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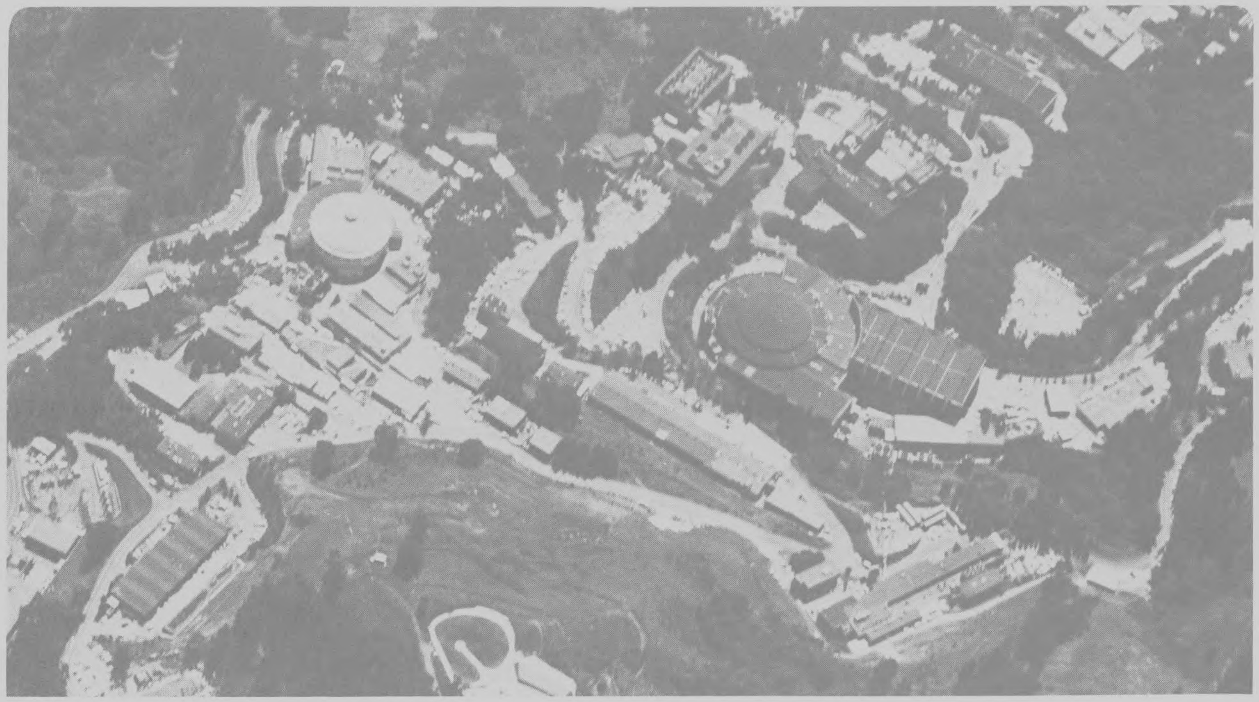

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Symmetries on the Space of (2,2) Superstring Vacua and Automorphism Groups of Calabi-Yau Manifolds *

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Abstract

Symmetry groups on the space of (2,2) string vacua for $c = 3, 6, 9$ are discussed in the context of orbifoldized Landau-Ginzburg theories. A general method for finding the maximal symmetry groups on the moduli space of untwisted marginal operators is presented, by studying symmetries on the resolution of isolated singularities of superpotentials. Stabilizing subgroups of such symmetry groups are shown to correspond with automorphism groups of Calabi-Yau manifolds. In addition to our earlier work on this subject we present some new examples for $c = 9$ (2,2) vacua. Subsequently we discuss modular transformations that relate small volume target-spaces to large ones.

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1. Introduction.

In a perturbative formulation of string theory one considers string backgrounds as two-dimensional conformal field theories (for review see [1]). One refers to such string backgrounds as string *vacua*. For space-time supersymmetric string theories it turns out that the vacua correspond to $N = 2$ superconformal field theories (SCFTs) [2, 3]. The (2,2) vacua that will be considered in this paper, are those for which both left- and right-handed SCFTs have $N = 2$ world sheet SUSY.

A property used in the study of the space of possible string vacua is the possibility to deform a given vacuum continuously. One calls the coupling parameters of such continuous deformations the *moduli* of the string vacuum. Thus, in general, to each CFT a moduli space of continuous deformations is attached. This space is denoted by $\mathcal{M}$. By construction, each deformation of the vacuum will lead to a geometrically different target space. For example, in a circle compactification one may change continuously the radius, R, corresponding to just one real modulus.

An important aspect of the moduli spaces of string vacua is that they respect a number of surprising symmetries. An example of such a symmetry is the well known duality symmetry in circle compactifications, which relates a circle with radius R to one with radius $1/R$ [4]. This duality is an automorphism of the conformal field theory, which gives rise to an isomorphism of two models with *distinct* underlying geometry. The duality symmetry was generalized to d-dimensional ($d > 1$) toroidal compactifications [5, 6, 7, 8, 9] and to toroidal orbifolds [10, 11, 12]. In these cases, the symmetry groups are non-abelian, and contain elements relating small volumes target spaces to large ones. Such 'modular' transformations are important in constructing an effective low energy field theory [7, 13, 14, 12, 15] and possible applications to cosmology [16].

So far, however, there is no systematic way to classify these symmetry groups. In [17] an approach to this problem is discussed using a description of (2,2) vacua in terms of 'orbifoldized' $N = 2$ Landau-Ginzburg (LG) models (this approach was used later also in [18]). [1] It turns out that such models can be used to describe (2,2) vacua corresponding to target Kähler manifolds with vanishing first Chern class [19, 20, 21, 22, 23, 24, 25, 26]. These LG models are characterized by superpotentials which are weighted homogeneous polynomials with isolated singularities. Using the mathematical theory of isolated singularities one may obtain insight in the structure of the symmetry groups mentioned above.

In these proceedings we review the technique discussed in [17] and discuss some new examples for $c = 6$ and $c = 9$ compactifications. In addition to this, we discuss transformations that relate small volume compactifications with large volume compactifications. Before we discuss our results in detail, we review some of the properties of the (2,2) vacua that we will consider in this work. For most of the notation and general background on LG theories relevant for (2,2) vacua we refer to the reviews [22, 26].

The LG models that we will consider are characterized by superpotentials $W(X_i)$ that are quasi-homogeneous functions with isolated singularities. The weight q_i of the scalar chiral superfield X_i is its $U(1)$-charge at the critical point. Examples of such superpotentials are

$$W = \sum_{i=1}^{n} X_i^{l_i}, \tag{1.1}$$

where $l_i = 1/q_i$. Such superpotentials correspond to Gepner models of the type $\prod_{i=1}^{n} k_i$, where $k_i = l_i - 2$ is the level of a minimal $N = 2$ SCFT using the A-type modular invariant [3].

Let us next briefly recall some of the properties an $N = 2$ SCFT should have in order to serve as a vacuum of string theory. Essential for a string-like interpretation are two conditions which come from the relation between the left- and right-handed properties of string states and their $U(1)$-charges. String states have holomorphic and antiholomorphic (z and $\bar{z}$ dependent) parts. We denote the chiral holomorphic and chiral anti-holomorphic states by (c, c). Their anti-chiral partners, which are obtained by complex conjugation in *field space*, are denoted

[1] The study of the symmetries in the $c = 3$ cases was done already in [27, 28], although in an opposite way: the known symmetries in the $(2, 2)$ $c = 3$ toroidal-orbifolds were used in order to find the symmetries of the $c = 3$ LG moduli.

by (a, a). In addition there are (a, c) states, and their conjugated partners (c, a). In order to serve as a vacuum of the superstring the $U(1)$ charges in the NS-sector must be integral. Furthermore, for a space-time interpretation the theory should have a central charge which is a multiple of three.

The condition that the central charge be a multiple of three permits one to derive an algebraic equation describing the target space. It arises in the path integral of the LG theory as delta-function constraint, $\delta(W)$, evaluated in a weighted projective space, WCP^{n-1}. [2] This procedure is described in detail for the deformations of the models (1.1) in [20, 22], where it is claimed that this hypersurface is a candidate target space on which the string propagates. It is also possible to factor out the target manifold by a discrete automorphism group. Such orbifoldizing (not to be confused with the first orbifoldization of the LG theory) of the target space gives rise to a different string vacuum.

The addition of marginal deformations to a given vacuum, corresponds to a change in the geometry of the target space. A superstring vacuum has two different types of moduli; those associated with complex structure deformations and those associated with the Kähler structure deformations. Naively, the complex structure deformations are given by the marginal operators of the (c, c) ring. The marginal deformations in the (c, c) ring come from the (1,1) operators, where the notation denotes the left- respectively right-handed $U(1)$-charge, and describe perturbations of the superpotential. Some of the moduli in the (c, c) ring come from the untwisted sector, and correspond to a perturbation of the superpotential, while others come from the twisted sector of the orbifoldized LG model [25]. The Kähler structure deformations are represented by the moduli coming from the (a, c) ring. These operators always come from the twisted sectors of the orbifoldized LG theory.

The picture in which the (c, c) moduli correspond to complex structure deformations, while the (a, c) moduli correspond to Kähler structure deformations can be interchanged, that is, we can view the (c, c) moduli as deformations of

[2] The identification $X_i = e^{2\pi i q_i} X_i$ which give rise to the weighted projective space amounts to a twisting or 'orbifoldizing' of the original LG model by a product of cyclic groups. This twisting turns out to correspond to the generalized GSO-projection [25].

the Kähler structure (of a possibly different target space)[29, 30, 31]. This interchanging is a consequence of a symmetry $q_L \to -q_L$ of the $N = 2$ SCFT, where q_L denotes the left-handed $U(1)$-charge. This symmetry plays an important role in the study of symmetries on the moduli space of (2,2)-superstring vacua arising from orbifoldized LG models. In the general scheme we will describe this symmetry will be used to treat the discrete groups as acting on the Kähler structure moduli.

This concludes our brief review of the basics of LG theories that define (2,2) vacua at the critical point. In the next section we discuss the general scheme used in [17] to find physical symmetries on the moduli space of a given $N = 2$ LG theory. By construction, the symmetries found are those acting on the submoduli of (c, c) untwisted deformations. We will give some new examples in the c=6 (K3) and c=9 (CY) cases. In order to simplify the discussion we will further restrict to the surviving (c,c) untwisted moduli of an orbifoldized CY manifold. [3] Subsequently, in section 3 we will discuss the technique from a more geometrical point of view, leading to the conclusion that stabilizing sub groups of the symmetry groups correspond to automorphism groups of the target space, i.e. of Calabi-Yau (CY) manifolds. Finally in section 4 we comment on an application of our result involving a symmetry between small volume compactifications and large volume compactifications.

[3]The compactification on the quotient of a CY manifold by a discrete group is interesting as it reduces the number of generations.

2. Symmetries on the Untwisted (c,c) Moduli Space.

In this section we review the algorithm described in [17] for constructing symmetry groups acting on the moduli space of a given $N = 2$ LG model. After we discussed the general technique, we will present some new examples in the $c = 6$ and $c = 9$ case. For simplicity we discuss the deformations of a model of the type (1.1), described by the superpotential

$$W(X, a) = \sum_{i=1}^{n} X_i^{l_i} + \sum_{j=1}^{m} a_j \Phi_j(X), \tag{2.1}$$

where Φ_j is a (c, c) primary field of charge (1,1) (and therefore of super-dimension (1/2,1/2)); m is the number of such fields in the chiral ring and a_j is a complex parameter. We call the space of couplings $a = (a_1, ..., a_m)$ 'the a-moduli space'.

The a-moduli space is a subspace of the full moduli space of geometrically different target-spaces of the string theory. The physical symmetries of this subspace correspond to generalized duality transformations which relate target spaces which differ geometrically but for which the physical theory is the same.

Such symmetry transformations can be studied by performing certain field redefinitions of the chiral superfields X_i (and their complex conjugates). The idea is to look for those field redefinitions for which the kinetic term in the $N = 2$ LG action remains a Kähler potential, and for which the effect on the superpotential can be expressed as a transformation involving only the parameters a, i.e. $W(X, a)$ is changed to $W(X, a')$ (up to an overall factor). As the physical theory is left unchanged, we conclude that the points a and a' in the moduli space are physically equivalent. We will call such a transformation $a \to a'$ a 'modular' transformation. In principle one may obtain in this way all the generators of such 'modular' transformations and the group thus generated will certainly be a subgroup of the full symmetry group acting on the moduli space of $N = 2$ SCFTs.

Let us explain this idea in more detail. For a given LG theory, consider the transformation

$$X_i'^{n_i} = U_{ij} X_j^{n_j}. \tag{2.2}$$

The $U(1)$ symmetry in the $N=2$ superconformal algebra restricts the powers n_i to be such that the charges $q(X_k^{n_k})$ are equal for all k. Under the field redefinition the kinetic term K is transformed to K', which must be a Kähler potential. Any non-singular linear transformation on the fields X_i, mixing only fields of the same charge, changes K to a new Kähler potential. [4] The universality class of the flow of the kinetic term is preserved since the operator corresponding to the difference $K' - K$ is an irrelevant perturbation. Thus, we are interested in transformations of the form

$$X_i' = U_{ij} X_j; \qquad \overline{X}_i' = \overline{U}_{ij} \overline{X}_j, \tag{2.3}$$

where U is a non-singular matrix, and the indices i,j run over all chiral scalar superfields with the same $U(1)$ charge. Some of the solutions for U are diagonal matrices consisting just of phases

$$U_{ij} = \delta_{ij}\alpha_i, \qquad \text{s.t. } (\alpha_i)^{l_i} = 1, \tag{2.4}$$

(there is no summation on the index i).

In another simple case, U is a permutation matrix, permuting different superfields X_i with themselves. The property of the phase and permutation transformation is that whatever the superpotential in (2.1) is (i.e., no matter what the values of a_j are), the kinetic term and the part of W describing the Gepner model are invariant. Only the moduli parameters, a, are transformed into a'. In the case where U is of the form (2.4), the parameters a' are related to a by phases. The point $a=0$ is a fixed point of U. Such symmetries of the Gepner's models were described in reference [32]. In the permutation case, the a' parameters are related to the original ones by permutations. Even though the permutation symmetry is obvious in the superpotential framework, its physical consequences are not trivial. For example, a permutation may relate a scale transformation or a deformation of the complex structure, on a toroidal orbifold, with a transformation which blows up orbifold singularities [17].

[4] For example, if one chooses $K = \sum_{i=1}^{n} \overline{X}_i X_i$, it is transformed to $K' = \sum_{i,j} (U^\dagger U)_{ij} \overline{X}_i X_j$ which is Kähler. In principle, there might be non-linear field redefinitions such that K' is a Kähler potential, but we do not consider those here.

Other solutions for U are possible if we start at a point in the moduli space which is a Gepner model of the type k^N. [5] For that let us discuss the conditions on the matrices U coming from the requirement that the transformed superpotential $W(X')$ corresponds to an untwisted (c,c) deformation of the original Gepner model. The deformed superpotential of the Gepner model k^N is given by (1.1,2.1)

$$W(X,a) = \sum_{i=1}^{N} X_i^{k+2} + \sum_{j=1}^{m} a_j \Phi_j. \tag{2.5}$$

Let us perform a linear transformation on the superfields X_i. In order to study symmetries on the a-moduli, we should make sure that the new superpotential, $W'(X,a')$ has the same form as the original W, i.e.

$$W(X',a) = W'(X,a') = C(a)(\sum_{i=1}^{N} X_i^{k+2} + \sum_{j=1}^{m} a'_j \Phi_j) \tag{2.6}$$

This means the the deformations in W' are still described by marginal operators from the chiral ring. The factor $C(a)$ can be eliminated by rescaling the superfields.

What are the conditions one gets on the entries of U? Expressing X'_i in $W(X')$ in terms of the superfields X_i, gives N terms of the form X_i^{k+2}, $i = 1, ..., N$, with coefficients $A_i(U,a)$. We also obtain $N(N-1)$ terms of the form $X_i^{k+1} X_j$, $i \neq j$, with coefficients $A_{ij}(U,a)$. The coefficients of all the terms in the new superpotential are functions of the entries of the matrix U and of the original moduli parameters, a. In order to satisfy (2.6), the coefficients $A_i(a)$ should be equal to $C(a)$. This preserves the part of the superpotential describing the Gepner model. Hence we have at most $N-1$ independent equations coming from this condition. Furthermore, we impose that the coefficients $A_{ij}(U,a)$ will vanish. This guarantees that we only get terms which appear in the (c,c) ring, and gives rise to $N(N-1)$ equations. All together, one finds N^2-1 complex equations:

$$\begin{aligned} A_i(U,a) = A_{i+1}(U,a), &\quad i = 1, ..., N-1; \\ A_{jk}(U,a) = 0, \quad j \neq k, &\quad j,k = 1, ..., N. \end{aligned} \tag{2.7}$$

[5]The discussion can be generalized to deformations of a model $\prod_i k_i^{N_i}$ by considering each factor $k_i^{N_i}$ separately.

Let us next determine some of the properties that the matrix U should have. The matrix U can be written as $U = \gamma U'$, where γ is a complex number and $\det(U') = 1$. The number γ gives an irrelevant factor γ^{k+2} between W' and K. Thus, we are left with the $N^2 - 1$ complex parameters of the matrix U'. The equations in (2.7) are not linear equations, so one does not get in general a unique solution; a set of discrete solutions is possible. In the rest of this section we give examples of non-trivial solutions (in addition to the phase and permutation ones) in the $c = 6$ and $c = 9$ cases.

2.1. An Example in the $c = 6$ case.

The (2,2) string vacua, for which $c = 6$, correspond geometrically to either a complex two-dimensional torus or a K3-surface (which is also a projective complex two-dimensional surface). We will discuss the K3 compactification. The space of K3 compactifications, viewed as string backgrounds, is simply connected and isomorphic to the homogeneous space $O(20,4)/(O(20)\times O(4))$. There is a discrete group of symmetries acting on this space which is conjectured to be isomorphic to $O(20,4,\mathbf{Z})$ [33]. We would like to consider the 'physical' moduli space obtained by modding out this discrete symmetry group. The $c = 6$ Gepner models form a finite set of points on that space. They are fixed by the phase transformations of the type (2.4), but they are transformed in general to different points under the action of other elements in $O(20,4,\mathbf{Z})$. However, this action will in general destroy the invariants of the singularity of the superpotential, hence dividing out this group from the moduli spaces is not very useful in the context of LG models.

As a first step, therefore, we want to divide by a sub-group which preserves the invariants of the singularity. There are two different types of elements in such a group. The first type leaves the superpotential invariant for any a, although it acts non-trivially on $O(20,4)/(O(20) \times O(4))$ (analogous to $\Gamma(2)$ and $\Gamma(3)$ in the $c = 3$ case [27, 17]). Such elements generate a sub-group closely related to the monodromy of the singularity. A discussion on those physical symmetries is presented in section 3. Below we will discuss an explicit example concerning the symmetries which act nontrivially on the a-moduli.

Consider a family of K3 surfaces parametrized by the superpotential

$$\begin{aligned} W(X_1,X_2,X_3) = \\ & X_1^6+X_2^6+X_3^6+a_1(X_1^3X_2^3+X_1^3X_3^3+X_2^3X_3^3) \\ & +a_2X_1^2X_2^2X_3^2+a_3(X_1^4X_2X_3+X_2^4X_1X_3+X_3^4X_1X_2). \end{aligned} \tag{2.8}$$

The 6-dimensional (c,c) moduli space in (2.8) is the space which is invariant under the twist group generated by the elements $\mathrm{diag}(1,\alpha,\alpha^2)$ and P_{123} (a cyclic permutation). It can be thus considered as the untwisted moduli of the K3 surface modded out by the above twist group. The transformation

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix} = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{2.9}$$

where $\alpha^3=1$, takes $W(X_1,X_2,X_3,a)$ to $W'(X_1',X_2',X_3')$ which is explicitly

$$\begin{aligned} W' = & X_1^6+X_2^6+X_3^6 \\ & +\left(\frac{90+9a_1-6a_2+15a_3}{3+3a_1+a_2+3a_3}\right)(X_1^4X_2X_3+X_2^4X_1X_3+X_3^4X_1X_2) \\ & +\left(\frac{60-21a_1+2a_2+6a_3}{3+3a_1+a_2+3a_3}\right)(X_1^3X_2^3+X_1^3X_3^3+X_2^3X_3^3) \\ & +\left(\frac{270+27a_1+9a_2-18a_3}{3+3a_1+a_2+3a_3}\right)X_1^2X_2^2X_3^2, \end{aligned} \tag{2.10}$$

up to an overall irrelevant factor. We thus arrive at the following conclusion: The LG model described by the superpotential (2.8) is equivalent upon the field redefinition (2.9) to the model with superpotential (2.10).

The complete group of symmetries acting on the a-moduli is generated by field redefinitions of the type U_P (permutations), U_T (phases) and U_S (presented in (2.9)). The group of symmetries which leaves the (a_1,a_2,a_3) moduli invariant is isomorphic to $\mathcal{T} \ltimes \mathcal{S}_3$, where $\mathcal{T}$ is the tetrahedral group and $\mathcal{S}_n$ is the symmetric group of n elements. The point $(a_1,a_2,a_3)=(-10,0,0)$ is a fixed point, and thus the group $\mathcal{T} \ltimes \mathcal{S}_3$ corresponds to an automorphism group of the K3 surface defined at this point. We will elaborate on this in section 3.

The transformation of the type U_S plays a special role, as it relates finite parameters a to infinite ones. Combined with the other symmetries, and motivated by an explicit study in the $c = 3$ cases [27, 17], we conjecture that such symmetries relate small volumes of the target-space to large ones. This will be discussed in section 4.

2.2. Examples for which $c = 9$

For $c = 9$ (2,2) vacua, the target space is conjecturally either a torus or a Calabi-Yau (CY) manifold [3]. The moduli space of CY compactifications is disconnected. That is, we consider a possible equivalence on the boundary of each simply connected component as disconnected, so each simply-connected component defines a family of CY manifolds with the same topology. In the following we will describe two examples of the symmetries acting on the (c, c) untwisted moduli which survive the action of a twist group G_t. (Other examples involving phase and permutation symmetries on the untwisted sub-moduli space surviving the action of a twist group on a CY manifold are presented in [18]).

Consider the (c, c) untwisted deformations of the LG theory corresponding to the Gepner model 6^4, which are invariant under $G_t = \mathbf{Z}_2^2 \ltimes \mathbf{Z}_2^2$ generated by the elements: diag$(-1, 1, 1, 1)$, diag$(1, 1, -1, 1)$, P_{12} and P_{34}, where P_{ij} is a permutation of i and j. The surviving a-moduli space is 24-dimensional, and the superpotential is given by

$$\begin{aligned} W = {} & X_1^8 + X_2^8 + X_3^8 + X_4^8 \\ & + a_1 X_1^2 X_2^2 (X_1^4 + X_2^4) + a_2 X_1^2 X_2^2 X_3^2 X_4^2 + a_3 X_1^2 X_2^2 (X_1^2 + X_2^2)(X_3^2 + X_4^2) \\ & + a_4 X_1^2 X_2^2 (X_3^4 + X_4^4) + a_5 X_3^2 X_4^2 (X_3^2 + X_4^2)(X_1^2 + X_2^2) \\ & + a_6 X_3^2 X_4^2 (X_3^4 + X_4^4) + a_7 X_3^2 X_4^2 (X_1^4 + X_2^4) \\ & + a_8 (X_1^2 + X_2^2)(X_3^2 + X_4^2)(X_1^4 + X_2^4) + a_9 (X_1^2 + X_2^2)(X_3^2 + X_4^2)(X_3^4 + X_4^4) \\ & + a_{10} (X_1^4 + X_2^4)(X_3^4 + X_4^4) + a_{11} X_3^4 X_4^4 + a_{12} X_1^4 X_2^4 . \end{aligned} \tag{2.11}$$

The a-moduli space in (2.11) is invariant under the action of a symmetry group generated by G_t, $P_{13}P_{24}$, $\mathrm{diag}(\sqrt{i},\sqrt{i},1,1)$ and S, where

$$S=\frac{1}{\sqrt{c(a)}}\begin{pmatrix} s & 0 \\ 0 & c(a)I \end{pmatrix}; \qquad s=\frac{1}{\sqrt{-2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $c(a)$ is an a dependent factor which is chosen such that W' contains $X_1^8+X_2^8+X_3^8+X_4^8$ up to an overall factor. (If one extends G_t to include also the element $P_{13}P_{24}$, then on the invariant 14-dimensional a-moduli space $c(a)=1$). The presence of the transformation of the type S is again an evidence for symmetries relating small volumes of the target-space to large ones. This transformation relates finite parameters a to divergent ones.

The second example we consider is the group of symmetries acting on the a-moduli of deformations of the model $1\cdot 16^3$, which is invariant under $G_t = Z_3 \ltimes Z_3$ generated by $\mathrm{diag}(1,\alpha,\alpha^2)$ ($\alpha^3=1$)and P_{123} acting on the superfields X_i, $i=1,2,3$, corresponding to the 16^3 factor. The invariant untwisted (c,c) moduli space is 46-dimensional. The deformed superpotential is given by

$$\begin{aligned} W &= Y^3+X_1^{18}+X_2^{18}+X_3^{18} \\ &+a_1A^6+a_2A^5B+a_3A^4B^2+a_4A^4C+a_5A^4Y+a_6A^3B^3 \\ &+a_7A^3BC+a_8A^3BY+a_9A^2B^2C+a_{10}A^2B^2Y+a_{11}A^2B^4+a_{12}A^2CY \\ &+a_{13}AB^5+a_{14}AB^3C+a_{15}AB^3Y+a_{16}ABC^2+a_{17}ABCY \\ &+a_{18}B^4C+a_{19}B^4Y+a_{20}B^2C^2+a_{21}B^2CY+a_{22}C^3+a_{23}C^2Y, \end{aligned} \tag{2.12}$$

where

$$A=X_1X_2X_3, \qquad B=X_1^3+X_2^3+X_3^3, \qquad C=X_1^3X_2^3+X_1^3X_3^3+X_2^3X_3^3.$$

The group of symmetries acting on the a-moduli in (2.12) is isomorphic to $(\mathcal{S}_3 \ltimes \mathcal{T})\times \mathbf{Z}_3$. The $\mathbf{Z}_3$ corresponds to a phase redefinition of the field Y ($Y\to\alpha Y$); the subgroup $\mathcal{S}_3$ corresponds to permutations of the fields X_i; the tetrahedral group is generated by the $\mathbf{Z}_3$ phase redefinition $\mathrm{diag}(\alpha,1,1)$, and by the element

$$S=\frac{1}{\sqrt{-3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix}.$$

The transformation S acting on the superfields X_i should be followed by a rescaling of the field Y, $Y \rightarrow c(a)Y$. The a dependant factor $c(a)$ is chosen such that W' contains $Y^3 + X_1^{18} + X_2^{18} + X_3^{18}$ up to an overall factor. It is again remarkable that the transformation S relates finite parameters a to infinite ones.

The general scheme can be used to study the symmetries on the moduli of Gepner models taken in any type of modular invariance. For example, we studied the symmetries on the moduli space of the model $1 \cdot 16_E^3/(\mathbf{Z}_3 \ltimes \mathbf{Z}_3)$. The label E means that each $k = 16$ factor is in the type E_7 modular invariance. The twist group $\mathbf{Z}_3 \ltimes \mathbf{Z}_3$ is chosen such that the effective number of generations is 3 [34], and thus the models are phenomenologically interesting. Among the symmetries, we find one which takes small volumes to big ones. It acts on a sub-space of the CY manifold, which is isomorphic to a complex torus, in a similar way as in the flat case. A detailed study of this example will be presented in [35].

3. Symmetry Groups on the a-Moduli Space from Singularity Theory.

It is the aim of this section to show that the theory of resolving singularities may be of use for a more general study of the symmetries on the moduli space of (2,2)-string vacua. Although much of the techniques we will discuss are in the context of $c = 6$ theories, they are in fact more general and could be applied, at least in principle, to the $c = 9$ theories as well.

In the first subsection we will apply some ideas of singularity theory to obtain a useful description of the a-moduli space of the untwisted (c, c) deformations of a given LG theory. For this we will closely follow [17]. Subsequently, we will use this description to discuss the symmetry groups introduced in the previous section from a general point of view. In particular we will show that sub groups of the symmetry groups discussed in section 2 which stabilize certain deformations correspond to automorphism groups of CY manifolds. In the case the CY-manifold is a K3 surface we may use a classification theorem of automorphism groups of these surfaces, to identify (large) symmetry groups of certain (2,2)-vacua.

3.1. The moduli space of untwisted (c, c) deformations of a given LG-theory.

Let us explain briefly the description of the moduli space of untwisted (c, c) deformations. The moduli space of $c = 6$ (2,2) vacua which is formed by the untwisted marginal deformations of a LG model can be described as a symmetric sub-space, of the homogeneous space $\mathcal{M} = O(20, 4)/(O(4) \times O(20))$, modded out by a discrete symmetry group which contains the monodromy group Γ. The group Γ is a topological invariant of the singularity of the superpotential in the LG model, so it is invariant under arbitrary smooth deformations of the superpotential. This can be used to reformulate any smooth deformation of the superpotential in terms of the marginal operators that appear in the (c, c) ring. The monodromy group gives rise to a fundamental domain for the marginal deformations of the superpotential, on which the symmetry group meant in section 2 acts.

To see this explicitly we will concentrate on a particular set of singularities summarized in tables 1 and 2 that are relevant in the $c = 6$ case. Tables 1-2 give a complete list of all the isolated singularities which can be compactified to a K3 surface in $\mathbf{C}^3$ which is described by a single polynomial equation involving only four variables (see e.g.[36, 37]). Henceforth we will restrict to LG superpotentials that consist of only four fields (we exclude $c = 6$ models defined as tensor products $c = 3$ models, which correspond to toroidal orbifolds). The singularities in table 1 are the so-called *exceptional modality-one* singularities in the classification by Arnold [38]. For both tables the central charge of the corresponding LG-theory is $\frac{N+2}{N}$. The chiral ring of these LG theories all contain one field with dimension greater than one and none of dimension equal to one. In other words: the spectrum of these LG theories does not contain marginal operators, i.e. there are no physical moduli in any of these theories. However, there exists a mathematical construction, called *the compactification* of the singularity which gives rise to new LG-theories which *do* contain physical moduli. The compactification of the singularities in tables 1-2 lead to LG-theories that correspond at the critical point to $N = 2$ SCFTs on K3 surfaces. In particular, one may obtain in this way all Gepner models that are known to be defined on K3 surfaces, including type $D-$modular invariants. They are thus labelled by some of the polynomials in tables 1-2.

Since the compactification of a singularity is tightly related to its resolution we will explain first some generalities on resolutions of isolated simple singularities focussing on the role of the monodromy group. Let us denote the polynomial defining the singularity by $f(X_1, X_2, X_3)$. To *resolve* or *unfold* a singularity means to change f analytically in 'all possible ways', so that the singularities that will appear as a result of this are of a simpler nature. Such analytical deformations preserve the (analytical) structure of the singularity. For example, consider the polynomial

$$f = X_1^2 + X_2^3 + X_3^7 \tag{3.1}$$

which is the modality-one singularity K_{12} in table 1. A priori there is no reason to expect that a generic resolution will correspond to an operator that appears in the chiral ring of the superpotential. However, a result in [39] on resolving isolated singularities states that *all* possible deformations of a given superpotential can always be written in terms of linear combinations of operators that appear in the chiral ring. Thus, for the example in (3.1) it is enough to consider

$$f_a = X_1^2 + X_2^3 + X_3^7 + \sum_{\substack{i=0,\, j=0,1 \\ k=0,\cdots 5}} a_{ijk} X_1^i X_2^j X_3^k, \tag{3.2}$$

which depends on 12 complex parameters and corresponds to elements in the chiral ring of the LG model with superpotential (3.1).

It is customary [40] to describe the above resolution in terms of the hypersurfaces Σ_a defined as

$$\Sigma_a = \{(X_1, X_2, X_3) \mid f_a = 0\}\,, \qquad \Sigma_0 \equiv f_0 = 0. \tag{3.3}$$

The union of all of these surfaces constitutes a fiber space, denoted as

$$\pi : \Sigma \longrightarrow S \tag{3.4}$$

and turns out to be equivalent with the resolution (3.2). The base space S is the affine space of parameters a_{ijk}. The fiber $\pi^{-1}(a)$ is isomorphic to the surface Σ_a.

The deformation is invariant under multiplication with the non-zero complex numbers, $\mathbf{C}^*$. Furthermore, there is always one element a_{ijk} for each of the polynomials in table 1-2 which corresponds to the single irrelevant operator in the LG theory. Restricting to the relevant operators, and taking into account the $\mathbf{C}^*$ symmetry, we let the space

$$S_a^+ \equiv S^+ - \{0\}/\mathbf{C}^* \tag{3.5}$$

be the base space parameterizing all (untwisted) relevant perturbations. (Recall that there are no marginal deformations in this case.) In S^+ we will consider two sub-spaces

$$S_{reg}^+ \equiv \Big\{a \in S^+ \mid \; \Sigma_a \text{ non-singular}\Big\} \tag{3.6}$$

$$S_{sim}^+ \equiv \Big\{a \in S^+ \mid \; \Sigma_a \text{ has only simple singularities}\Big\}. \tag{3.7}$$

The moduli space S_a^+ has a rich topology, so in order to describe it we need some invariant objects. An example of such an invariant turns out to be the monodromy group. Formally, the monodromy group of a singularity is defined by the embedding of the fundamental group of S_{reg}^+ into the integral homology of the surface Σ_a. Fortunately, due to a result in [41], the only non-trivial homology group turns out to be $H_2(\Sigma_{a_0}, \mathbf{Z}) \simeq \mathbf{Z}^\mu$, which is for the singularities in table 1-2 an even symmetric integral lattice, referred to as the *Milnor lattice*, denoted by L_a. Its dimension, i.e. the dimension of the integral homology group of Σ_a is given by the Milnor number μ and corresponds to the minimal number of independent deformations of the singularity necessary to describe its resolution [40].

In some cases the monodromy group has a more concrete definition, namely as the group generated by pseudo-reflections acting on a basis $\{e_i\}_{i=1}^{i=\mu}$ in L_a:

$$s_{e_i}(x) = x + q(x, e_i)e_i, \qquad x \in L_a, \tag{3.8}$$

where q denotes the bilinear form on L_a. It satisfies $q(e_i, e_i) = -2$. Hence $s_{e_i}(e_i) = -e_i$ and $s_{e_i}(x) = 0$ for all $x \perp e_i$. The vectors $\{e_i\}$ are the so-called vanishing cycles of the singularity [40]. The group Γ is an infinite group for the singularities of tables 1 and 2. One has

$$\Gamma \subseteq \mathrm{Aut}(L_a), \tag{3.9}$$

where $\mathrm{Aut}(L_a)$ denotes the full group of isometries of the lattice L_a. The equality occurs exactly for the singularities of table 1 [40, 42].

It follows [39] that the monodromy group is a topological invariant of the singularity and hence invariant under smooth deformations of the superpotential. One may use this fact to find a fundamental domain for these deformations. For this one introduces a period map ϕ_ω:

$$\phi_\omega : S_{reg}^+ \longrightarrow L_{\mathbf{C}}$$

$$\phi_\omega(a)\,(\gamma) \doteq \oint_{\gamma(a)} \omega(a) \qquad \text{for each } a \in S_{reg}^+, \tag{3.10}$$

where $L_{\mathbf{C}}$ denotes the complexification of the Milnor lattice. The holomorphic two-form on Σ_a is defined as

$$\omega(a) = \text{P.R.} \frac{dX_1 \wedge dX_2 \wedge dX_3}{f_a(X_1, X_2, X_3)}. \tag{3.11}$$

The notation 'P.R.' means to take the Poincaré residue of the meromorphic three-form $\frac{dX_1 \wedge dX_2 \wedge dX_3}{f_a(X_1, X_2, X_3)}$. We refer the reader to [43, page 147] for an explanation of this map.

Now it turns out [44] that the period map extends naturally to a multivalued holomorphic ramified map

$$\phi_\omega : S^+_{sim} \longrightarrow L_{\mathbf{C}}. \tag{3.12}$$

Let D denote the component containing the image of S^+_{sim} under the map ϕ induced by the period map. The monodromy group Γ acts on D discontinuously, so the space D/Γ is a well defined complex space. This is called the fundamental domain for the lattice L_a. As is shown in [40] the map ϕ factorizes over $\mathbf{C}^*$, i.e. it extends to a map

$$\tilde{\phi} : S^+_{sim}/\mathbf{C}^* \longrightarrow D/\Gamma. \tag{3.13}$$

Using this period map one can show [44] that for the singularities of tables 1-2

$$S^+_{sim}/\mathbf{C}^* \subseteq D/\Gamma, \tag{3.14}$$

where the equality occurs for only three singularities of the type K_{12}, W_{12}, U_{12} appearing in table 1. So we may describe the moduli space of the affine surfaces Σ_a by way of the fundamental domain D/Γ.

In order to find a description of the moduli space of the compactified surfaces $\overline{\Sigma}_a$ one has to study the compactification of D/Γ, which as we mentioned earlier, arises through an embedding of S^+_{sim} in $S^+_a \equiv S^+ - \{0\}/\mathbf{C}^*$. In general the compactification of Σ_a is obtained via a particular resolution, consisting of the addition of extra polynomials. In general such a resolution looks like

$$\overline{f_a} = f_0(X_i) + \sum_{\substack{\alpha \\ q(\text{Pol}_\alpha)<1}} a^\alpha \text{Pol}_\alpha(X_i) Y^{n_\alpha} \tag{3.15}$$

where Pol_α denotes a set of weighted homogeneous polynomials labelled by α in the variables X_i, $i = 1, 2, 3$, of degree at most N, where N is the Coxeter number associated with the singularity listed in the tables 1-2, i.e. it is the least common multiple of the weights of the original polynomial f. The new variable is denoted by Y. Its power n_α depends on the set of divisors of N which appears in $q(\mathrm{Pol}_\alpha) = n_\alpha/N$ (in the corresponding $N = 2$ SCFT, q is the $U(1)$ charge). The coefficients a^α are normalized using the $\mathbf{C}^*$ symmetry. The precise form of the polynomials follows from the requirement that the resulting surface describes a *compact* CY surface, i.e. a K3 surface in the $c = 6$ case. All the fields of dimension less than one for the original uncompactified singularity become all marginal fields for the compactified singularity. In this way all the singularities of tables 1-2 give rise after compactification to K3 surfaces [45, 44].

Let us illustrate this for the example (3.2). The compactification of Σ_a corresponds to a particular deformation of Σ_0 by adding a term X_4 to the polynomial f_a

$$\overline{f_a} = X_1^2 + X_2^3 + X_3^7 + a_{000}X_4^{42} + \sum_{\substack{i=0, j=1\ldots5;\\ i=1, j=0\ldots4}} a_{ijk}X_2^i X_3^j X_4^k \tag{3.16}$$

where $k = 42 - 14i - 6j$. Ruling out the possibility of having $a_{000} = 0$ and using the $\mathbf{C}^*$-symmetry to normalize $a_{000} = 1$, we restrict to the space S_a^+ defined in (3.5). Thus the addition of the term X_4^{42} amounts to an embedding

$$S_{sim}^+ \subset S_a^+. \tag{3.17}$$

The surface defined by

$$\overline{f} = X_1^2 + X_2^3 + X_3^7 + X_4^{42} = 0 \tag{3.18}$$

is known to describe a K3 surface which has only simple isolated singularities. If one uses $\overline{f}$ as a superpotential in a LG model one finds that the content of the chiral ring coincides with that of the spectrum of chiral states of the $c = 6$ Gepner model $1 \cdot 5 \cdot 40$. Note that the terms of the sum in (3.16) correspond to the *marginal* deformations in the untwisted (c, c) chiral ring of the deformed

Gepner model $1 \cdot 5 \cdot 40$. The space of

$$\{a_{ijk} | i = 0, j = 1...5; \quad i = 1, j = 0...4; \quad k = 42 - 14i - 6j\}$$

is what we called in section 2 the a-moduli space. The fact that one has to add a polynomial X_4 of degree 42 follows from the details of the compactification of the affine surface Σ_a turning it into a K3 surface. In this example above there is only one coefficient ($a = 1$) in (3.15) so that $n = N = 42$.

In general the equation $\overline{f_a} = 0$ at $Y = 0$ describes a curve in $\mathbf{CP}^2$ which has at most isolated simple singularities. That is, the surfaces $\overline{\Sigma}_a$ have for all a the same singularities, implying that there exists a uniform resolution

$$\pi' : \overline{\Sigma} \longrightarrow S^+ - \{0\}/\mathbf{C}. \tag{3.19}$$

This describes a holomorphic family of K3 surfaces all having the same singularity structure. Because of this, one may describe the compactification alternatively in terms of a compactification of the space D/Γ. From [44] one learns that the compactification of D/Γ is obtained by adding the rational boundary components of D. The compactification of D/Γ is denoted by $(D/\Gamma)^* = D^*/\Gamma$. D^* is a symmetric space in the coset $O(20,4)/(O(20) \times O(4))$. S_a^+ is embedded in D^*/Γ by a map ψ, which is an extension of the period map ϕ introduced earlier, such that the following diagram is commutative

$$\begin{array}{ccc} S^+_{sim} & \xrightarrow{\phi} & D/\Gamma \\ \downarrow & & \downarrow \\ S^+_a & \xrightarrow{\psi} & D^*/\Gamma \end{array} \tag{3.20}$$

The embedding ψ turns out to be an isomorphism [40, 44] for the three singularities of the type K_{12}, W_{12}, U_{12} in table 1.

The space D^*/Γ is the submoduli space of K3 surfaces which are of the same type as $\overline{\Sigma}_a$. We thus conclude that the space D^*/Γ corresponds to the moduli space formed by the (untwisted) marginal deformations of the LG theory defined by a compactification of a given singularity of tables 1-2. This moduli space is a sub space of the complete moduli space of all K3 surfaces denoted by $\mathcal{M}$. That is

$$S_a^+ \subset \mathcal{M}.$$

The space $\mathcal{M}$ is well known. We recall from [46] that $\mathcal{M}$ is the simply connected space,

$$\mathcal{M} \simeq O(19,3)/(O(19) \times O(3)). \tag{3.21}$$

In other words: the homology lattice $H_2(\overline{\Sigma}, \mathbf{Z})$ is the same for all K3 surfaces $\overline{\Sigma}$. The latter is known to be the even self-dual Lorentzian lattice L_{K3} of rank 22

$$L_{K3} = 2(-E_8) \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.22}$$

where E_8 denotes the self-dual lattice associated with the Dynkin diagram of the Lie algebra of type E_8. The isometry group of this lattice, i.e. its automorphism group, is known to be the discrete group $O(19,3,\mathbf{Z})$ [46]. We thus conclude that the Milnor lattice of any of the compactified singularities in tables 1-2 is embedded into L_{K3}. The results in [40, 44, 45], described above, now imply that this embedding can be alternatively described by using the period map ψ, i.e. we have an embedding

$$S_a^+ \subseteq D^*/\Gamma \subset \mathcal{M}, \tag{3.23}$$

where Γ is the monodromy group of the affine surface Σ_a. In fact, Γ is contained in the automorphism group of L_{K3}. The action of Γ on the symmetric subspace $D^* \subset \mathcal{M}$ relates *different* points which are physically equivalent: under the action of Γ the superpotential of the corresponding LG model remains the same. That is, adding an untwisted marginal deformation the monodromy does not change. Of course the monodromy has in general a *non* trivial effect on the kinetic term of the LG-model. However, as the monodromy of the superpotential remains the same, the change of the kinetic term corresponds to a perturbation of the kinetic term which becomes irrelevant at the critical point (see e.g. [47]).

3.2. Physical symmetries on S_a^+ and automorphism groups of K3 surfaces.

Let us now turn to the problem of finding symmetry groups on the moduli space S_a^+ of untwisted marginal deformations of a (2,2) vacuum. The description of the moduli space in terms of the monodromy group turns out to be very useful to obtain a geometrical understanding of these groups. Recall that in section 2 we presented an algorithm for finding the generators of such symmetry groups using certain field redefinitions of the LG theory. For most of the potentials it is in general not so easy to identify these groups. In this subsection we will describe these groups in a more geometrical way which allows one to identify certain sub groups as automorphism groups of CY manifolds. Again we will restrict ourselves to the $c = 6$ superpotentials, i.e. to K3 surfaces, since in this case the complete classification of automorphism groups is known [48]. However, most of the arguments hold for the $c = 9$ superpotentials as well.

To describe possible symmetry groups on the moduli space of untwisted deformations we will use the moduli space D^*/Γ. We start with a more precise description of the Milnor lattice $\overline{L}_a$ of the compactified singularity using the Hodge decomposition of the two dimensional cohomology of of a K3 surface $\overline{\Sigma}$:

$$H^2(\overline{\Sigma}, \mathbf{C}) = H^{2,0}(\overline{\Sigma}) + H^{1,1}(\overline{\Sigma}) + H^{0,2}(\overline{\Sigma}), \tag{3.24}$$

with

$$\dim H^{2,0}(\overline{\Sigma}) = \dim H^{0,2}(\overline{\Sigma}) = 1, \qquad \dim H^{1,1}(\overline{\Sigma}) = 19. \tag{3.25}$$

Any K3 surface is completely determined by the unique holomorphic two-form ω (i.e. a form without poles or zeros) of which we saw an example in the previous sub-section. This two-form is an element in $H^{2,0}(\overline{\Sigma})$.

Next we define the lattice $S_{\overline{\Sigma}}$ as

$$S_{\overline{\Sigma}} = H^2(\overline{\Sigma}, \mathbf{Z}) \cap H^{1,1}(\overline{\Sigma}),$$

that is,

$$S_{\overline{\Sigma}} = \left\{ x \in H^2(\overline{\Sigma}, \mathbf{Z}) | x \cdot H^{2,0}(\overline{\Sigma}) = 0 \right\}. \tag{3.26}$$

This lattice is referred to as the Picard lattice [49, 50]. Together with it we define its orthogonal complement by

$$T_{\overline{\Sigma}} = S_{\overline{\Sigma}}^{\perp} \qquad \text{in } H^2(\overline{\Sigma}, \mathbf{Z}). \tag{3.27}$$

Obviously,

$$S_{\overline{\Sigma}} \oplus T_{\overline{\Sigma}} = L_{K3}, \tag{3.28}$$

where L_{K3} is given in (3.22). It is known [49, 50] that any K3 surface is uniquely determined by its Picard lattice.

The lattice $S_{\overline{\Sigma}}$ has signature $(1, k)$, $k \leq 20$ so that $T_{\overline{\Sigma}}$ has signature $(2, 19 - \text{rank}\, S_{\overline{\Sigma}})$. It is not hard to see that the lattice $\overline{L_a}$ which was defined in the previous section as homology lattice formed by the vanishing cycles of the compactified singularity is given as the Poincaré dual of the lattice $S_{\overline{\Sigma}}$.

Let us now explain the relevance of the automorphism group of a K3 surface in the discussion of the physical symmetries acting on the moduli space S_a^+. Let $\text{Aut}(S_{\overline{\Sigma}})$ denote the automorphism group of the Picard lattice of a K3 surface $\overline{\Sigma}$. Denote by $\tilde{\Gamma}$ its subgroup generated by pseudo-reflections (defined in (3.8)). This group contains in particular the monodromy group Γ introduced in the previous section. To be more precise: the quotient $\tilde{\Gamma}/\Gamma$ fixes the 'point' $Y = 0$ in (3.15) added at infinity by which we compactified the surface Σ_a.

A well known fact [49] states that the automorphism group of the K3 surface can be expressed as the quotient

$$\text{Aut}(\overline{\Sigma}) \simeq \text{Aut}(S_{\overline{\Sigma}})/\tilde{\Gamma}. \tag{3.29}$$

We will assume that $\text{Aut}(\overline{\Sigma})$ is a finite group. By definition $\text{Aut}(\overline{\Sigma})$ leaves invariant (up to a complex phase) the holomorphic two-from ω which defines the K3 surface. To be more precise: for any element $\omega \in H^{2,0}(\overline{\Sigma})$ we have

$$g^*\omega = \alpha(g)\omega, \tag{3.30}$$

with $\alpha(g) \in \mathbf{C}^*$ and $g \in \text{Aut}(\overline{\Sigma})$. (The star above g denotes the pullback). If we denote the kernel of α by $G_{\overline{\Sigma}}$, i.e. the elements g for which $\alpha(g) = 1$, then we have that

$$\mathrm{Aut}(\overline{\Sigma}) \simeq G_{\overline{\Sigma}} \ltimes \mathbf{Z}_M, \tag{3.31}$$

where it is known [49] that M can be at most equal to 66. One can show rather easily, using the results in [51] that for each of the polynomials in table 1-2 M is at most equal to the Coxeter number N. (For the models corresponding to type A-modular invariants, N is the least common multiple of the powers in the defining polynomial.) If the three powers are co-prime it turns out that $\mathrm{Aut}(\overline{\Sigma})$ is the group of phases $\mathbf{Z}_N$ discussed in [32].

We now come to the main point. Let us consider a particular K3 surface arising via the compactification of one of the singularities in tables 1-2, and fix a parameter $a_0 \in S_a^+$, corresponding to a particular deformation. The action of the automorphism group $\mathrm{Aut}(\overline{\Sigma}_{a_0})$ on the holomorphic two-form $\omega(a_0)$ defined in (3.30) is induced from an action on $\mathbf{CP}^n$ [48]. (That is, it corresponds to a redefinition of the fields X_i). The only transformations which preserve $\omega(a_0)$, which we consider, are linear transformations on the fields X_i. [6] It then follows from the definition (3.11) of ω, that in terms of the LG model, $\mathrm{Aut}(\overline{\Sigma}_{a_0})$ is an example of a sub-group of *physical* symmetries acting on the theory by a field redefinition. [7]

The action of the automorphism group $\mathrm{Aut}(\overline{\Sigma}_{a_0})$ on the holomorphic two-form $\omega(a_0)$ can be lifted to the whole space of deformations S_a^+; the group $\mathrm{Aut}(\overline{\Sigma}_{a_c})$ acts non-trivially on the two-forms $\omega(a)$ associated to deformations parameterized by $a \in S_a^+$, $a \neq a_0$. The deformation for which $a = a_0$ corresponds to the fixed point of this group, which is the automorphism group of the K3 surface defined by the two-form $\omega(a_0)$.

In section 2 we considered the example (2.8) where we found explicitly that the group $\mathrm{Aut}(\overline{\Sigma}) \simeq \mathcal{T} \ltimes S_3$ acts on the moduli space S_a^+ with fixed point $(a_1, a_2, a_3) = (-10, 0, 0)$. The non-abelian group $G_{\overline{\Sigma}} \simeq M_9$ of order 72 is contained in the maximal automorphism group of the K3 surface $\sum_{i=1}^{3} X_i^6 - 10(X_1^3X_2^3 + X_1^3X_3^3 +$

[6]In principle, there may be non-linear transformations which preserve ω. However, all the examples studied in [48] turn out to correspond to linear unitary field redefinitions.

[7]Vacua for which $\mathrm{Aut}(\overline{\Sigma}_{a_0})$ is not trivial are important as they usually correspond to models with an extended symmetry.

$X_2^3 X_3^3) = 0$ in $\mathbf{CP}^2$ (see e.g. [48]). This is the sub-group of field redefinitions preserving ω, i.e. preserving the superpotential and of determinant 1.

The above found relation between the automorphism groups of K3 surfaces and the group of transformations described in section 2 by field redefinitions is in fact generalized to all possible a-moduli spaces. The groups generated by the field redefinitions described in section 2 contain the automorphism groups of *any* K3 surfaces that appears as a possible deformation of the unperturbed surface. The automorphism groups correspond to the stabilizing subgroups in the group obtained from field redefinitions, acting non-trivially on the a-moduli of untwisted (c, c) marginal deformations.

Remarkably, the classification of symplectic automorphisms of a K3 surface is known [48]. It follows that each possible group $G_{\overline{\Sigma}}$ is isomorphic to one of the 11 groups (or subgroups thereof) listed in table 3, shown at the end of this section. For the notation of these groups we refer to [48, 52]. The blank entries in the fourth column correspond to K3 surfaces that are not obtained via compactification of one of the singularities in tables 1-2. (See [48] for the corresponding surfaces.) The first K3 surface is closely related to the compactification of the singularity K_{12} in tables 1-2 (see [38, 40, 53]) Lines 4,5 and 7 correspond to the deformed Gepner model 2^4. The group F_{384} is generated by phase transformations and permutations. The group M_{20} contains the transformation of the type S. Line 10 is the example mentioned above, which we discussed in detail in section 2. The full automorphism group is obtained by taking the semi-direct product with the cyclic group $\mathbf{Z}_M$ as in (3.31). So we conclude that the groups $\mathrm{Aut}(\overline{\Sigma})$ are subgroups of the 'modular' group obtained from field redefinitions in section 2.

By construction, the groups $G_{\overline{\Sigma}}$ are all sub-groups of $O(19, 3, Z)$ and therefore of $O(20, 4, \mathbf{Z})$ as well. This concludes the discussion on the relation between subgroups of the symmetry groups acting on the moduli space of untwisted marginal deformations of a $(2, 2)$ $c = 6$ vacuum and automorphism groups of $K3$ surfaces. In principle the same argument holds also for the $c = 9$ case, however, much less is known about the structure of automorphism groups of CY manifolds (work on this subject is in progress [35]).

No.	$G_{\overline{\Sigma}}$	order	K3 surface
1	$L_2(7)$	168	$X_1^3X_2 + X_2^3X_3 + X_3^3X_1 + X_4^4 = 0$
2	$\mathcal{A}_6$	360	$\sum_1^6 X_i = \sum_1^6 X_i^2 = \sum_1^6 X_i^3 = 0$
3	$\mathcal{S}_5$	120	$\sum_1^5 X_i = \sum_1^6 X_i^2 = \sum_1^5 X_i^3 = 0$
4	M_{20}	960	$\sum_1^4 X_i^4 + 12\prod_1^4 X_i = 0$
5	F_{384}	384	$\sum_1^4 X_i^4 = 0$
6	$\mathcal{A}_{4,4}$	288	
7	T_{192}	192	$\sum_1^4 X_i^4 - 2i\sqrt{3}(X_1^2X_2^2 + X_3^2X_4^2) = 0$
8	H_{192}	192	
9	N_{72}	72	
10	M_9	72	$\sum_1^3 X_i^6 - 10(X_1^3X_2^3 + X_2^3X_3^3 + X_3^3X_1^3) = 0$
11	T_{48}	48	

Table 3: *Finite automorphism groups of K3 surfaces which are nontrivial physical symmetry groups.*

4. Transformations Relating Small and Large Volume Compactifications.

In this section we discuss a particular symmetry on the moduli space of (2,2)-vacua which relates small radius compactifications with large ones. For definiteness we restrict here to $c = 6$ superpotentials. In order to reformulate the moduli problem for $c = 6$ LG theories in terms of metrics we use a known relation [54] between the metric and the holomorphic two-form on the K3- surface $\overline{\Sigma}_a$. We remind the reader that a Kähler metric $g_{i\bar{j}}$ is a CY metric (i.e. Ricci-flat) if

$$R_{i\bar{j}} = \frac{\partial^2}{\partial_i \bar{\partial}_{\bar{j}}} \log(\det(g_{i\bar{j}})) = 0, \tag{4.1}$$

and correspondingly defines a unique class (up to isomorphism) in $H^{1,1}(\overline{\Sigma}_a)$. Now, for K3 surfaces (i.e. compact CY manifolds in complex dimension 2) this implies that a Kähler metric on it is Ricci-flat if and only if there exists a positive constant c such that [54]

$$\phi \wedge \phi = c\omega_a \wedge \overline{\omega}_a, \tag{4.2}$$

where ω_a is the unique global holomorphic two-form characterizing the K3 surface, as defined in (3.11), and $\phi = \mathrm{Im} g_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}}$ is the Kähler form on $\overline{\Sigma}_a$. Now let us consider a deformed K3 surface defined by the two-form ω_a. The volume of the surface corresponding to the metric $g_{i\bar{j}}$ is given by

$$Vol(g_{i\bar{j}}(a)) = \int_{\overline{\Sigma}_a} \omega_a \wedge \overline{\omega}_a \tag{4.3}$$

which is explicitly depending on the moduli $a \in S_a^+$ of the K3 surface. Note that this is *not* the volume form obtained from the embedding of the surface into some projective space: the induced metric coming from the embedding is *not* Ricci-flat and gives rise to a volume form which can not be written in terms of a holomorphic and anti-holomorphic two-form.

Let us next find generators in the symmetry group on the moduli space of a LG model that act nontrivially on the volume in (4.3). In particular, we will be interested in the possibility of finding modular transformations that transform 'small volumes to large ones'. We will follow the example in (2.8), with the action of an S transformation on the a-moduli. However, the same arguments will give the same conclusion in other cases where a type S transformation relates finite values of a to infinite ones.

The limit $a_1 \to \infty$ in (2.8) is equivalent to $(a_1, a_2, a_3) = (-7, 9, 3)$ as can be found using (2.10). We will denote the superpotential for given a parameters by $W_{a_1,a_2,a_3}(X_1, X_2, X_3)$. The holomorphic two-form (3.11) in both cases is the same, up to an overall factor $9/a_1$ at the limit $a_1 \to \infty$. This can be seen as follows. At the limit $a_1 \to \infty$, the superpotential is effectively

$$W_{a_1,a_2,a_3}(X_1, X_2, X_3) \to a_1(X_1^3X_2^3 + X_1^3X_3^3 + X_2^3X_3^3). \tag{4.4}$$

On the other hand,

$$W_{-7,9,3}(X_1', X_2', X_3') = 9(X_1'^3X_2'^3 + X_1'^3X_3'^3 + X_2'^3X_3'^3), \tag{4.5}$$

where in (4.5) the X_i' are related to the X_1 by the type S transformation (2.9). The Jacobian of the transformation (2.9) is 1, and thus the factor $9/a_1$ is established. Using (4.3) we conclude that the ratio between the volumes of the two corresponding models diverges, and thus small volumes are related to large ones.

Tables of Singularities

In the following tables N denotes the dual Coxeter number of the singularity which equals the number of the least common multiple in the case the singularity is of the form $\sum_i X_i^{p_i}$. In the last collumn we have given the number of untwisted (c,c) deformations. It is computed for the *compactified* singularity, i.e. after adding a term with power equal to N. These tables can be found e.g. in [37].

N	Superpotential	Type	Λ
24	$X_1^2X_3 + X_2^3 + X_3^4$	Q_{10}	8
18	$X_1^2X_3 + X_2^3 + X_2X_3^3$	Q_{11}	9
15	$X_1^2X_3 + X_2^3 + X_3^5$	Q_{12}	10
16	$X_1^2X_3 + X_2X_3^2 + X_2^4$	S_{11}	9
13	$X_1^2X_3 + X_2X_3^2 + X_1X_2^3$	S_{12}	10
12	$X_1^3 + X_2^3 + X_3^4$	U_{12}	10
30	$X_1^3X_2 + X_2^5 + X_3^2$	Z_{11}	9
22	$X_1^3X_2 + X_1X_2^4 + X_3^2$	Z_{12}	10
18	$X_1^3X_2 + X_2^6 + X_3^2$	Z_{13}	11
20	$X_1^4 + X_2^5 + X_3^2$	W_{12}	10
16	$X_1^4 + X_1X_2^4 + X_3^2$	W_{13}	11
42	$X_1^2 + X_2^3 + X_3^7$	K_{12}	10
30	$X_1^3 + X_1X_2^5 + X_3^2$	K_{13}	11
24	$X_1^3 + X_2^8 + X_3^2$	K_{14}	12

Table 1. *The Exceptional Modality 1 Superpotentials with* $c = 3(\frac{N+2}{N})$.

N	Superpotential	Type	Λ
18	$X_1^9 + X_2^3 + X_3^2$	$R_{269}(J_{3,0})$	14
14	$X_1^7 + X_2^3 X_1 + X_3^2$	$R_{247}(Z_{1,0})$	13
12	$X_1^4 X_2 + X_2^3 + X_1 X_3^2$	$R_{245}(Q_{2,0})$	12
12	$X_1^6 + X_2^4 + X_3^2$	$R_{236}(W_{1,0})$	13
10	$X_1^5 + X_2^2 X_3 + X_1 X_3^2$	$R_{234}(S_{1,0})$	12
9	$X_2^3 + X_3^3 + X_1 X_2^3$	$R_{233}(U_{1,0})$	12
10	$X_1^5 + X_2^5 + X_3^2$	$R_{225}(N_{16})$	14
8	$X_1^4 + X_2^4 + X_2 X_3^2$	$R_{223}(V_{1,0})$	13
12	$X_1^{12} + X_2^3 + X_3^2$	$R_{146}(J_{4,0})$	18
10	$X_1^{10} + X_1 X_2^3 + X_3^2$	$R_{135}(Z_{2,0})$	17
9	$X_1^9 + X_2^3 + X_1 X_3^2$	$R_{134}(Q_{3,0})$	16
8	$X_1^8 + X_2^4 + X_3^2$	$R_{124}(X_{2,0})$	17
7	$X_1^7 + X_1 X_2^3 + X_1 X_3^2 + X_2^2 X_3$	$R_{123}(S_{2,0}^*)$	16
6	$X_1^6 + X_2^3 + X_3^3$	R_{122}	16
5	$X_1^5 + X_2^5 + X_1 X_3^2$	R_{112}	16
6	$X_1^6 + X_2^6 + X_3^2$	R_{113}	19
4	$X_1^4 + X_2^4 + X_3^4$	R_{111}	19

Table 2. *Remaining superpotentials that embed in* $\mathbf{C}^3$ *with* $c = 3(\frac{N+2}{N})$.

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