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THE PERMUTATION GROUP AND THE COUPLING OF n SPIN- $\frac{1}{2}$ ANGULAR MOMENTA

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ABSTRACT

The classic problem of constructing the sharp spin states for n spin- $\frac{1}{2}$ particles by simultaneously classifying the states by their irreducible transformation properties under both SU(2) and S_n is solved explicitly by recognizing that these states are a special case of the boson polynomials of U(n).

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I. Introduction

Much of this workshop has dealt with the rôle of the permutation group in elucidating the classical structure of complex molecules. The rôle of the permutation group in the quantum theory of molecules, while certainly not ignored, has perhaps received less emphasis. I wish therefore to discuss the rôle of the permutation group in a simple but important problem in quantum mechanics.

The problem I have in mind is that of constructing the quantum spin states of a n electron system such that (i) the states have sharp total spin angular momentum; and (ii) the states transform irreducibly under permutations of the electrons. The importance of this special result in implementing the Pauli principle is well known.

The solution to this problem has, of course, been given by numerous authors, using a variety of techniques (a few of the references are given below, Ref. 1-4). H. Weyl (5) had already given the structure of the general result in 1928, but he did not give the explicit answer. My only excuse in presenting this result anew is to illustrate the novelty and explicitness of results which can be obtained as special cases of the boson polynomials discussed in Prof. Biedenharn's talk (6). This classic problem is also a nice one for illustrating a cooperative rôle of two group structures: The symmetric group S_n and the quantum mechanical rotation group SU(2).

II. Formulation of the Problem

In order to give a precise statement of the problem, we first summarize the properties of the irreducible representations of SU(2) and S_n.

A. Irreducible representations of the quantum mechanical rotation group.

The Lie algebraic and Hilbert space properties of the total angular momentum $\vec{J} = (J_1, J_2, J_3)$ of a physical system (J_i is the component of $\vec{J}$ relative to an inertial frame of reference) may be summarized as follows:

(1). Each J_i is a linear Hermitian operator on a separable Hilbert space H. The components of $\vec{J}$ satisfy the commutation relations

$$\vec{J} \times \vec{J} = i\, \vec{J} \,. \qquad (1)$$

(2). H contains a subspace H_j of dimension $2j+1$, $j \in \{0,1/2,1,\ldots\}$, which is invariant and irreducible under the action of J_1, J_2, J_3. The space H_j possesses an orthonormal basis

$$\{\,|jm\rangle \;\big|\; m = j, j-1, \ldots, -j\} \qquad (2)$$

such that the vectors in this basis are simultaneous eigenvectors of the square of total angular momentum, $J^2 = J_1^2 + J_2^2 + J_3^2$, and of J_3, that is,

$$J^2|jm\rangle = j(j+1)\,|jm\rangle\ , \tag{3}$$

$$J^3|jm\rangle = m|jm\rangle\ ; \tag{4}$$

furthermore, the action of J_1 and J_2 on this basis is expressed by

$$(J_1 \pm iJ_2)|jm\rangle = [(j\mp m)(j\pm m+1)]^{1/2}|jm\pm 1\rangle\ . \tag{5}$$

The irreducible representation of SU(2) carried by the space H_j has the following properties.

Let $R(\phi,\hat{n})$ denote a rotation of the Euclidean space R^3 about an axis specified by the unit vector $\hat{n}$ by an angle ϕ. The Hilbert space H_j then undergoes a transformation onto itself which is given up to ± sign by the unitary operator

$$U = e^{-i\phi\hat{n}\cdot\vec{J}}\ . \tag{6}$$

In particular, the transformation U of the basis (2) is

$$U|jm\rangle = \sum_{m'} D^j_{m'm}(U)\,|jm'\rangle\ , \tag{7}$$

where U denotes the 2x2 unitary matrix

$$U = e^{-i\phi(\hat{n}\cdot\vec{\sigma})/2} = 1\cos\frac{\phi}{2} - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\phi}{2} \tag{8}$$

in which $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ denotes the Pauli matrices. $D^j_{m'm}$ denotes a function which maps each $U \epsilon$ SU(2) to the complex numbers and is given explicitly by

$$D^j_{m'm}(U) = [(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2} \times \sum_{\boxed{\alpha}} \frac{(u^1_1)^{\alpha^1_1}(u^1_2)^{\alpha^1_2}(u^2_1)^{\alpha^2_1}(u^2_2)^{\alpha^2_2}}{(\alpha^1_1)!\ (\alpha^1_2)!\ (\alpha^2_1)!\ (\alpha^2_2)!}\ , \tag{9}$$

where

$$U = \begin{pmatrix} u^1_1 & u^2_1 \\ u^1_2 & u^2_2 \end{pmatrix}. \tag{10}$$

The summation is over all nonnegative integers α^j_i which have the fixed row and column sums indicated by the notation (cf. Ref. 6)

$$\begin{array}{|cc|l} \alpha_1^1 & \alpha_1^2 & j+m' \\ \alpha_2^1 & \alpha_2^2 & j-m' \\ \hline j+m & j-m & \end{array} \quad . \qquad (11)$$

Letting $D^j(U)$ denote the $(2j+1)\mathrm{x}(2j+1)$ matrix representation of U on the space H, we have: *The group of matrices*

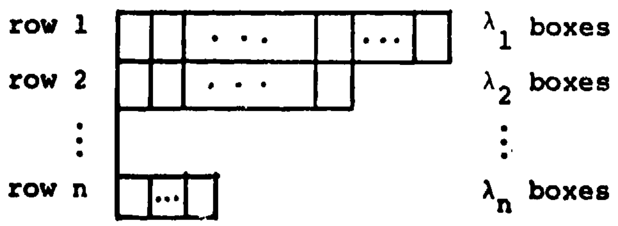
$$D^j = \{ D^j(U) | U \varepsilon SU(2) \} \qquad (12)$$

is an irreducible unitary representation of SU(2). Furthermore, letting $j = 0, 1/2, 1, \ldots,$ we obtain all the inequivalent irreducible unitary representations of SU(2).

B. Irreducible representations of the symmetric group

A great deal has already been said in this conference about the irreducible representations of the symmetric group, S_n, including an explicit construction of the Yamanouchi real orthogonal representations given in Prof. Biedenharn's talk. We will recall here three basic results:

(i) The irreducible representations of S_n are in one-to-one correspondence with the partitions $[\lambda_1\ \lambda_2 \ldots \lambda_n]$, $\sum_i \lambda_i = n$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, of n into not more than n nonzero parts. Each such partition $[\lambda]$ also defines a *Young frame*



(13)

Thus, an irreducible representation of S_n may be denoted by $\Gamma^{[\lambda]}$ and the set of all (inequivalent) irreducible representations by $\{\Gamma^{[\lambda]} | [\lambda]$ is a partition of n$\}$.

(ii) The basis vectors of a linear vector space $V^{[\lambda]}$ which carries an irreducible representation of S_n are in one-to-one correspondence with the set of *standard Young tableaux* of shape $[\lambda]$. (A standard Young tableaux is a Young frame in which the n boxes have been "filled in" with the integers 1,2,...,n without repetition and such that the sequence of integers appearing in any row or any column is strictly increasing when read from left to right across the row and from top to bottom down the column.)

The Yamanouchi symbol (y) of a standard Young tableau is the sequence of integers $(y) = (y_1, y_2, \ldots, y_n)$ in which y_{n-s+1} is the number of the row in which integer s occurs. Different standard tableaux have different Yamanouchi symbols. Thus, a basis of $V^{[\lambda]}$ may be denoted by

$$\left\{ |[\lambda];(y)\rangle \,\middle|\, \begin{array}{l}(y) \text{ is the Yamanouchi symbol of a standard} \\ \text{Young tableau of shape } [\lambda]\end{array} \right\} \qquad (14)$$

(iii) Each permutation $P \in S_n$ defines a mapping of $V^{[\lambda]}$ onto $V^{[\lambda]}$; hence,

$$P: \; |[\lambda];(y)\rangle \rightarrow \sum_{(y')} \Gamma^{[\lambda]}_{(y')(y)}(P)\,|[\lambda];(y')\rangle \qquad (15)$$

Thus, the irreducible representation of S_n carried by $V^{[\lambda]}$ is

$$\Gamma^{[\lambda]} = \left\{ \Gamma^{[\lambda]}(P) \,\middle|\, P \in S_n \right\} \quad . \qquad (16)$$

C. Coupling of n kinematically independent angular momenta

The Hilbert space for describing the union of two physical systems, when considered as a single physical system, is the tensor product of the Hilbert spaces of the individual systems. The space of interest for the determination of the states of the total angular momentum, $\vec{J} = \sum_{k=1}^{n} \vec{J}(k)$, of n individual physical systems labelled by $1,2,\ldots,n$, where system k is in the angular momentum state j_k, is the tensor product space $\mathcal{H}_{(j_1\ldots j_n)}$ defined by

$$\mathcal{H}_{(j_1\ldots j_n)} = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \ldots \otimes \mathcal{H}_{j_n} \quad . \qquad (17)$$

A basis of this space is

$$\left\{ |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes \ldots \otimes |j_n m_n\rangle \,\middle|\, \text{each } m_k = j_k, \ldots, -j_k \right\} \quad , \qquad (18)$$

and the dimension of the space is

$$\dim \mathcal{H}_{(j_1\ldots j_n)} = \prod_k (2j_k+1) \quad . \qquad (19)$$

Let $i_1, i_2, \ldots, i_n$ denote a rearrangement of the integers $1,2,\ldots,n$. We define the action of the permutation

$$P = \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix} \qquad (20)$$

on the space $H(j_1 \ldots j_n)$ to be the linear mapping

$$P: \quad H(j_1 \ldots j_n) \to H(j_{i_1} \ldots j_{i_n}) \tag{21}$$

defined explicity by

$$P: \quad |j_1 m_1\rangle \otimes \ldots \otimes |j_n m_n\rangle \to |j_{i_1} m_{i_1}\rangle \otimes \ldots \otimes |j_{i_n} m_{i_n}\rangle \ . \tag{22}$$

For the problem at hand, we require only the special case $j_1 = \ldots = j_n = 1/2$ and will denote the corresponding tensor product space by H:

$$H = H_{\frac{1}{2}} \otimes \ldots \otimes H_{\frac{1}{2}} \tag{23}$$

with basis

$$\left\{ |\tfrac{1}{2}\, m_1\rangle \otimes \ldots \otimes |\tfrac{1}{2}\, m_n\rangle \,\middle|\, m_k = \tfrac{1}{2}, -\tfrac{1}{2} \right\} \tag{24}$$

so that

$$\dim H = 2^n \ . \tag{25}$$

In this case, we have

$$P: \quad H \to H \ , \text{ each } P\varepsilon S_n \ . \tag{26}$$

Thus, H is the carrier space of a representation Γ of S_n:

$$\Gamma = \{\Gamma(P) \,|\, P\varepsilon S_n\} \ , \tag{27}$$

where each $\Gamma(P)$ is a $2^n \times 2^n$ permutation matrix.

The unitary rotation (7) of the single electron states $|\frac{1}{2},\frac{1}{2}\rangle$, $|\frac{1}{2},-\frac{1}{2}\rangle$ takes the form

$$U|\tfrac{1}{2},\tfrac{1}{2}\rangle = u_1^1|\tfrac{1}{2},\tfrac{1}{2}\rangle + u_2^1|\tfrac{1}{2},-\tfrac{1}{2}\rangle \ , \tag{28}$$

$$U|\tfrac{1}{2},-\tfrac{1}{2}\rangle = u_1^2|\tfrac{1}{2},\tfrac{1}{2}\rangle + u_2^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle \ .$$

Under a rotation $R(\phi,\hat{n})$ of the composite n-particle system in R^3, the tensor product space H undergoes the transformation (up to ± sign) given by

$$U: H \to H \ ,$$

$$U: |\tfrac{1}{2}m_1\rangle \otimes \ldots \otimes |\tfrac{1}{2}m_n\rangle \to U|\tfrac{1}{2}m_1\rangle \otimes \ldots \otimes U|\tfrac{1}{2}m_n\rangle \ . \tag{29}$$

Thus, H is the carrier space of the representation

$$D = \{D(U) = U \otimes \ldots \otimes U \,|\, U\varepsilon SU(2)\} \tag{30}$$

of SU(2) (⊗ here designates the matrix direct product).

The actions of the operators P and *U* commute an *H* and correspondingly the matrix representations given by Eqs. (27) and (30) also commute:

$$\Gamma(P)D(U) = D(U)\Gamma(P), \text{ each } P\varepsilon S_n, \text{ each } U\varepsilon SU(2) \ . \tag{31}$$

Accordingly, *H is the carrier space of the representation*

$$\{D(U)\Gamma(P)\,|\,P\varepsilon S_n, U\varepsilon SU(2)\} \tag{32}$$

of *the direct product group* SU(2) × S_n.

We may now give a precise statement of the problem to be solved: *Split the space H into a direct sum of carrier spaces of irreducible representations of* SU(2) × S_n *or, equivalently, reduce the representation* (32) *of* SU(2) × S_n *into its irreducible components.*

III. Solution to the Problem

Since the announced purpose of this talk was to demonstrate that the complete solution to the problem stated above could be obtained as a special case of the boson polynomials discussed in the previous talk (Ref. 6), we proceed directly to that result.

Consider then the U(n)*U(n) boson state vectors in which the double Gel'fand patterns are specialized in the following manner:

$$\left|\begin{pmatrix} & & (\mu) & & \\ \frac{n}{2}+j & & \frac{n}{2}-j & & 0 \cdots 0 \\ & \frac{n}{2}+j & & \frac{n}{2}-j & 0 \\ & & \frac{n}{2}+j & \frac{n}{2}-j & \\ & & \frac{n}{2}+m & & \end{pmatrix}\right\rangle$$

$$= \sum_{k_1\cdots k_n}{}_{0}^{1} \left\langle\begin{matrix} 2j & 0 \\ & j+m \end{matrix}\right| \left\langle\begin{matrix} i_1 \\ 1 \quad 0 \\ k_n \end{matrix}\right\rangle \left\langle\begin{matrix} i_2 \\ 1 \quad 0 \\ k_{n-1} \end{matrix}\right\rangle \cdots \left\langle\begin{matrix} i_n \\ 1 \quad 0 \\ k_1 \end{matrix}\right\rangle \left|\begin{matrix} 0 & & 0 \\ & 0 & \end{matrix}\right\rangle$$

$$\cdot\, a^1_{2-k_1}\, a^2_{2-k_2} \cdots a^n_{2-k_n}|0\rangle \ , \tag{33}$$

where

(i) (μ) is any lexical Gel'fand pattern with weight $(1,1,\ldots,1)$;

(ii) $(i_1 i_2 \ldots i_n)$ is the sequence of 0's and 1's such that under the identification 0 = 2 the sequence becomes the Yamanouchi symbol of the standard Young tableau of shape $[\frac{n}{2}+j, \frac{n}{2}-j]$ that corresponds to the Gel'fand pattern (μ);

(iii) the symbol $\left|\begin{matrix} 2j & 0 \\ \multicolumn{2}{c}{j+m} \end{matrix}\right\rangle$ denotes the basis vector $|jm\rangle$ of standard angular momentum theory;

(iv) the action of each of the four fundamental Wigner operators

$$\left\langle \begin{matrix} & i & \\ 1 & & 0 \\ & k & \end{matrix} \right\rangle \qquad (i,\ k = 0,1)$$

on an arbitrary basis vector $|jm\rangle$ is obtained from the pattern calculus rules. For convenience, we state these results:

$$\begin{aligned}
\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle |jm\rangle &= \left[\frac{j+m+1}{2j+1}\right]^{1/2} |j+\tfrac{1}{2}, m+\tfrac{1}{2}\rangle , \\
\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle |jm\rangle &= \left[\frac{j-m+1}{2j+1}\right]^{1/2} |j+\tfrac{1}{2}, m-\tfrac{1}{2}\rangle , \\
\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle |jm\rangle &= -\left[\frac{j-m}{2j+1}\right]^{1/2} |j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle , \\
\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle |jm\rangle &= \left[\frac{j+m}{2j+1}\right]^{1/2} |j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle .
\end{aligned} \tag{34}$$

Remark. Each i_k in Eq. (33) may assume the values 0 or 1 independently. Quite remarkably, the matrix element of the string of Wigner operators appearing in Eq. (33) is automatically zero unless the sequence $(i_1 i_2 \ldots i_n)$ is the Yamanouchi symbol for (μ) as explained in (ii) above.

Consider next the transformation properties of the boson state vectors (33) (as discussed in Ref. 6).

Under the unitary transformation of the n×n boson matrix $A = (a_i^j)$ given by

$$A \to \tilde{V} A, \; V = \begin{pmatrix} U & 0 \\ 0 & I_{n-2} \end{pmatrix}, \; U \epsilon SU(2) \; , \tag{35}$$

we see that the basis vectors (33) undergo the transformation (7).

Consider also the unitary transformation of A given by

$$A \to AI_P \quad , \tag{36}$$

where I_P is the permutation matrix

$$I_P = [e_{i_1} e_{i_2} \ldots e_{i_n}] \tag{37}$$

in which e_i denotes a unit column vector with 1 in row i and zeroes elsewhere, and P denotes the permutation

$$P = \begin{pmatrix} 1 & 2 & \ldots n \\ i_1 & i_2 & \ldots i_n \end{pmatrix} \; . \tag{38}$$

Then, from the results of Ref. 6, we find that under the transformation (36) of A, the basis vectors (33) undergo the transformation (15) in which

$$[\lambda] = [\tfrac{n}{2} + j \;\; \tfrac{n}{2} - j \;\; 0 \ldots 0] \; , \tag{39}$$

and (y) is the Yamanouchi symbol corresponding to $(i_1 i_2 \ldots i_n)$ as described under (ii) above. (The irreducible representation $\Gamma^{[\lambda]}$ is then the Yamanouchi real orthogonal representation.)

Thus, the orthonormal basis vectors (33) corresponding to fixed n and j $(0 \leq j \leq n/2)$, but with $m = j, \ldots, -j$ and (μ) running over all lexical patterns of weight $(1,1,\ldots,1)$, are the basis vectors of a vector space which carries the irreducible representation

$$D^j \otimes \Gamma^{[\frac{n}{2}+j \frac{n}{2} - j]} \tag{40}$$

of $SU(2) \times S_n$.

Observe next that since the boson operators $a_i^j, j = 1,2,\ldots,n$ and $i = 3,4,\ldots,n$ do not occur in the basis vectors (33), we can put $a_i^j = 0$ for $j = 1,\ldots,n$ and $i = 3,\ldots,n$. [The general boson polynomial is then zero unless the lower pattern is of the form occurring in the left-hand side of Eq. (33).] If we now let A denote the $2 \times n$ matrix

$$A = \begin{pmatrix} a_1^1 & a_1^2 \ldots a_1^n \\ a_2^1 & a_2^2 \ldots a_2^n \end{pmatrix} \; , \tag{41}$$

then the transformation of A corresponding to Eqs. (35) and (36) is

$$A \rightarrow \tilde{U}AI_P, \quad U\varepsilon SU(2), \quad P\varepsilon S_n \quad . \tag{42}$$

Comparing this transformation of the boson matrix A with the transformations (26) and (29) of H, we see that the appropriate way to make the transcription from the boson basis vectors (33) to basis vectors of H is to make the replacement

$$a^1_{2-k_1} a^2_{2-k_2} \cdots a^n_{2-k_n} \left| 0 \right\rangle \rightarrow \left| \begin{matrix} 1 & 0 \\ k_1 & \end{matrix} \right\rangle \otimes \left| \begin{matrix} 1 & 0 \\ k_2 & \end{matrix} \right\rangle \otimes \cdots \otimes \left| \begin{matrix} 1 & 0 \\ k_n & \end{matrix} \right\rangle \quad , \tag{43}$$

where

$$\left| \begin{matrix} 1 & & 0 \\ & 1 & \end{matrix} \right\rangle = \left| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle , \quad \left| \begin{matrix} 1 & & 0 \\ & 0 & \end{matrix} \right\rangle = \left| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle \quad . \tag{44}$$

Let us summarize the results obtained above. We introduce the simpler notation $|(i_1 i_2 \cdots i_n);\, jm\rangle$ for the basis vectors obtained from Eq. (33) by the replacement (43):

$$\left|(i_1 i_2 \cdots i_n);\, jm\right\rangle = \sum_{k_1 \cdots k_n}{}^{1}_{0} \left\langle \begin{matrix} 2j & 0 \\ j+m & \end{matrix} \right| \left\langle \begin{matrix} & i_1 & \\ 1 & & 0 \\ & k_n & \end{matrix} \right\rangle \cdots \left\langle \begin{matrix} & i_n & \\ 1 & & 0 \\ & k_1 & \end{matrix} \right\rangle \left| \begin{matrix} 0 & & 0 \\ & 0 & \end{matrix} \right\rangle$$

$$\cdot \left| \begin{matrix} 1 & 0 \\ k_1 & \end{matrix} \right\rangle \otimes \cdots \otimes \left| \begin{matrix} 1 & 0 \\ k_n & \end{matrix} \right\rangle \quad . \tag{45}$$

Then the set of vectors

$$\left\{ \left|(i_1 i_2 \cdots i_n);\, jm\right\rangle \;\middle|\; \begin{matrix} (i_1 i_2 \cdots i_n) \text{ is a Yamanouchi symbol} \\ (0=2) \text{ of a standard Young tableau of} \\ \text{shape } [\tfrac{n}{2} + j \;\; \tfrac{n}{2} - j];\ m = j, j-1, \ldots, -j \end{matrix} \right\} \tag{46}$$

is an orthonormal basis of a subspace $H(\frac{n}{2} + j,\ \frac{n}{2} - j)$ of H which is a carrier space of the irreducible representation

$$D^j \otimes \Gamma^{[\frac{n}{2} + j \;\; \frac{n}{2} - j]} \tag{47}$$

of $SU(2) \times S_n$. The explicit transformation properties are:

$$U \left|(i_1 \cdots i_n);\, jm\right\rangle = \sum_{m'} D^j_{m'm}(U) \left|(i_1 \cdots i_n);\, jm'\right\rangle \quad . \tag{48}$$

$$P{:}\left|(i_1 \cdots i_n);\, jm\right\rangle \rightarrow \sum_{(i'_1 \cdots i'_n)} \Gamma^{[\frac{n}{2} + j \;\; \frac{n}{2} - j]}_{(i'_1 \cdots i'_n),(i_1 \cdots i_n)}(P) \left|(i'_1 \cdots i'_n);jm\right\rangle . \tag{49}$$

The solution to the problem posed in Section II is now completed with the result:

$$H = \sum_j \bullet H\left(\frac{n}{2} + j,\ \frac{n}{2} - j\right) \quad , \tag{50}$$

where the sum is over

$$j = \frac{n}{2},\ \frac{n}{2} - 1, \ldots,\ \frac{1}{2} \text{ or } 0 \quad . \tag{51}$$

Equation (50) may also be proved by appealing to the known properties of the boson polynomials: The set of boson polynomials

$$B\begin{pmatrix} & & (m') & & \\ m_{1n} & m_{2n} & 0 \ \ldots & 0 & \\ & m_{1n} \quad m_{2n} & & 0 & \\ & m_{1n} \quad m_{2n} & & & \\ & & m_{11} & & \end{pmatrix}(A) \tag{52}$$

corresponding to all allowed patterns (m') and m_{11} and to all partitions $[m_{1n}m_{2n}]$ of N spans the space of all polynomials in the 2n bosons a_1^j, a_2^j $(j = 1,2,\ldots,n)$ which are homogeneous of degree N. For the case at hand, we have $N = n = m_{1n} + m_{2n}$ and $2j = m_{1n} - m_{2n}$ so that $m_{1n} = \frac{n}{2} + j$ and $m_{2n} = \frac{n}{2} - j$, where all partitions $[m_{1n}m_{2n}]$ of n are obtained by letting j run over $\frac{n}{2}$, $\frac{n}{2} - 1, \ldots, \frac{1}{2}$ or 0. [Observe that this result is applicable to the space H since the mapping (43) can be reversed.]

One could, of course, also prove Eq. (50) by a dimensionality check, it being necessary then to verify that

$$\sum_j \dim D^j \cdot \dim \Gamma^{[\frac{n}{2}+j,\frac{n}{2}-j]} = 2^n \quad , \tag{53}$$

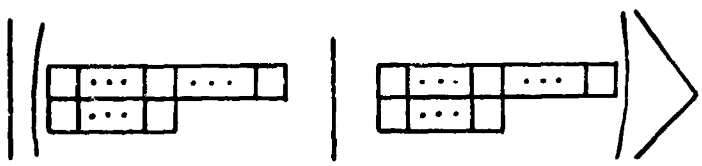
where

$$\dim D^j = (2j+1) \quad , \tag{54}$$

$$\dim \Gamma^{[\frac{n}{2}+j,\frac{n}{2}-j]} = \frac{2(2j+1)}{2j+2+n} \begin{pmatrix} n \\ \frac{n}{2} - j \end{pmatrix} \quad . \tag{55}$$

Remark. Once one knows the result in the form of Eq. (45), it is certainly not necessary to use anything as powerful as the general $U_n * U_n$ boson polynomials to obtain it, since one may prove the validity of Eq. (45) directly. The boson polynomials, however, yield many useful results when specialized in various ways, and our purpose here was to illustrate one such case of physical interest. We hope that the simplicity of the final result justifies our presenting it here.

In concluding, I would like to give one more notation for the basis vectors (45) which illustrates most vividly the interrelationship between SU(2) and S_n. A similar notation has been employed by Rota and collaborators (7) in their studies in combinatorics. This notation employs two Young frames of the same shape $[\frac{n}{2} + j \; \frac{n}{2} - j]$:



(56)

The Young frame on the right is then filled in with 1,2,...,n to obtain a standard Young tableau; the Young frame on the left is filled in with 1's and 2's to obtain a standard Weyl tableau for SU(2).

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