Hamiltonian Field Description of

Two-Dimensional Vortex Fluids and

Guiding Center Plasmas

by

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Abstract

The equations that describe the motion of two-dimensional vortex fluids and guiding center plasmas are shown to possess underlying field Hamiltonian structure. A Poisson bracket which is given in terms of the vorticity, the physical although noncanonical dynamical variable, casts these equations into Heisenberg form. The Hamiltonian density is the kinetic energy density of the fluid. The well-known conserved quantities are seen to be in involution with respect to this Poisson bracket. Expanding the vorticity in terms of a Fourier-Dirac series transforms the field description given here into the usual canonical equations for discrete vortex motion. A Clebsch potential representation of the vorticity transforms the noncanonical field description into a canonical description.

I. Introduction

This paper is concerned with the Hamiltonian field formulation of the equations which describe the advection of vorticity in a two-dimensional fluid. These equations have received a great deal of attention in the past thirty years and are believed to model the large scale motions which occur in atmospheres and oceans. They have also arisen in the study of plasma transport perpendicular to a uniform magnetic field, the so-called guiding center plasma.[1,2] (For recent reviews see Refs. 3 and 4.)

It has been known for some time that a system of discrete vortex (or charge) filaments possesses a Hamiltonian description.[5] The equations of motion are

$$k_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i} \qquad k_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{1}$$

where k_i is the circulation of the i^{th} vortex which has coordinates x_i and y_i. The Hamiltonian, H, is the interaction energy and for an unbounded fluid has the form

$$H = \frac{-1}{2\pi} \sum_{i>j} k_i k_j \ln R_{ij},$$

where $R_{ij}{}^2 = (x_i-x_j)^2 + (y_i-y_j)^2$. The variables x_i and y_i are canonically conjugate. The formulation we describe here is a <u>field</u> formulation which possesses this underlying discrete dynamics.

In Sec. II we briefly review some aspects of finite degree of freedom Hamiltonian dynamics. The emphasis here is placed on the Lie algebraic properties of the Poisson bracket. This is used as a framework in which to explain the "constructive" approach to Hamiltonian dynamics. Such an approach frees one from the prejudice that a system need be in canonical variables to be Hamiltonian. This section is then concluded by the extension of these notions to infinite degree of freedom or Hamiltonian density systems. In Sec. III we present a Poisson bracket that renders the vortex equations into Heisenberg form. This formulation is novel in that it is noncanonical. In the remainder of this section we discuss involutivity of the well-known constants of motion for this system, Fourier space representation and truncation. In Sec. IV we expand the vorticity in a Fourier-Dirac series which, upon substitution into the Poisson bracket of Sec. III, yields the canonical discrete vortex description of the Introduction. Following this we introduce Clebsch potentials which also bring the Poisson bracket into canonial form. Finally, we obtain a spectral description where complex conjugate pairs are canonically conjugate. A quartic interaction Hamiltonian is obtained.

II. Constructive Hamiltonian Dynamics

The standard approach[6] to a Hamiltonian description is via a Lagrangian description. One constructs a Lagrangian on physical bases and through the Legendre transformation (assuming convexity) obtains the Hamiltonian and the following 2N (where N is the number of degrees of freedom) first order ordinary differential equations:

$$\dot{q}_k = [q_k, H] \quad ; \qquad \dot{p}_k = [p_k, H] \qquad k = 1,2,\ldots N \quad . \tag{2}$$

Here the Poisson bracket has the form

$$[f,g] = \sum_{k=1}^{N} \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right) = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} \quad . \tag{3}$$

The last equality of Eq. (4) follows from the substitutions,

$$z^i = \begin{cases} q_k & \text{for } i = k = 1,2\ldots N \\ p_k & \text{for } i = N + k = (N+1)\ldots \quad 2N \end{cases}$$

and

$$\left(J^{ij}\right) = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{4}$$

where I_N is the $N \times N$ unit matrix. We assume the repeated index summation convention here and henceforth. The quantity J^{ij} is known as the Poisson tensor or the cosymplectic form. It is not difficult to show that it transforms as a contravariant tensor under a change of coordinates. Those transformations which preserve its form, and hence the form of the Eqs. (2), the equations of motion, are canonical.

The constructive approach differs from the above in that one is not concerned with any underlying action principle nor with (initially at least) the necessity of canonical variables. The emphasis is placed on the algebraic properties of the Poisson bracket. A system need not have the canonical form of Eqs. (2) with Eq. (3) to be Hamiltonian. To make the idea more precise we introduce a few mathematical concepts. The quantities on which the Poisson bracket acts are differentiable functions defined on phase space. The collection of all such functions is a vector space (call it Ω) under addition and scaler multiplication. The Poisson bracket is a bilinear function which maps $\Omega \times \Omega$ to Ω. Also note that the Poisson bracket possesses the following two important properties: (i) $[f,g] = -[g,f]$ for every $f,g \in \Omega$ and (ii) the Jacobi identity, i.e., $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$ for every $f,g,h \in \Omega$. A vector space together with such a bracket defines a Lie algebra. Property (i) requires that the Poisson tensor be antisymmetric and property (ii) requires the following:

$$S^{ijk} \equiv J^{i\ell} \frac{\partial}{\partial z^\ell} J^{jk} + J^{j\ell} \frac{\partial}{\partial z^\ell} J^{ki} + J^{k\ell} \frac{\partial}{\partial z^\ell} J^{ij} = 0 \quad . \tag{5}$$

One can show that S^{ijk} transforms contravariantly; hence if it vanishes identically in one coordinate frame it does so in all. Similarly antisymmetry is coordinate independent. The covariance of properties (i) and (ii) suggests the converse outlook: if a system of equations possesses the form

$$\dot{z}^i = \tilde{J}^{ij} \frac{\partial H}{\partial z^j} \qquad i,j = 1,2\ldots 2N \tag{6}$$

where $\tilde{J}^{ij}$ is antisymmetric and fulfills the Jacobi requirement, but is not of the form of Eq. (4), then it is Hamiltonian. This outlook is justified by a theorem due to Darboux (1882) which states that assuming $\det(J^{ij}) \neq 0$ (locally) canonical coordinates can be constructed. (The proof of this theorem may be found in Refs. 7, 8, and 9.) Hence in order for a system to be Hamiltonian it is only necessary for it be representable in Heisenberg form with a Poisson bracket that makes Ω into a Lie algebra. The constructive approach simply amounts to constructing Poisson brackets with the appropriate properties.

The rigorous generalization of the above ideas to infinite dimensional systems requires the language of functional analysis and the differential geometry of infinite dimensional manifolds. (See Ref. 7, Ch. V and Refs. 10 -

15.) This of course is not our purpose here; rather we simply parallel the above. The Poisson bracket for a set of field equations usually has the following form:[6]

$$[\hat{F},\hat{G}] = \sum_{k=1}^{N} \int \left(\frac{\delta\hat{F}}{\delta\eta_k}\frac{\delta\hat{G}}{\delta\pi_k} - \frac{\delta\hat{G}}{\delta\eta_k}\frac{\delta\hat{F}}{\delta\pi_k}\right) d\tau \tag{7}$$

where the integration is taken over a fixed volume. The quantities on which the bracket acts are now functionals, such as the integral of the Hamiltonian density [e.g., Eq.(13)]. The functional derivative is defined by

$$\frac{d\hat{F}}{d\varepsilon}(\eta_k + \varepsilon w)\Big|_{\varepsilon=0} = \int \frac{\delta\hat{F}}{\delta\eta_k} w d\tau \equiv \langle\frac{\delta\hat{F}}{\delta\eta_k}|w\rangle$$

where the bra-ket notation is used to indicate the inner product $\langle f|g\rangle = \int fg\, d\tau$. In terms of this notation Eq. (7) becomes

$$[\hat{F},\hat{G}] = \langle\frac{\delta\hat{F}}{\delta u^i} \mid O^{ij} \frac{\delta\hat{G}}{\delta u^j}\rangle \tag{8}$$

where the 2N quantities η_k and π_k are as previously the 2N indexed quantities u^i. The canonical cosymplectic density has the form

$$(O^{ij}) = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \quad .$$

In noncanonical variables the quantity (O^{ij}) may depend upon the variables u^i, and further it may contain derivatives with respect to the independent variables. In general antisymmetry of Eq. (8) requires that the (O^{ij}) be an anti-self-adjoint operator. The Jacobi identity places further restrictions, analogous to Eq. (5), on this quantity. We defer a discussion of this to the

Appendix where the Jacobi identity for the bracket we present [Eq. (15)] is proved. The extension of the Darboux theorem to infinite dimensions has been proved by J. Marsden.[14] For a discussion pertinent here see Ref. 15.

III. Noncanonical Poisson Bracket

The equations under consideration are the following:

$$\omega_t = - \underline{v} \cdot \nabla \omega \tag{9}$$

$$\nabla \cdot \underline{v} = 0 \quad . \tag{10}$$

Here we use the usual Euclidian coordinate system with uniformity in the z direction (which has unit vector $\hat{z}$). The quantity $\omega(\underline{x},t) = \hat{z} \cdot \nabla \times \underline{v}(\underline{x},t)$, where $\underline{x} \equiv (x,y)$, is the vorticity and $\underline{v}$ is the flow velocity such that $\underline{v} \cdot \hat{z} = 0$. (For the guiding center plasma ω corresponds to the charge density and $\underline{v}$ to the $\underline{E} \times \underline{B}$ drift velocity.) For an unbounded fluid $\underline{v}$ can be eliminated from Eq. (9) by[16]

$$\underline{v} = \int \omega(\underline{x}') \, \underline{M} \, (\underline{x}|\underline{x}') \, d\tau' \quad , \tag{11}$$

where we display only the arguments necessary to avoid confusion. Here $\underline{M} = \hat{z} \times \nabla K(\underline{x}|\underline{x}')$ and $K(\underline{x}|\underline{x}')$ is the Green function for Laplace's equation in two dimensions,

$$K(\underline{x}|\underline{x}') = \frac{1}{2\pi} \ln \sqrt{(x-x')^2 + (y-y')^2} \quad .$$

The integration in Eq. (11) is over the entire x-y plane; $d\tau \equiv dxdy$. In this form Eq. (10) is satisfied manifestly. Equation (9) becomes

$$\omega_t = -\int \omega(\underline{x}') \, \underline{M}(\underline{x}|\underline{x}') \, d\tau' \cdot \nabla \, \omega(\underline{x}) \quad . \tag{12}$$

Equations (9) and (10) are known to possess conserved densities; that is, quantities which satisfy an equation of the form $\rho_t + \nabla\cdot\underline{J} = 0$, consistent with Eq. (12). Clearly any function of ω is conserved. In addition, the kinetic energy is conserved which is the natural choice for the Hamiltonian. With the density (mass) set to unity we have

$$\hat{H}\{\omega\} = \int \frac{v^2}{2} \, d\tau = \frac{1}{2} \int \underline{M}(\underline{x}|\underline{x}') \cdot M(\underline{x}|\underline{x}'') \, \omega(\underline{x}') \, \omega(\underline{x}'') \, d\tau \, d\tau' \, d\tau''$$

$$= -\frac{1}{2} \int K(\underline{x}|\underline{x}') \, \omega(\underline{x}') \, \omega(\underline{x}) \, d\tau \, d\tau' \quad . \tag{13}$$

The functional derivative of Eq. (13) is the following:

$$\frac{\delta\hat{H}}{\delta\omega} = -\int K(\underline{x}|\underline{x}') \, \omega(x') \, d\tau' \quad . \tag{14}$$

We introduce the Poisson bracket[17]

$$[\hat{F},\hat{G}] = \int \omega(\underline{x}) \left\{\frac{\delta\hat{F}}{\delta\omega}, \frac{\delta\hat{G}}{\delta\omega}\right\} d\tau \quad , \tag{15}$$

where $\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}$. One observes that the discrete vortex Poisson bracket is nestled inside the field Poisson bracket. In Sec. IV we will see how to regain the discrete bracket from this field bracket. It is not difficult to show from Eqs. (14) and (15) that

$$\omega_t = [\omega, \hat{H}] = -\int \omega \underline{M}\, d\tau' \cdot \nabla\omega \quad .$$

Clearly this bracket is antisymmetric by virtue of the antisymmetry of the discrete bracket. We prove the Jacobi identity in the Appendix.

We note by examination of Eq. (15) that any two functionals of ω are in involution; that is, if $\hat{F}_i\{\omega\} = \int F_i(\omega)d\tau$ (for i=1,2) are two such functionals where the F_i are arbitrary functions of ω, then

$$[\hat{F}_1, \hat{F}_2] = 0 \quad .$$

Also, substitution of any such $\hat{F}_i$ and $\hat{H}$ [Eq. (13)] into Eq. (15) and integration by parts yields

$$[\hat{F}_i, \hat{H}] = 0 \quad .$$

In particular, we see (when $F_i = \omega^2$) that the enstrophy commutes with the Hamiltonian.

The close relationship between this functional Hamiltonian formulation and the conventional formulation of Sec. II is seen by Fourier expanding the vorticity in a unit box with periodic boundary conditions,

$$\omega = \sum_{\underline{k}} \omega_{\underline{k}}\, e^{i\underline{k}\cdot\underline{x}} \quad , \tag{16}$$

where $\underline{k} = (k_1, k_2)$. The reality of ω implies $\omega_{\underline{k}}^* = \omega_{-\underline{k}}$. If we suppose for the moment that $\omega(\underline{x})$ depends upon some additional independent variable μ, then we have the following for some functional[18] $\hat{F}$:

$$\frac{\partial\hat{F}}{\partial\mu} = \int \frac{\delta\hat{F}}{\delta\omega}\frac{\partial\omega}{\partial\mu}\, dx\, dy \quad . \tag{17}$$

From this we see for $\mu = \omega_k$ upon Fourier inversion that

$$\frac{\delta\hat{F}}{\delta\omega} = \frac{1}{(2\pi)^2}\sum_{\underline{k}} \frac{\partial\hat{F}}{\partial\omega_{\underline{k}}} e^{-i\underline{k}\cdot\underline{x}} \quad , \tag{18}$$

where the $\hat{F}$ on the left hand side is treated as a functional of ω while the $\hat{F}$ of the right hand side is to be regarded as a function of the variables $\omega_{\underline{k}}$. Substituting Eqs. (16) and (18) into Eq. (15) yields,

$$[\hat{F},\hat{G}] = \sum_{\underline{k},\underline{\ell}} \frac{\omega_{\underline{k}+\underline{\ell}}}{(2\pi)^2}\, \hat{z}\cdot(\underline{\ell}\times\underline{k})\, \frac{\partial\hat{F}}{\partial\omega_{\underline{k}}}\frac{\partial\hat{G}}{\partial\omega_{\underline{\ell}}} \quad .$$

The Hamiltonian becomes

$$\hat{H} = 2\pi^2 \sum_{\underline{\ell}} \frac{|\omega_{\underline{\ell}}|^2}{\ell^2} \quad ,$$

and the equations of motion are

$$\dot{\omega}_{\underline{k}} = \sum_{\underline{\ell}} \frac{\hat{z}\cdot(\underline{k}\times\underline{\ell})}{\ell^2}\,\omega_{\underline{\ell}}\,\omega_{\underline{k}-\underline{\ell}} = \sum_{\underline{\ell}} J_{\underline{k},\underline{\ell}}\frac{\partial\hat{H}}{\partial\omega_{\ell}} \quad , \tag{19}$$

where $J_{k,\ell}$, the cosymplectic form, is

$$J_{\underline{k},\underline{\ell}} = \frac{\hat{z}\cdot(\underline{\ell}\times\underline{k})}{(2\pi)^2}\,\omega_{\underline{k}+\underline{\ell}} \quad . \tag{20}$$

Clearly, Eq. (19) is of the form of the finite degree of freedom equations, Eqs. (6), of Sec. II except here the sum ranges to infinity. The form Eq.

(20) is obviously antisymmetric and it is not difficult to verify Eq. (5).

At first, one might think that a truncation of the $J_{\underline{k},\underline{\ell}}$ would yield a finite Hamiltonian system which to some accuracy would mimic the original. Unfortunately, the process of truncation destroys the Jacobi identity. One must seek a change of variables which allows truncation. Canonical variables are suited for this purpose and in the next section we discuss this.

IV. Canonical Descriptions

As was noted in Sec. III, the Poisson bracket for the discrete vortex picture is embedded in that for the field. To see the connection between the two, we expand the vorticity (distributed vorticity) as follows:

$$\omega(\underline{x}) = \sum_i k_i \, \delta(\underline{x}-\underline{x}_i) \quad , \tag{21}$$

where $\delta(\underline{x})$ is the Dirac delta function, the k_i are constants and ω obtains its t dependence through the $\underline{x}_i$. Then using Eq. (17) we obtain the identity

$$\frac{\partial \hat{F}}{\partial x_i} = k_i \frac{\partial}{\partial x} \frac{\delta \hat{F}}{\delta \omega} \Bigg|_{\underline{x} = (x_i, y_i)} \tag{22}$$

where the functional $\hat{F}$ on the left hand side is now to be regarded as a function of the variables x_i and y_i. Similarly we obtain the relation for $\partial\hat{F}/\partial y_i$. Substituting Eqs. (21) and (22) into Eq. (15) yields

$$[\hat{F},\hat{G}] = \sum_j \frac{1}{k_j} \left(\frac{\partial \hat{F}}{\partial x_j} \frac{\partial \hat{G}}{\partial y_j} - \frac{\partial \hat{F}}{\partial y_j} \frac{\partial \hat{G}}{\partial x_j} \right) \quad . \tag{23}$$

Further, if we substitute Eq. (21) into the Hamiltonian, Eq. (13), we obtain

$$\hat{H} = \frac{-1}{4\pi} \sum_{i,j} k_i k_j \ln R_{ij} \quad .$$

Since this is singular along the diagonal i=j, we remove the self-energy of each vortex and obtain the usual result

$$\hat{H} = \frac{-1}{2\pi} \sum_{i>j} k_i k_j \ln R_{ij} \quad . \tag{24}$$

Equations (23) and (24) reproduce the Eqs. (1). Hence, we see that expansion of ω in a Fourier-Dirac series is a particular way of discretizing, a way which allows truncation without destroying the Hamiltonian structure. We now discuss another approach.

The cosymplectic form, Eq. (20), suggests by its linearity in $\omega_{\underline{k}+\underline{\ell}}$ that a quadratic change of variables (i.e., $\omega \sim \phi^2$, where ϕ is the new variable) is needed in order to achieve canonical form. Such a transformation [given by Eq. (31)] removes the nonlinearty present in the Poisson bracket and places it in the Hamiltonian [Eq. (30)]. Enroute to arriving at this result we introduce a Clebsch potential representation of the vorticity,[21]

$$\omega = \frac{\partial\psi}{\partial x}\frac{\partial\chi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\chi}{\partial x} \quad . \tag{25}$$

This substitution transforms the Poisson bracket, Eq. (15), into canonical form. Clearly, Eq. (25) is not uniquely invertable. We have the local gauge condition that any function $\bar{\psi}$, such that $\bar{\psi}_x \chi_y - \bar{\psi}_y \chi_x = 0$, can be added to ψ (and likewise for χ).

The chain rule for functional differentiation yields

$$\frac{\delta\hat{F}}{\delta\psi} = \nabla \cdot \left(\frac{\delta\hat{F}}{\delta\omega}\,\hat{z} \times \nabla\chi\right)$$

$$\frac{\delta\hat{F}}{\delta\chi} = -\nabla \cdot \left(\frac{\delta\hat{F}}{\delta\omega}\,\hat{z} \times \nabla\psi\right) \quad , \tag{26}$$

where on the left $\hat{F}$ is now regarded as a functional of ψ and χ. The canonical Poisson bracket for χ and ψ is

$$[\hat{F},\hat{G}] = \int \left(\frac{\delta\hat{F}}{\delta\psi}\frac{\delta\hat{G}}{\delta\chi} - \frac{\delta\hat{G}}{\delta\psi}\frac{\delta\hat{F}}{\delta\chi}\right) d\tau \quad ,$$

which upon substitution of Eqs. (26) yields the bracket Eq. (15). Clearly ψ and χ satisfy

$$\dot{\psi} = \frac{\delta\hat{H}}{\delta\chi} \quad ; \qquad \dot{\chi} = -\frac{\delta\hat{H}}{\delta\psi} \quad . \tag{27}$$

Upon Fourier transformation Eqs. (27) become

$$\dot{\psi}_{\underline{k}} = \frac{1}{(2\pi)^2}\frac{\partial\hat{H}}{\partial\chi_{-\underline{k}}} \quad ; \qquad \dot{\chi}_{\underline{k}} = -\frac{1}{(2\pi)^2}\frac{\partial\hat{H}}{\partial\psi_{-\underline{k}}} \quad . \tag{28}$$

We now introduce the field variable ϕ_k as follows:

$$\psi_{\underline{k}} = \frac{1}{2\pi}\left(\frac{\phi_{\underline{k}} + \phi^*_{-\underline{k}}}{\sqrt{2}}\right) \quad ; \qquad \chi_{\underline{k}} = \frac{-i}{2\pi}\left(\frac{\phi_{\underline{k}} - \phi^*_{-\underline{k}}}{\sqrt{2}}\right) \quad .$$

(This form maintains the reality condition for $\psi_{\underline{k}}$ and $\chi_{\underline{k}}$.) In terms of these variables Eqs. (28) become

$$i\dot{\phi}_{\underline{k}} = \frac{\partial\hat{H}}{\partial\phi^*_{\underline{k}}} \quad ; \qquad i\dot{\phi}^*_{\underline{k}} = -\frac{\partial\hat{H}}{\partial\phi_{\underline{k}}} \quad , \tag{29}$$

and the Hamiltonian has the form

$$\hat{H} = \sum_{\underline{\ell}+\underline{m}=\underline{s}+\underline{t}} S_{\underline{\ell},\underline{m},\underline{s},\underline{t}}\, \phi^*_{\underline{\ell}}\, \phi^*_{\underline{m}}\, \phi_{\underline{s}}\, \phi_{\underline{t}} \quad , \tag{36}$$

where the matrix elements $S_{\underline{\ell},\underline{m},\underline{s},\underline{t}}$ are

$$S_{\underline{\ell},\underline{m},\underline{s},\underline{t}} = \frac{1}{16\pi^2}\left[\frac{\hat{z}\cdot(\underline{t}\times\underline{\ell})\;\hat{z}\cdot(\underline{m}\times\underline{s})}{|\underline{\ell}-\underline{t}|\;|\underline{m}-\underline{s}|} + \frac{\hat{z}\cdot(\underline{s}\times\underline{\ell})\;\hat{z}\cdot(\underline{m}\times\underline{t})}{|\underline{\ell}-\underline{m}|\;|\underline{m}-\underline{t}|}\right] \quad .$$

The quadratic transformation mentioned above is

$$\psi_{\underline{k}} = \sum_{\underline{t}=\underline{k}+\underline{\ell}} \frac{i\hat{z}\cdot(\underline{t}\times\underline{\ell})}{(2\pi)^2}\, \phi_{\underline{t}}\phi^*_{\underline{\ell}} \quad . \tag{37}$$

Hence, we see the connection between Clebsch potentials and our bracket. This transformation allows discretization and truncation while not destroying the Hamiltonian structure.

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Appendix

Here we generalize the method used by P. Lax[22] for the Gardner bracket, to prove the Jacobi identity for Eq. (15). We suppose $F[u]$ is a functional of the variable u. Recall the functional derivative is defined by

$$\frac{d}{d\varepsilon} \hat{F}\{u + \varepsilon\omega\} \Big|_{\varepsilon=0} = \langle \frac{\delta \hat{F}}{\delta u} | \omega \rangle \quad .$$

We denote $\hat{G} = \langle \frac{\delta \hat{F}}{\delta u} | \omega \rangle$. $\hat{G}$ can again be regarded as a functional of u. Performing a second variation we obtain

$$\frac{d}{d\eta} \hat{G}\{u + \eta z\} \Big|_{\eta=0} = \langle \frac{\delta^2 \hat{F}}{\delta u^2} z | w \rangle$$

where the symbol $\delta^2\hat{F}/\delta u^2$ is used to denote an operator acting on z. By the equality of mixed partial derivatives this operator is self-adjoint,

$$\langle \frac{\delta^2 \hat{F}}{\delta u^2} z \mid w \rangle = \langle z \mid \frac{\delta^2 \hat{F}}{\delta u^2} w \rangle \quad .$$

Let us now take the variation of a bracket $[\hat{F},\hat{G}]$ defined by

$$[\hat{F},\hat{G}] = \langle \frac{\delta \hat{F}}{\delta u} \mid O \frac{\delta \hat{G}}{\delta u} \rangle \quad ,$$

where the operator O is anti-self-adjoint.

$$\frac{d}{d\varepsilon} [\hat{F},\hat{G}] \{u + \varepsilon w\} \Big|_{\varepsilon=0} = \langle \frac{\delta [\hat{F},\hat{G}]}{\delta u} \mid w \rangle$$

$$= \langle \frac{\delta^2 \hat{F}}{\delta u^2} w \mid O \frac{\delta \hat{G}}{\delta u} \rangle + \langle \frac{\delta \hat{F}}{\delta u} \mid O \frac{\delta^2 \hat{G}}{\delta u^2} w \rangle + \langle \frac{\delta \hat{F}}{\delta u} \mid \frac{\delta O}{\delta u} \frac{\delta \hat{G}}{\delta u} \rangle \quad ,$$

In this expression the first two terms are straight forward, the last comes from any dependence the operator O may have upon u; i.e.,

$$\frac{d}{d\varepsilon} O\ (u + \varepsilon w)\ \Big|_{\varepsilon=0} \equiv \frac{\delta O_w}{\delta u} \quad .$$

Isolating w we obtain

$$\frac{\delta[\hat{F},\hat{G}]}{\delta u} = \frac{\delta^2\hat{F}}{\delta u^2} \circ \frac{\delta\hat{G}}{\delta u} - \frac{\delta^2\hat{G}}{\delta u^2} \circ \frac{\delta\hat{F}}{\delta u} + T\ (\frac{\delta\hat{F}}{\delta u}, \frac{\delta\hat{G}}{\delta u}) \qquad \text{(A-1)}$$

where the operator T comes from removing $\delta O_w/\delta u$ from w. T is antisymmetric in its arguments.

The Jacobi identity is

$$S = \langle\frac{\delta\hat{E}}{\delta u} \mid O\ \frac{\delta[\hat{F},G]}{\delta u}\rangle + \langle\frac{\delta\hat{F}}{\delta u} \mid O\ \frac{\delta[\hat{G},\hat{E}]}{\delta u}\rangle + \langle\frac{\delta\hat{G}}{\delta u} \mid O\ \frac{\delta[\hat{E},\hat{F}]}{\delta u}\rangle = 0 \quad . \qquad \text{(A-2)}$$

Inserting Eq. (A-1) into (A-2) and using the self-adjointness of $\frac{\delta^2\hat{F}}{\delta u^2}$ and the anti-self-adjointness of O, we obtain

$$S = \langle\frac{\delta\hat{E}}{\delta u} \mid O\ T\ (\frac{\delta\hat{F}}{\delta u}, \frac{\delta\hat{G}}{\delta u})\rangle + \langle\frac{\delta\hat{F}}{\delta u} \mid O\ T\ (\frac{\delta\hat{G}}{\delta u}, \frac{\delta\hat{E}}{\delta u})\rangle + \langle\frac{\delta G}{\delta u} \mid O\ T\ (\frac{\delta E}{\delta u}, \frac{\delta F}{\delta u})\rangle = 0 \ . \qquad \text{(A-3)}$$

This equation is the functional equivalent of Eq. (5).

Now consider the bracket, Eq. (15). We obtain

$$\frac{\delta[\hat{F},\hat{G}]}{\delta\omega} = [\ \frac{\delta\hat{F}}{\delta\omega}, \frac{\delta\hat{G}}{\delta\omega}\ ] + \text{operator terms,}$$

where the first term is $T\left(\frac{\delta \hat{F}}{\delta\omega}, \frac{\delta \hat{G}}{\delta\omega}\right)$ and the remaining terms as shown above do not enter Eq. (A-3). Hence,

$$S = \int \omega \left\{\frac{\delta \hat{E}}{\delta\omega}, \left\{\frac{\delta \hat{F}}{\delta\omega}, \frac{\delta \hat{G}}{\delta\omega}\right\}\right\} d\tau + \text{cyc} \quad .$$

Clearly, S vanishes by virtue of the Jacobi identity for the discrete bracket.

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16. This formula is not unique. Without damage to the forthcoming Hamiltonian structure one may add to $\underline{v}$ the cross product of $\hat{z}$ with the gradient of an arbitrary harmonic function.

17. Clearly, this bracket is noncanonical. It appears that the first use of a noncanonical bracket was by C. S. Gardner (Ref. 18) where a bracket for the Korteweg-deVries equation was given. Recently brackets for an ideal fluid and MHD were obtained in Ref. 19. Brackets for the Maxwell-Vlasov and Poisson-Vlasov equations were obtained in Ref. 20. (See Ref. 15 for generalization.) We note that the Poisson bracket given here [Eq. (15)] is precisely that for the one-dimensional Vlasov-Poisson equations if one

replaces the vorticity by the distribution function and the phase space (x,y) by (x,v), although the Hamiltonians are different. This is the subject of, P. J. Morrison, Princeton Plasma Phys. Lab Report #1788 (1981), to be published.

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