Proceedings, 1983 Particle Accelerator Conference, 3/21-23/83, Santa Fe, N.M. IEEE Trans. on Nucl. Sci.

BNL 32724

BNL--32724

CONF-830311--18

DE83 010158

LONGITUDINAL INSTABILITIES OF LONG GAUSSIAN BUNCHES*

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Abstract

We present an overview of the longitudinal instabilities of Gaussian bunches subject to a harmonic RF potential. Our emphasis is on the behavior of long bunches having lengths greater than the wavelength of the perturbing electromagnetic fields. We exhibit the crossover between the dominance of the synchrotron modes and the coasting-beam-like distortions of the bunch distribution, which occurs as the real or imaginary part of the coherent oscillation frequency becomes large compared to the synchrotron oscillation frequency. For a narrow band impedance the growth rate of the coasting-beam-like modes is determined by the average beam current, and for a broad band impedance the growth rate is determined by the peak current. We discuss the transition between these two regimes by considering the growth rate as a function of the bandwidth of the impedance.

Mathematical Formalism

Our starting point is the treatment of coherent instabilities developed by Wang and Pellegrini,[1] and our notation follows that of Krinsky and Wang[2] appearing in these proceedings. We shall confine our attention to the case of Gaussian bunches subject to a harmonic RF potential, $U_0(\phi) = \omega_s^2\phi^2/2$, where ω_s is the angular frequency of the synchrotron oscillations, and ϕ is the azimuthal angular coordinate relative to a synchronous particle with angular revolution frequency ω_0 and energy E_0. Our interest is in the conditions required for the line charge density $\lambda(\phi,t)$ to exhibit a coherent oscillation of frequency Ω, i.e. for $\lambda(\phi,t)$ to have the form:

$$\lambda(\phi,t) = \rho_0(\phi) + \rho(\phi)\exp(-i\Omega t). \tag{1}$$

Here, $\rho_0(\phi)$ denotes the line charge density of the unperturbed bunch and $\rho(\phi)\exp(-i\Omega t)$ is the coherent perturbation.

Let us suppose there to be M equally spaced bunches each containing N/M particles. We assume the distribution function corresponding to the unperturbed bunches is

$$\Psi_0(J) = \frac{Ne}{2\pi\omega_s L^2}\exp(-J/\omega_s L^2), \tag{2}$$

where L is the bunch length in radians, e the electric charge of a particle, and the action-variable J is related to the synchrotron oscillation amplitude r via $J = \omega_s r^2/2$. Introducing the Fourier transform ρ_n of the perturbation $\rho(\phi)$ and using the linearized Vlasov equation, the coherent oscillations are found to be described by the infinte set of linear equations:[1]

$$\sum_n T_{mn}\,\rho_n = \rho_m, \tag{3}$$

where the summation is restricted to $n = Mj + s$, for integer j varying from $j = -\infty,\ldots,\infty$, and fixed symmetric multibunch mode number $s = 0,1,2,\ldots,M-1$. Taking $\psi_0(J,\theta) = r\cos\theta$ in Eq. (21) of ref.[2], the matrix element T_{mn} is found to be given by

$$T_{mn} = \frac{i\kappa}{1-\exp(2\pi iQ)}\frac{Z_n}{n\omega_s}\int_0^\infty dJ\,\frac{d\Psi_0}{dJ}\int_{-2\pi}^{0} d\theta'\exp(-iQ\theta')$$

$$\times\frac{\partial}{\partial\theta'}\int_{-2\pi}^{0} d\theta\exp(inr\cos(\theta+\theta')-imr\cos\theta), \tag{4}$$

where $Z_n \equiv Z_n(n\omega_0+\Omega)$ denotes the longitudinal impedance, and we have defined

$$Q = \Omega/\omega_s, \tag{5}$$

$$2\pi\kappa = e\alpha\omega_0^3/2\pi E_0\,, \tag{6}$$

with α being the momentum compaction.

The representation of the matrix element T_{mn} given in Eq. (4) can be simplified by employing the two integrals:

$$\int_{-2\pi}^{0} d\theta\exp(ia\cos(\theta+\theta')-ib\cos\theta)=2\pi J_0(\sqrt{a^2+b^2-2ab\cos\theta'}), \tag{7}$$

$$\int_0^\infty r^2 dr\exp(-ar^2)J_1(br)=(b/4a^2)\exp(-b^2/4a), \tag{8}$$

where $J_k(x)$ is the k-th order Bessel function. In this manner, we find the following representation for T_{mn} involving only a single integration:

$$T_{mn}=\frac{-i\alpha\omega_0^2 eI_{av}}{2\pi E_0\omega_s^2L^2}\frac{Z_n}{n}\exp(-(|m|-|n|)^2L^2/2)H(mnL^2,Q), \tag{9}$$

with $I_{av} = Ne\omega_0/2\pi$ the average current, and

$$H(x,Q) = -x\exp(-|x|)\,A(x,Q) \tag{10}$$

$$[1-\exp(2\pi iQ)]A(x,Q)=\int_{-2\pi}^{0} d\theta\sin\theta\exp(-iQ\theta+x\cos\theta). \tag{11}$$

The relationship between this integral representation and the conventional expansion in synchrotron modes is estabished by using in Eq. (11) the generating function for I_k - Bessel functions, yielding

$$A(x,Q) = -\frac{1}{x}\sum_{k=-\infty}^{\infty}\frac{kI_k(x)}{k-Q}\,. \tag{12}$$

The synchrotron mode expansion of Eq. (12) is useful when one synchrotron mode dominates, however, when many synchrotron modes contribute, then the integral representation of Eq. (11) is more appropriate.

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*Research supported by the U.S. Department of Energy.

Short Wavelength Limit

Let us consider the case when the wavelengths of the perturbing electromagnetic fields are short compared to the bunch length. Using the integral representation of Eqs. (9) - (11), we derive an asymptotic expression for T_{mn} valid when mL and nL are large compared to unity. This representation explicitly shows the cross-over behavior between synchrotron mode oscillations and coasting beam like perturbations of the line charge density.

To proceed we rewrite Eq. (11) defining A(x,Q) in the form:

$$xA(x,Q) = -\exp(x) + \frac{Q}{\sin\pi Q}\int_0^{\pi} d\theta\, \cos Q(\pi-\theta)\, \exp(x\cos\theta). \quad (13)$$

We restrict our attention to the case when

$$|Q| << |x| \text{ and } |\mathrm{Im}Q| << \sqrt{|x|} \quad , \qquad (14)$$

and we find for $|x| \to \infty$:

$$H(x,Q) = 1-(Q\cot\pi Q)c(x,Q)-Qs(x,Q), \quad x>o, \qquad (15a)$$

$$H(x,Q) = \frac{-Q}{\sin\pi Q}\, c(-x,Q) \qquad , \quad x<o, \qquad (15b)$$

where we have defined

$$c(x,Q)+is(x,Q) = \int_0^{\infty} d\theta\, \exp(-x\theta^2/2 + iQ\theta) \ . \qquad (16)$$

Now we note that

$$c(x,Q) = \sqrt{\pi/2x}\, \exp(-Q^2/2x) \qquad (17)$$

and

$$1 + iQc(x,Q) - Qs(x,Q) = h(Q/\sqrt{x}) \ , \qquad (18)$$

where the dispersion integral h(x) is defined by[1]

$$h(x) = \int_0^{\infty} \theta d\theta\, \exp(-\theta^2/2 + ix\theta) \qquad (19)$$

$$= -\int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{2\pi}}\ \frac{\exp(-\zeta^2/2)}{(\zeta - x)^2} \ . \qquad (20)$$

Employing Eqs. (17) and (18) in Eqs. (15a, b), we find

$$H(x,Q) = h(Q/\sqrt{x}) - \sqrt{\frac{\pi}{2}}(i+\cot\pi Q)\frac{Q}{\sqrt{x}}\exp(-Q^2/2x), x>o, \quad (21a)$$

$$H(x,Q) = -\sqrt{\frac{\pi}{2}}\frac{Q}{\sqrt{|x|}\ \sin\pi Q}\exp(-Q^2/2|x|) \ , \ x<o \ . \quad (21b)$$

The desired asymptotic representation of T_{mn} follows from using Eqs. (21a,b) with $x = mnL^2$ in Eq. (9). The poles at Q = integer of the function cot πQ in Eq. (21a), correspond to the synchrotron oscillation modes which dominate for slow blowup. When $\mathrm{Im}Q >> 1$ or $|\mathrm{Re}Q| >> \sqrt{x}$, the second term in Eq. (21a) becomes negligible, due to the factors $(i + \cot\pi Q)$ and $\exp(-Q^2/2x)$, respectively, leaving the coasting beam type of behavior exhibited by $h(Q/\sqrt{x})$.

In the case of a coasting beam different revolution modes m and n are not coupled. On the other hand, these modes are coupled for a bunched beam by the matrix T_{mn}. When mL and nL are large in magnitude, and the growth rate is fast, $\mathrm{Im}Q >> 1$, then Eq. (21b) shows that the coupling between modes with mn < o becomes negligible. Hence, there is no coupling between the slow and fast waves. It is also seen from Eq. (21b) that for mL and nL large in magnitude and $|\mathrm{Re}Q| >> \sqrt{|mn|}\ L$, the slow and fast waves decouple, even when the growth rate is slow.

It therefore follows that when $|mn|\ L^2 >>1$ and either $|\mathrm{Re}Q| >> \sqrt{|mn|}\ L$ or $\mathrm{Im}Q >>1$, one has

$$T_{mn} = \frac{-i\alpha\omega_o^2\, eI_{av}}{2\pi E_o\omega_s^2 L^2}\,\frac{Z_n}{n}\,\beta(m-n)h\left(\frac{Q}{L\sqrt{mn}}\right), \ mn>o, \qquad (22a)$$

$$T_{mn} = 0 \qquad , \ mn<o \ , \qquad (22b)$$

where $\beta(n) = \exp(-n^2L^2/2)$ is the Fourier transform of the unperturbed bunch density, $\rho_o(\phi) = \exp(-\phi^2/2L^2)$.

Narrow Band Impedance

As an example we consider a resonant impedance with bandwidth so narrow that it may be approximated by

$$Z_n = Z_{n_o}\delta_{n,n_o} + Z^*_{n_o}\ \delta_{n,-n_o} \quad , \qquad (23)$$

where δ_{n,n_o} is the Kronecker delta function. Although the approximation of Eq. (23) violates causality, as embodied in the Kramers-Kronig dispersion relations, it is useful as an illustration of the formalism under study in this note. We take $n_oL >>1$, and we suppose that either ReQ or ImQ are large so that Eqs. (22a,b) are valid. It then follows from Eqs. (3) and (22b) that the coherent frequency Ω is determined by $1 = T_{n_o,n_o}$, which upon using Eq. (22a) becomes

$$1 = \frac{-i\alpha\omega_o^2 eI_{av}}{2\pi E_o\sigma^2}\,\frac{Z_{n_o}}{n_o}\, h\left(\frac{\Omega}{|n_o|\sigma}\right) , \qquad (24)$$

where $\sigma = \omega_s L$ is the spread in revolution frequency among particles in a bunch. Eq. (24) has the form of a dispersion relation for a coasting beam with current I_{av}.

The Fourier transform of the perturbation to the line charge density is seen from Eq. (3) to be $\rho_m = T_{mn_o}$. Using Eq. (22a) and performing the inverse Fourier transform, we find that the perturbation $\rho(\phi)$ is given up to a multiplicative constant by

$$\rho(\phi) = \frac{\partial}{\partial\phi}\left(\exp(in_o\phi)\ \rho_o(\phi)\right) \ . \qquad (25)$$

Broad Band Impedance

Consider a high-frequency broad-band impedance satisfying $Z_n = Z_{n_o}$, for $|n-n_o|<\Delta$, where $n_o >> \Delta >> 1/L$. Since the range of the wake field, $1/\Delta$, is short compared to the bunch length L, and certainly short compared to the spacing between bunches, we can ignore interaction between bunches. To ease the notation we assume one bunch to be in the ring (M = 1). For $n_o - \Delta < m,n < n_o + \Delta$, we approximate Eq. (22a) by

$$T_{mn} = \frac{-i\alpha\omega_o^2 eI_o}{2\pi E_o\sigma^2}\,\frac{Z_{n_o}}{n_o}\ h(\Omega/n_o\sigma)\beta(m-n) \quad , \qquad (26)$$

where I_0 is the average current of the single bunch. To proceed we approximate the infinite set of linear equations (3) by a finite subset,

$$\rho_m = \sum_{n=n_0-\Delta}^{n_0+\Delta} T_{mn}\,\rho_n \quad , \quad |m-n_0| \le \Delta \quad . \tag{27}$$

In this case we make use of the representation for T_{mn} given in Eq. (26), and our problem reduces to solving the following eigenvalue problem:

$$\Lambda^\Delta v_m = \sum_{n=n_0-\Delta}^{n_0+\Delta} \beta(m-n)\, v_n \quad . \tag{28}$$

The coherent frequency is determined in terms of an eigenvalue Λ^Δ by the dispersion relation:

$$1 = \frac{-i\alpha\omega_0^2\, e I_0 \Lambda^\Delta}{2\pi E_0 \nu^2} \frac{Z_{n_0}}{n_0} h(\Omega/n_0\sigma) \quad . \tag{29}$$

Since $\beta(m-n)$ is sharply peaked about $m = n$, the peak width being of order $1/L \ll \Delta$, we expect that the largest eigenvalues do not depend strongly upon the cutoff value Δ. Therefore, they should be closely approximated by the eigenvalues of the easier problem which results when $\Delta \to \infty$. In this case the eigenfunctions of Eq. (28) become $v_n(\zeta) = \exp(-in\zeta)$ and the corresponding eigenvalues $\Lambda^\infty(\zeta) = \Sigma\beta(n)\exp(in\zeta)$, where $0 \le \zeta < 2\pi$ parametrizes the different eigensolutions. When the bunch length is short compared to the ring circumference, $L \ll 1$, then[1] the largest eigenvalue $\Lambda^\infty_{max} = \sqrt{2\pi/L} = I_{peak}/I_0$, where I_{peak} is the peak current of the bunch. To illustrate the rate of convergence as $\Delta \to \infty$, we plot in Fig. 1, the ratio $\Lambda^\Delta_{max}/\Lambda^\infty_{max}$, as a function of $L\Delta$. It is seen that when $L\Delta > 3$, the ratio is greater than 90%. Hence, for $L\Delta > 3$, it is a good approximation to replace $I_0\Lambda^\Delta$ by I_{peak} in the dispersion relation of Eq. (29).

To gain some insight into the nature of the perturbed line charge density, we take as an approximation to the eigenvectors of Eq. (28),

$$\rho_n = \begin{cases} \exp(-in\zeta) & |n-n_0| \le \Delta \\ 0 & |n-n_0| > \Delta \end{cases} \tag{30}$$

The perturbation to the line charge density is

$$\rho(\phi) = \sum_{n=n_0-\Delta}^{n_0+\Delta} \exp(in(\phi-\zeta)) = \exp(in(\phi-\zeta)) f_\Delta(\phi-\zeta) \quad , \tag{31}$$

where

$$f_\Delta(\phi-\zeta) = \frac{\sin[(\Delta+\frac{1}{2})\,(\phi-\zeta)]}{\sin[(\phi-\zeta)/2]} \quad . \tag{32}$$

We see that $\rho(\phi)$ is a plane wave modulated by the function $f_\Delta(\phi-\zeta)$, which is sharply peaked about $\phi = \zeta$ with peak width of order $1/\Delta$. The detailed structure within the peak depends on the short distance behavior of the wake field, which has been ignored in our approximate treatment, and hence is outside the scope of our discussion.

Let us close by commenting on the attempt made by Messerschmid and Month[3] to describe the microwave instability. Their approach was based upon the ansatz, $\rho(\phi) = \exp(in_0\phi)\,\rho_0(\phi)$, where $\rho_0(\phi)$ is the unperturbed bunch density. This has the form of a plane wave modulated by a shape function, however, the shape function is always taken to be $\rho_0(\phi)$ independent of the bandwidth, Δ, of the impedance. Our discussion shows that this is incorrect, and that the shape function should have a peak width of order $1/\Delta$, the range of the wake field. This local behavior is closely related to the peak current dependence[4] of the coherent frequency[3] for $\Delta \gg 1/L$. The ansatz of Messerschmid and Month[3] is more appropriate to the case of a narrow band resonant impedance with bandwidth $\Delta \ll 1/L$. Then the coherent frequency depends on the average current [see Eq. (24)] and the perturbed density is approximately as given in Eq. (25).

References

1. J. M. Wang and C. Pellegrini, Proc. XI Int. Conf. on H. E. Accel., Geneva (1980), p. 554.
2. S. Krinsky and J. M. Wang, "Longitudinal Instabilities with a Non-Harmonic RF Potential", preceeding article.
3. E. Messerschmid and M. Month, Nucl. Instrum. and Methods 136, (1976), p. 1.
4. D. Boussard, CERN LABII/RF/INT/75-2 (1975).

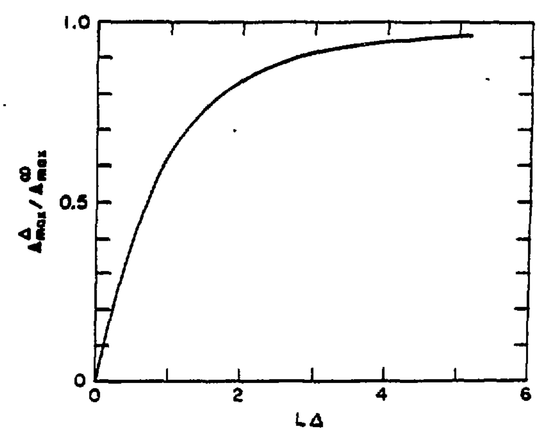

Fig. 1. The ratio of the largest eigenvalues corresponding to finite bandwidth Δ and infinite bandwidth, plotted against $L\Delta$, where L is the bunch length in radians

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