold compactification.

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