231953

UCRL-JC 128121

UCRL-JC-

PREPRINT

Transversely Isotropic Elasticity and Poroelasticity Arising from Thin Isotropic Layers

James G. Berryman

Lawrence Livermore National Laboratory, Livermore, California 94550

This paper was prepared for submittal to the
Proceedings of the Third International Conference
on Theoretical and Computational Acoustics,
Newark, New Jersey,
July 14-18, 1997

July 1997

Lawrence Livermore National Laboratory

Transversely Isotropic Elasticity and Poroelasticity Arising from Thin Isotropic Layers

James G. Berryman
Lawrence Livermore National Laboratory
P. O. Box 808 L-202
Livermore, CA 94551-9900

Abstract

Since the classic work of Postma [1955] and Backus [1962], much has been learned about elastic constants in vertical transversely isotropic (VTI) media when the anisotropy is due to fine layering of isotropic elastic materials. However, new results are still being discovered. For example, the P-wave anisotropy parameter c_{11}/c_{33} lies in the range $\frac{1}{4} \leq c_{11}/c_{33} \leq \langle \lambda + 2\mu \rangle \langle 1/(\lambda + 2\mu) \rangle$, when the layers are themselves composed of isotropic elastic materials with Lamé constants λ and μ and the vertical average of the layers is symbolized by $\langle \cdot \rangle$. The lower bound corrects a result of Postma. For porous layers, a connected solid frame forms the basis of the elastic behavior of a poroelastic medium in the presence of confining forces, while connected pores permit a percolating fluid (if present) to influence the mechanical response of the system from within. For isotropic and anisotropic poroelastic media, we establish general formulas for the behavior of transversely isotropic poroelasticity arising from laminations of isotropic components. The Backus averaging method is shown to provide elementary means of constructing general formulas. The results for confined fluids are then compared with the more general Gassmann [1951] formulas that must be satisfied by any anisotropic poroelastic medium and found to be in complete agreement. Such results are important for applications to oil exploration using AVO (amplitude versus offset) since the presence or absence of a fluid component, as well as the nature of the fluid, is the critical issue and the ways in which the fluid influences seismic reflection data still need to be better understood.

1 Introduction

Two primary goals of seismic reflection data processing are: (1) to image geologic structure and (2) to provide information about lithology for interpretation. The process used to achieve the second goal is made complex by the fact that the same seismic velocity may result from several different combinations/mixtures of materials in the earth. The resulting questions of uniqueness make it necessary to explore the possible range of seismic velocities that can occur within the set of circumstances deemed mostly likely to occur in the earth at the site of interest.

Fine horizontal layering (*i.e.*, layers with thickness small compared to the wavelength of the seismic wave) is known to cause vertical transverse isotropy (VTI) – wherein wave speeds vary with angle in such media, but are uniquely determined by the angle from the vertical. Efforts in this area are represented in the literature by work of Postma [1955], Backus [1962], Berryman [1979], Schoenberg and Muir [1987], Anderson [1989], and many others. There has continued to be some doubt about the range of anisotropy parameters possible in such media. Here I will correct an error of Postma [1955] and show that the P-wave anisotropy parameter c_{11}/c_{33} can be a factor of 2 smaller than previously supposed. I also obtain a simple upper bound on this parameter in terms of layer elastic parameters.

Then, in order to explore the area of most interesting applications of such results, we consider percolation phenomena in fluid-saturated porous media, where two distinct sets of percolating continua intertwine. A connected solid frame forms the basis of the elastic behavior of a poroelastic medium in the presence of external confining forces, while connected pores permit a percolating fluid (if present) to influence the mechanical response of the system from within.

There is a great deal of current interest in the anisotropy of Earth materials, and especially so when there is fluid present in pores and fractures in the Earth. Fluids of economic interest to the oil industry are typically oil, gas, and water, while fluids of interest in environmental remediation applications are generally the same. Environmental concerns often center around fluid contaminants which may be in the form of oil or gas, or could be other undesirable organic materials in ground water. Brines or steam may be used to flush other fluids out of the ground, whether for economic purposes or for environmental cleanup. Thus, it is important to understand the role of pore fluids in determining effective constants of such materials, and the fine layering or laminate model of earth materials plays a significant role in the analysis.

In this work, I study some simple means of estimating the effects of fluids on elastic constants and in particular we will derive formulas for anisotropic poroelastic constants using a straightforward generalization of the method of Backus [1962] for determining the effective constants of a laminated elastic material. There has been some prior work in this area by Norris [1993], Gurevich and Lopatnikov [1995], among others. One distinction between these earlier approaches and mine arises from the desire to understand the transition from elastic analysis to poroelastic whereas the earlier work in this area has started with poroelasticity as given and then applied a generalization of Backus' approach to the lamination analysis. Finally, I want to mention that methods similar to the ones to be presented here could as easily be applied in the same context to the problem of determining percolation for fluid flow or effective fluid permeability (Darcy's constant) and that would be of some interest in these applications as well.

2 Wave Propagation in Anisotropic Elastic Media

First, I will introduce the notation needed in the later analysis.

In tensor notation, the relationship between components of stress σ_{ij} and strain $u_{k,l}$ is given by

$$\sigma_{ij} = c_{ijkl}u_{k,l}, \tag{1}$$

where c_{ijkl} is the adiabatic stiffness tensor, and repeated indices on the right hand side of (3) are summed. In (1), u_k is the kth Cartesian component of the displacement vector $\mathbf{u}$, and $u_{k,l} = \partial u_k/\partial x_l$. Whereas for an isotropic elastic medium the stiffness tensor has the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{2}$$

depending on only two parameters (the Lamé constants, λ and μ), this tensor can have up to 21 independent constants for general anisotropic elastic media. The stiffness tensor has pairwise symmetry in its indices such that $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{ijlk}$, which will be used later to simplify the resulting equations.

The general equation of motion for wave propagation through an anisotropic elastic medium is given by

$$\rho\ddot{u}_i = \sigma_{ij,j} = c_{ijkl}u_{k,lj}, \tag{3}$$

where $\ddot{u}_i$ is the second time derivative of the ith Cartesian component of the displacement vector $\mathbf{u}$ and ρ is the density (assumed constant). Equation (3) is a statement that the product of mass times acceleration of a particle is determined by the internal stress force $\sigma_{ij,j}$.

A commonly used simplification of the notation for elastic analysis is given by introducing the strain tensor, where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

Then, using one version of the Voigt convention, in which the pairwise symmetries of the stiffness tensor indices are used to reduce the number of indices from 4 to 2 using the rules $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, 23 or $32 \rightarrow 4$, 13 or $31 \rightarrow 5$, and 12 or $21 \rightarrow 6$, I have

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & 2c_{44} & & \\ & & & & 2c_{55} & \\ & & & & & 2c_{66} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \tag{5}$$

Although the Voigt convention introduces no restrictions on the stiffness tensor, I have chosen to limit discussion to the form in (5), which is not completely general. Of the 36 coefficients (of which 21 are generally independent), I choose to treat only those cases for which the 12 coefficients shown (of which nine are generally independent) are nonzero. This form includes all orthorhombic, cubic, hexagonal, and isotropic systems, while excluding triclinic, monoclinic, trigonal, and some tetragonal systems, since each of the latter contains additional off-diagonal constants that may be nonzero. Nevertheless, I restrict the discussion to (5) or to the still simpler case of transversely isotropic (TI) materials.

For TI materials, $c_{11} = c_{22} \equiv a$, $c_{12} \equiv b$, $c_{13} = c_{23} \equiv f$, $c_{33} \equiv c$, $c_{44} = c_{55} \equiv l$, and $c_{66} \equiv m$. There is also one further constraint on the constants that $a = b + 2m$, following from rotational symmetry in the x_1x_2-plane. In such materials, (5) may be replaced by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} a & b & f & & & \\ b & a & f & & & \\ f & f & c & & & \\ & & & 2l & & \\ & & & & 2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}, \tag{6}$$

in which the matrix has the same symmetry as hexagonal systems and of which isotropic symmetry is a special case (having $a = c = \lambda + 2\mu$, $b = f = \lambda$, and $l = m = \mu$).

Recall that the equation of motion may be written as

$$\rho \ddot{u}_i = c_{ijkl} u_{k,lj}. \tag{7}$$

After Fourier transforming in both space and time [*i.e.*, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u} \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$, where $\mathbf{k}$ is the wavevector and ω is the angular frequency], I find

$$(\rho\omega^2 \delta_{ik} - c_{ijkl} k_j k_l) u_k = 0, \tag{8}$$

which provides three equations for the components of displacement u_1, u_2, u_3. These equations can be solved if and only if the determinant of the coefficients vanishes, which implies

$$det(\rho\omega^2 \delta_{il} - c_{ijkl} k_j k_l) = 0. \tag{9}$$

The left hand side of this equality would be a perfect square for all values of k_1^2 and k_3^2 if $f + l \equiv 0$, while the right hand side would be a perfect square if the quantity defined as $A \equiv (a - l)(c - l) - (f + l)^2 = 0$. The first case with $f + l = 0$ will virtually never happen because both f and l are normally positive quantities. The second case with $A = 0$ can occur for some types of anisotropic media, but I show later that this cannot occur for finely layered media. Nevertheless, if $A = 0$, then the dispersion relations of (14) reduce to

$$\rho\omega_p^2 = ak_1^2 + ck_3^2 \tag{17}$$

for $\omega_p = \omega_+$ and

$$\rho\omega_{sv}^2 = l(k_1^2 + k_3^2) = lk^2 \tag{18}$$

for $\omega_{sv} = \omega_-$, showing that the P-wave surface for velocity squared is elliptical if $a \neq c$, while the SV-wave surface for velocity squared is circular and therefore isotropic. The dispersion relation (15) shows that the SH-wave surface for velocity squared is always an ellipse as long as $l \neq m$. I call A the anellipticity parameter because, if $A \neq 0$, then the dispersion relations for qP- and qSV-waves are anelliptical (*i.e.*, something other than elliptical) in shape.

Phase velocities are obtained as a function of angle from these expressions by first defining the wavevector angle θ such that

$$\mathbf{k} = k(\sin\theta\hat{x}_1 + \cos\theta\hat{x}_3). \tag{19}$$

Then, the phase velocity vector for each type of wave is given in general by

$$\mathbf{v}_{ph} = v_{ph}(\sin\theta\hat{x}_1 + \cos\theta\hat{x}_3), \tag{20}$$

where $v_{ph} \equiv \omega/k$.

The group velocity is defined by

$$\mathbf{v}_{gr} = \frac{\partial\omega}{\partial k_1}\hat{x}_1 + \frac{\partial\omega}{\partial k_3}\hat{x}_3 = v_{gr}(\sin\phi\hat{x}_1 + \cos\phi\hat{x}_3), \tag{21}$$

where the group angle ϕ is determined by

$$\tan\phi = \frac{\partial\omega/\partial k_1}{\partial\omega/\partial k_3}. \tag{22}$$

One other angle is particularly important, since it is the one that is most easily measured, and that is the angle of particle motion ψ for a wave passing a particular point in space. The particle motion is given by the displacement vector $\mathbf{u}$, so

$$\mathbf{u} = u(\sin\psi\hat{x}_1 + \cos\psi\hat{x}_3), \tag{23}$$

where

$$\tan\psi = u_1/u_3. \tag{24}$$

3 Averaging Thin Layers for Low Frequency Behavior

Backus [1962] presents an elegant method of producing the effective constants for a thin layered medium composed of either iostropic or anisotropic elastic layers. For simplicity, I assume that the layers are isotropic, in which case equation (5) becomes

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \quad (25)$$

The key idea presented by Backus is that these equations can be rearranged into a form where rapidly varying coefficients multiply slowly varying stresses or strains. By doing so, I arrive at the following equation

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ -e_{33} \\ e_{23} \\ e_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \frac{4\lambda(\lambda+\mu)}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & & & \\ \frac{2\lambda\mu}{\lambda+2\mu} & \frac{4\lambda(\lambda+\mu)}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & & & \\ \frac{\lambda}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & -\frac{1}{\lambda+2\mu} & & & \\ & & & \frac{1}{2\mu} & & \\ & & & & \frac{1}{2\mu} & \\ & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}, \quad (26)$$

which can be averaged essentially by inspection. Equation (26) can be viewed as a Legendre transform of the original equation, to a different set of dependent/independent variables in which both vectors have components with mixed physical significance, some being stresses and some being strains. Otherwise these equations are completely equivalent to the original ones.

Performing the layer average, while assuming the variation is along the z or x_3 direction, I find, using the notation of (6),

$$\begin{pmatrix} <\sigma_{11}> \\ <\sigma_{22}> \\ -<e_{33}> \\ <e_{23}> \\ <e_{31}> \\ <\sigma_{12}> \end{pmatrix} = \begin{pmatrix} \left\langle\frac{4\lambda(\lambda+\mu)}{\lambda+2\mu}\right\rangle & \left\langle\frac{2\lambda\mu}{\lambda+2\mu}\right\rangle & \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & & & \\ \left\langle\frac{2\lambda\mu}{\lambda+2\mu}\right\rangle & \left\langle\frac{4\lambda(\lambda+\mu)}{\lambda+2\mu}\right\rangle & \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & & & \\ \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle & -\left\langle\frac{1}{\lambda+2\mu}\right\rangle & & & \\ & & & \left\langle\frac{1}{2\mu}\right\rangle & & \\ & & & & \left\langle\frac{1}{2\mu}\right\rangle & \\ & & & & & \langle 2\mu\rangle \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}$$

$$= \begin{pmatrix} a-f^2/c & b-f^2/c & f/c & & & \\ b-f^2/c & a-f^2/c & f/c & & & \\ f/c & f/c & -1/c & & & \\ & & & 1/2l & & \\ & & & & 1/2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}, \quad (27)$$

which can then be solved to yield the expressions

$$a = \left\langle\frac{\lambda}{\lambda+2\mu}\right\rangle^2 \left\langle\frac{1}{\lambda+2\mu}\right\rangle^{-1} + 4\left\langle\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right\rangle, \quad (28)$$

$$b = \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} + 2 \left\langle \frac{\lambda\mu}{\lambda + 2\mu} \right\rangle, \tag{29}$$

$$c = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1} \tag{30}$$

$$f = \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1}, \tag{31}$$

$$l = \left\langle \frac{1}{\mu} \right\rangle^{-1} \tag{32}$$

and

$$m = \langle \mu \rangle . \tag{33}$$

One very important fact that is known about these equations is that they reduce to isotropic results with $a = c$, $b = f$, and $l = m$ if the shear modulus is a constant, regardless of the behavior of λ.

4 Review of Known Inequalities for the Elastic Constants

Since the stress-strain relation (6) is derivable from an energy functional, it is not hard to show that the matrix must be nonnegative or the material will be mechanically unstable. Nonnegativity of the matrice implies that all its principal minors must be nonnegative, which in turn implies the following inequalities:

$$a = b + 2m \geq 0, \qquad c \geq 0, \tag{34}$$

$$l \geq 0, \qquad m \geq 0, \tag{35}$$

and

$$(a^2 - b^2)/4m = b + m \geq 0, \qquad ac - f^2 \geq 0, \tag{36}$$

and

$$\left[a(ac - 2f^2) - b(bc - 2f^2)\right]/4m = (b + m)c - f^2 \geq 0. \tag{37}$$

The second inequality in (36) follows from (37), (35), and the second inequality in (34) and is therefore often omitted from such listings. Similarly, the inequality for a follows from those for m and $b + m$. All of these inequalities must be satisfied regardless of the source of the anisotropy.

The formulas (28)-(33) can be used to derive some very simple relations among the constants. For example,

$$c \geq f \tag{38}$$

follows directly from (30) and (31), simply noting that $\lambda/(\lambda+2\mu) \le 1$ in every layer. The inequality

$$c \ge \frac{4}{3} l \tag{39}$$

is derived directly from the fact that

$$\left\langle \frac{1}{\lambda+2\mu} \right\rangle \le \left\langle \frac{1}{4\mu/3} \right\rangle = \frac{3}{4} \left\langle \frac{1}{\mu} \right\rangle, \tag{40}$$

which follows from the fact that the bulk modulus must be nonnegative in each layer so that $\lambda + 2\mu/3 \ge 0$ everywhere. Then, the two shear moduli must satisfy

$$l \le m \tag{41}$$

since

$$1 \le \langle \mu \rangle \left\langle \frac{1}{\mu} \right\rangle. \tag{42}$$

follows easily from the well-known Cauchy-Schwartz inequality $\langle \alpha\beta \rangle^2 \le \langle \alpha^2 \rangle \langle \beta^2 \rangle$ by setting $\alpha = \mu^{1/2}$ and $\beta = 1/\mu^{1/2}$. Equality applies in the Cauchy-Schwartz inequalities only when $\alpha = const \times \beta$, which implies in the present circumstances that μ must be constant for $l = m$. But this is precisely the condition mentioned earlier for the layer equations to be isotropic, so I exclude this case from consideration.

Another inequality can be derived from the formulas obtained for finely layered media. I showed earlier that the anellipticity parameter given by

$$A \equiv (a-l)(c-l) - (f+l)^2 \tag{43}$$

has the property that the dispersion relations for both qP- and qSV-waves are simple ellipses when $A = 0$ and are anelliptic otherwise. Using the results (28)-(33), A is shown to satisfy $A > 0$ for any fine layered transversely isotropic medium by noting that

$$A = cl \left[\left\langle \mu \left(\frac{\lambda+\mu}{\lambda+2\mu} \right) \right\rangle \left\langle \frac{1}{\mu} \left(\frac{\lambda+\mu}{\lambda+2\mu} \right) \right\rangle - \left\langle \frac{\lambda+\mu}{\lambda+2\mu} \right\rangle^2 \right] \ge 0. \tag{44}$$

The inequality follows again from the Cauchy-Schwartz inequality in another form somewhat more transparent for the present application $\langle \alpha \rangle^2 = \left\langle (\mu\alpha)^{1/2} (\alpha/\mu)^{1/2} \right\rangle \le \langle \mu\alpha \rangle \langle \alpha/\mu \rangle$. Equality again applies only when μ is identically constant. But, it was mentioned earlier that finely layered media are isotropic if μ does not vary, so $A > 0$ holds for all such finely layered media if their overall constants are anisotropic.

5 Range of the Anisotropy Parameters

From (15), we know that the SH-wave in finely layered VTI media has an elliptical surface for velocity squared. Furthermore,

$$m/l = \langle \mu \rangle \left\langle \frac{1}{\mu} \right\rangle \ge 1 \tag{45}$$

follows from (32), (33), and (42). So the horizontal shear wave velocity for SH-waves is always greater than or equal to the vertical velocity. I choose to define the ratio m/l as the SH-wave anisotropy parameter, and have the simple universal result that this parameter is always greater than or equal to unity.

The qP-wave does not always have an elliptical dispersion relation, but it is nevertheless always true that if $k_3 = 0$ then $\rho\omega_+^2 = ak_1^2$ and if $k_1 = 0$ then $\rho\omega_+^2 = ck_3^2$. Thus, I may define the P-wave anisotropy parameter to be a/c and seek to determine what the range of this parameter might be. Formula (28) for a may be rewritten as

$$a = \left\langle \frac{(\lambda+2\mu)^2 - \lambda^2}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1}, \tag{46}$$

which can be rearranged into the convenient and illuminating form

$$a = \langle \lambda+2\mu \rangle - \left[\left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \right] \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1}. \tag{47}$$

This formula is very instructive because the term in square brackets is again in Cauchy-Schwartz form, so this factor is nonnegative. Furthermore, the magnitude of this term depends principally on the fluctuations in the λ Lamé constant, largely independent of μ. Clearly, if $\lambda = \mathit{constant}$, then this factor vanishes identically, regardless of the behavior of μ. Large fluctuations in λ will tend to make this term large. If in addition I consider the combination

$$\frac{a}{c} - 1 = \left[\langle \lambda+2\mu \rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - 1 \right] - \left[\left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \right], \tag{48}$$

the first bracket on the right hand side is again in Cauchy-Schwartz form showing that it always makes a positive contribution unless $\lambda + 2\mu = \mathit{constant}$, in which case it vanishes. Similarly, the second term always makes a negative contribution unless $\lambda = \mathit{constant}$, in which case it vanishes.

If the finely layered medium is composed of only two distinct types of isotropic elastic materials and they appear in the layering sequence with equal spatial frequency, then I find that

$$\frac{a}{c} - 1 = (\mu_2 - \mu_1) \frac{(\lambda_2 - \lambda_1) + (\mu_2 - \mu_1)}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}. \tag{49}$$

This result agrees with Postma [1955] except for an obvious typographical error in the denominator of his published formula. This formula shows clearly that if $\mu_1 = \mu_2$ then the P-wave anisotropy parameter is identically equal to unity as expected. Also, if $\mu_1 \neq \mu_2$ but $\lambda_1 = \lambda_2$, then (49) implies $a/c \geq 1$, as we inferred from (48).

Now, I use this formula to deduce the smallest possible value of the right hand side of (49). The shear moduli must not be equal (for anisotropy), so without loss of generality I suppose that $\mu_2 > \mu_1$. Then, the numerator is seen to become negative by taking λ_2 towards negative values and $\lambda_1 \to +\infty$. The smallest value λ_2 can take is determined by the bulk modulus bound $\lambda_2 + \frac{2}{3}\mu_2 \geq 0$. So we may set $\lambda_2 = -\frac{2}{3}\mu_2$ in both the numerator and denominator. This choice also makes the factor $\lambda_2 + 2\mu_2 = \frac{4}{3}\mu_2$ as small as possible in the denominator, thus helping to magnify the effect of the negative numerator as much as possible. The result so far is that

$$\frac{a}{c} - 1 = \frac{3}{4}\left(\frac{\mu_2 - \mu_1}{\mu_2}\right)\left(\frac{-\lambda_1 + \mu_2/3 - \mu_1}{\lambda_1 + 2\mu_1}\right) \tag{50}$$

The parameter λ_1 may vary from $-\frac{2}{3}\mu_1$ to plus infinity. At $\lambda_1 = -\frac{2}{3}\mu_1$, the second expression in parentheses is positive, But, this expression is also a monotonically decreasing function of λ_1 and approaches -1 as $\lambda_1 \to +\infty$. Thus, the smallest value of the P-wave anisotropy parameter is given by

$$\frac{a}{c} = 1 - \frac{3}{4}\frac{\mu_2 - \mu_1}{\mu_2} \geq \frac{1}{4}. \tag{51}$$

This result differs by a factor of 2 from the corresponding result of Postma [1955], the earlier result being obtained improperly by allowing three of the four elastic constants to vanish and also using a physically motivated but unnecessary restriction that both λ_1 and λ_2 must be nonnegative. If we had used the nonnegativity constraint on the λ's, my result would have changed to

$$\frac{a}{c} = 1 - \frac{\mu_2 - \mu_1}{2\mu_2} \geq \frac{1}{2}, \tag{52}$$

which is the same *inequality* as that found by Postma, but his *equality* differed from that in (52) and was in fact improperly obtained.

As a final point about the formula (47), note that it implies in general that

$$a \leq \langle \lambda + 2\mu \rangle, \tag{53}$$

so I have a general upper bound on the P-wave anisotropy parameter stating that

$$\frac{a}{c} \leq \langle \lambda + 2\mu \rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle. \tag{54}$$

Before concluding this section, note one further identity for the P-wave anisotropy parameter. The general formula can be rearranged to give

$$\frac{a}{c} - 1 = 4\left[\left\langle \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\lambda + \mu}{\lambda + 2\mu} \right\rangle \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle\right]. \tag{55}$$

This formula is not in Cauchy-Schwartz form, but is nevertheless probably the simplest form of the result for this anisotropy parameter. In particular, it is easy to see from this form that if either $\mu = constant$ or $\lambda + \mu = constant$, then the right hand side vanishes identically. The first result is well-known and the second has been known since Postma's [1955] work to be true for two-constituent layered media [also see (49)]. The present result generalizes Postma's observation in this case.

6 Significance of Fluctuations in Lamé λ

Some further results of the type presented in the preceding section have been found recently by Anno [1997] and by Berryman, Grechka, and Berge [1997]. In all cases, it is found that fluctuations in the Lamé parameter λ play a key role in the analysis. In some sense this is inevitable for anisotropy due to layering because there are only two independent parameters, λ and μ, and the first result is that the anisotropy cannot exist if the shear modulus does not fluctuate. Thus, fluctuations in μ may be assumed from the outset, and the only question to be addressed is how fluctuations in λ affect the results.

It is very important to recognize however that poroelastic analysis shows the mechanical effect of fluids is negligible on the shear modulus μ but not negligible overall and therefore must be contained entirely in changes in Lamé λ. This fact provides the motivation to study effects of layering in poroelastic media containing fluids, which is the subject of the following sections.

7 Porous Elastic Materials Containing Fluids

Now I want to broaden my scope and consider materials composing the laminate are not homogeneous isotropic elastic materials, but rather elastic materials containing voids or pores. The pores may be either air-filled, or alternatively they may be partially or fully saturated with a liquid, a gas, or a fluid mixture. For simplicity, I suppose here that the pores are either air-filled or they are fully saturated with some other homogeneous fluid. When the porous layers are air-filled, it is generally adequate to assume that the analysis of the preceding section holds, but with the new interpretation that the Lamé parameters are those for the porous elastic medium in the absence of saturating fluids. The resulting effective constants λ_{dr} and μ_{dr} are then said to be those for "dry" — or somewhat more accurately "drained" — conditions. These constants are also sometimes called the "frame" constants, to distinguish them from the constants associated with the solid materials composing the frame, which are often called the "grain" or "mineral" constants.

One simplification that arises immediately is due to the fact that the presence of pore fluids has no mechanical effect on the shear moduli, so $\mu_{dr} = \mu$. There may be other effects on the shear moduli due to the presence of pore fluids, such as softening of cementing materials or expansion of interstitial clays, which I call "chemical" effects to distinguish them from the purely mechanical effects to be considered here. We neglect all such chemical effects in the following analysis. This means that the lamination analysis for the effective shear moduli (since it is uncoupled from the analysis involving λ) does not change in the presence of fluids. Thus, equations (32) and (33) continue to apply for the poroelastic problem, and we can therefore simplify our system of equations in order to focus on the parts of the analysis that do change in the presence of fluids.

The presence of a saturating pore fluid introduces the possibility of an additional control field and an additional type of strain variable. The pressure p_f in the fluid is the new field parameter that can be controlled. Allowing sufficient time for global pressure equilibration will permit us to consider p_f to be a constant throughout the percolating (connected) pore fluid, while restricting the analysis to quasistatic processes. The change ζ in the amount of fluid mass contained in the pores (see Berryman and Thigpen [1985]) is the new type of strain variable, measuring how much of the original fluid in the pores is squeezed out during the compression of the pore volume while including the effects of compression or expansion of the pore fluid itself due to changes in p_f. It is most convenient to write the resulting equations in terms of compliances rather than stiffnesses, so the basic equation to be considered takes the form:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{12} & -\beta \\ s_{12} & s_{11} & s_{12} & -\beta \\ s_{12} & s_{12} & s_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \tag{56}$$

The constants appearing in the matrix on the right hand side will be defined in the following two paragraphs. It is important to write the equations this way rather than using the inverse relation in terms of the stiffnesses, because the compliances s_{ij} appearing in (56) are simply related to the drained constants λ_{dr} and μ_{dr} in the same way they are related in normal elasticity, whereas the individual stiffnesses obtained by inverting the equation in (56) must contain coupling terms through the parameters β and γ that depend on the pore and fluid compliances. Thus, I find that

$$s_{11} = \frac{1}{E_{dr}} = \frac{\lambda_{dr} + \mu}{\mu(3\lambda_{dr} + 2\mu)} \tag{57}$$

and

$$s_{12} = -\frac{\nu_{dr}}{E_{dr}}, \tag{58}$$

where the drained Young's modulus E_{dr} is defined by the second equality of (57) and the drained Poisson's ratio is determined by

$$\nu_{dr} = \frac{\lambda_{dr}}{2(\lambda_{dr} + \mu)}. \tag{59}$$

When the external stress is hydrostatic so $\sigma = \sigma_{11} = \sigma_{22} = \sigma_{33}$, the equation (56) telescopes down to

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} 1/K_{dr} & -\alpha/K_{dr} \\ -\alpha/K_{dr} & \alpha/BK_{dr} \end{pmatrix} \begin{pmatrix} \sigma \\ -p_f \end{pmatrix}, \tag{60}$$

where $e = e_{11} + e_{22} + e_{33}$, $K_{dr} = \lambda_{dr} + \frac{2}{3}\mu$ is the drained bulk modulus, $\alpha = 1 - K_{dr}/K_m$ is the Biot-Willis parameter [Biot and Willis, 1957] with K_m being the bulk modulus of the solid minerals present, and Skempton's pore-pressure buildup parameter B [Skempton, 1954] is given by

$$B = \frac{1}{1 + K_p(1/K_f - 1/K_m)}. \tag{61}$$

New parameters appearing in (61) are the bulk modulus of the pore fluid K_f and the pore modulus $K_p = \alpha/\phi K_{dr}$ where ϕ is the porosity. The expressions for α and B can be generalized slightly by supposing that the solid frame is composed of more than one constituent, in which case the K_m appearing in the definition of α is replaced by K_s and the K_m appearing explicitly in (61) is replaced by K_ϕ [see Brown and Korringa, 1975; Rice and Cleary, 1976; Berryman and

Milton, 1991; Berryman and Wang, 1995]. This is an important additional complication [Berge and Berryman, 1995], but one that I will not pursue here.

Comparing (56) and (60), I find that

$$\beta = \frac{\alpha}{3K_{dr}} \tag{62}$$

and

$$\gamma = \frac{\alpha}{BK_{dr}}. \tag{63}$$

With all the constants defined now in terms of measureable quantities, I can continue with the analysis that generalizes the Backus [1962] approach to computing the layer averages. It should be clear at this point that the appropriate Legendre transformed equations are

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ -e_{33} \\ \zeta \end{pmatrix} = \begin{pmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & \nu/(1-\nu) & \beta E/(1-\nu) \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & \nu/(1-\nu) & \beta E/(1-\nu) \\ \nu/(1-\nu) & \nu/(1-\nu) & -(1-\nu-2\nu^2)/(1-\nu)E & \beta(1+2\nu)/(1-\nu) \\ \beta E/(1-\nu) & \beta E/(1-\nu) & \beta(1+2\nu)/(1-\nu) & -[\gamma - 2\beta^2 E/(1-\nu)] \end{pmatrix} \times \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}, \tag{64}$$

with the fast variables on the left and the slow variables (actually constant) in the vector on the right. Signs have been chosen so the matrix is symmetric. I have also dropped the subscript dr from the drained constants ν and E in (64) as there should be no confusion. Note that the 3×3 submatrix in the upper left is identical to that in (26) after the change in notation from λ,μ to E,ν is taken into account.

Once I have this equation, the averaging is trivial. If the assumed form of the resulting equations is taken — in analogy to (6) and consistent with the general structure of the matrix in (64) — to be

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix} = \begin{pmatrix} a & b & f & g \\ b & a & f & g \\ f & f & c & h \\ g & g & h & k \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix}, \tag{65}$$

then the resulting rearrangement of these equations is

$$\begin{pmatrix} <\sigma_{11}> \\ <\sigma_{22}> \\ -<e_{33}> \\ <\zeta> \end{pmatrix} = \begin{pmatrix} a-x & b-x & y & z \\ b-x & a-x & y & z \\ y & y & u & v \\ z & z & v & w \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}, \tag{66}$$

where

$$x = \frac{f^2k - 2fgh + cg^2}{ck - h^2}, \quad y = \frac{fk - gh}{ck - h^2}, \quad z = \frac{cg - fh}{ck - h^2}, \tag{67}$$

and

$$u = -\frac{k}{ck - h^2}, \quad v = \frac{h}{ck - h^2}, \quad w = -\frac{c}{ck - h^2}. \tag{68}$$

It is not difficult to check that these equations reduce correctly to the earlier ones if I first set $g = h = 0$ and then let $k \to 0$.

Now all the matrix elements appearing in (66) are obtained directly by averaging (64) and therefore are assumed known. I do not list all of these relations as they should be clear from the expressions already given, but to provide two examples note that

$$w = 2\left\langle \frac{\beta^2 E}{1-\nu} \right\rangle - \langle \gamma \rangle \quad \text{and} \quad z = \left\langle \frac{\beta E}{1-\nu} \right\rangle. \tag{69}$$

Given all these equations, it is then straightforward to invert for the desired final expressions:

$$a = \left\langle \frac{E}{1-\nu^2} \right\rangle + x, \tag{70}$$

$$b = \left\langle \frac{\nu E}{1-\nu^2} \right\rangle + x, \tag{71}$$

$$c = -\frac{w}{uw - v^2}, \tag{72}$$

$$f = cy + hz, \tag{73}$$

$$g = hy + kz, \tag{74}$$

$$h = \frac{v}{uw - v^2}, \tag{75}$$

and

$$k = -\frac{u}{uw - v^2}. \tag{76}$$

The order in which the computations are done in practice is this: first compute c, h, and k; next compute f and g; then compute x using (67); finally compute a and b.

The results show that, whereas transverse isotropy due to layering in elastic materials produces five independent constants (recall that $a = b + 2m$ in general for transverse isotropy), transverse isotropy due to layering in poroelastic materials results in eight independent constants ($a = b + 2m$ still holds for poroelasticity as is easily shown from our formulas). When performing the averaging based on (64), we see that all the new terms in the matrix depend on averages of the poroelastic constant β which is proportional to the Biot-Willis parameter and therefore related to effective stress [the relative importance of external and internal loading — see (60)]. However, only the new diagonal term w depends directly on the bulk modulus K_f of the pore fluid through γ. It follows that, when I solve for the effective constants, I find that w influences all these effective constants. So the presence of pore fluid can significantly affect the pressure dependence of such materials, while having little or no effect on the shear response.

This completes the analysis of the constants for transverse isotropy in poroelasticity arising from thin layering of isotropic elastic and porous materials. Next I check that these results are consistent with known general results for anisotropic poroelasticity [Gassmann, 1951; Brown and Korringa, 1975].

8 Relations for Anisotropy in Poroelastic Materials

Gassmann [1951] and Brown and Korringa [1975] have considered the problem of obtaining effective constants for anisotropic poroelastic materials when the pore fluid is confined within the pores. The confinement condition amounts to the constraint that the increment of fluid content $\zeta = 0$, while the external loading σ is changed and the pore-fluid pressure p_f is allowed to respond as necessary and equilibrate.

To provide a simple derivation of the Gassmann equation for anisotropic materials, I consider the anisotropic generalization of (56)

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & -\beta_1 \\ s_{12} & s_{22} & s_{23} & -\beta_2 \\ s_{13} & s_{23} & s_{33} & -\beta_3 \\ -\beta_1 & -\beta_2 & -\beta_3 & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \tag{77}$$

The shear terms are excluded as before since they do not interact mechanically with the fluid effects. This form is again not completely general in that it includes orthorhombic, cubic, hexagonal, and all isotropic systems, but excludes triclinic, monoclinic, trigonal, and some tetragonal systems that would have some nonzero off-diagonal terms in the full elastic matrix. Also, I have assumed that the material axes are aligned with the spatial axes. But this latter assumption is not significant for the derivation that follows. Such an assumption is important when properties of laminated materials having arbitrary orientation relative to the spatial axes need to be considered, but I do not treat this more general problem here.

Before proceeding, I want to discuss the significance of the matrix elements appearing in (77) briefly. In the so-called "jacketed test," a porous sample is enclosed in a thin jacketing material with a tube into the pore space to permit the fluid to flow freely in or out while maintaining constant fluid pressure. Then it is sufficient to consider the case with $p_f = 0$. It is possible under these circumstances, at least in principle, to make 12 independent measurements by varying σ_{ii}'s and measuring e_{jj}'s and ζ. In fact measurements of drained elastic compliances are commonly made in such a manner, but it is less common for the β_i's to be measured this way. To complete the measurements, a second common test — the so-called "unjacketed test" — is performed in which a uniform pressure field is applied to the sample so that $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p_f$. Then, by making measurements of the e_{jj}'s again as p_f varies, a set of solid material compliances κ_i

is measured, given essentially by row sums of the matrix in (77)

$$\kappa_i = \sum_{j=1}^{3} s_{ij} - \beta_i, \quad \text{for} \quad i = 1,2,3. \tag{78}$$

These three constants are expected to be directly related to the compliances of the mineral grains composing the porous frame; if the frame is microhomogeneous (*i.e.*, containing a single solid constitutent), the compliances κ_i will be the compliances of the mineral composing the frame such that $\sum_i \kappa_i = 1/K_m$, where K_m is the bulk modulus of the mineral. If the measurement apparatus is inadequate so that the β_i's could not be determined directly in the jacketed test, then we see from (78) that they can be determined by combining results from the jacketed and unjacketed measurements on the solid compliances. The remaining constant γ can again be measured (at least in principle) directly in the unjacketed test by making measurements on the changes in fluid content ζ. An alternative to these rather difficult measurements is the confined test which I describe next.

If the fluid is confined, then $\zeta \equiv 0$ in (77) and p_f becomes a linear function of σ_{11}, σ_{22}, σ_{33}. Eliminating p_f from the resulting equations, I obtain the general expression for the strain dependence on external stress under confined conditions:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \end{pmatrix} = \left[\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} - \gamma^{-1} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}$$

$$\equiv \begin{pmatrix} s^*_{11} & s^*_{12} & s^*_{13} \\ s^*_{12} & s^*_{22} & s^*_{23} \\ s^*_{13} & s^*_{23} & s^*_{33} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}. \tag{79}$$

The s_{ij}'s are fluid-drained constants, while the s^*_{ij}'s are the fluid-confined constants.

The fundamental result (79) was obtained earlier by both Gassmann [1951] and Brown and Korringa [1975], and may be written as

$$s^*_{ij} = s_{ij} - \frac{\beta_i \beta_j}{\gamma}, \quad \text{for} \quad i,j = 1,2,3. \tag{80}$$

This expression is just the anisotropic generalization of the well-known Gassmann equation for isotropic, microhomogeneous porous media. Equation (80) has often been written in a slightly different way, by making use of the formulas (78) to eliminate the β's in favor of the solid and drained compliances. The principal advantage of such an alternative formula is that all constants appearing explicitly can be obtained by measurements of porous frame strain, without resorting to the more difficult measurements of changes in pore-fluid content.

Now it is not difficult to see that the lamination formulas derived earlier in the paper satisfy these general conditions. This simple test provides one means of checking that I did the lamination analysis correctly and also provides a convenient means of summarizing the results.

9 Conclusions

In the present paper, I have discussed isotropic and anisotropic poroelastic media and established general formulas for the behavior of transversely isotropic elasticity and poroelasticity arising from laminations of isotropic components. The Backus [1962] averaging method is shown to provide elementary means of constructing general formulas. The results for confined fluids are then compared with the more general Gassmann [1951] formulas that must be satisfied by any anisotropic poroelastic medium and found to be in complete agreement. The transition from analysis of laminations of elastic materials to laminations of poroelastic materials in the presence of saturating pore fluids follows easily from the simple observation that certain variables are quasistatically constant across a layered medium and provides a very intuitive and mathematically transparent approach to obtaining formulas of current interest. Choosing to do the analysis in terms of compliances rather than stiffnesses also proved to be an important simplification for the poroelastic case.

Such results are especially important for applications to oil exploration using AVO (amplitude versus offset) since the presence or absence of the fluid component, as well as the precise nature of the fluid, is one of the most critical issues. For this reason, the ways in which the fluid can influence seismic reflection data need to be understood in more detail than has been possible in the past [Thomsen, 1993; Mukerji and Mavko, 1994].

Acknowledgments

I thank Patricia Berge and Vladimir Grechka for helpful technical discussions on poroelasticity and anisotropy, and Phil Anno for a preprint of his paper. Work was performed under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48 and supported specifically by the Geosciences Research Program of the DOE Office of Energy Research within the Office of Basic Energy Sciences, Division of Engineering and Geosciences, as part of the Advanced Computational Technology Initiative.

REFERENCES

D. L. Anderson, *Theory of the Earth*, Blackwell, Boston, 1989, pp. 317–319.

P. Anno, Equivalent anisotropy for finely-layered media, in *Proceedings of the Project Review, Center for Wave Phenomena, Colorado School of Mines*, Vail, Colorado, May 18–21, 1997, pp. 307–312.

G. E. Backus, Long-wave elastic anisotropy produced by horizontal layering, *J. Geophys. Res.*, **67**, 4427–4440 (1962).

P. A. Berge and J. G. Berryman, Realizability of negative pore compressibility in poroelastic composites, *ASME J. Appl. Mech.* **62**, 1053–1062 (1995).

J. G. Berryman, Long-wave elastic anisotropy in transversely isotropic media, *Geophysics* **44**, 896–917 (1979).

J. G. Berryman, V. Grechka, and P. A. Berge, Analysis of Thomsen parameters for finely layered VTI media, in *Proceedings of the Sixty-Seveth Annual Meeting of the Society of Exploration Geophysicists*, Dallas, Texas, November 2–7, 1997, in press.

J. G. Berryman and G. W. Milton, Exact results for generalized Gassmann's equations in composite porous media with two constituents, *Geophysics* **56**, 1950–1960 (1991).

J. G. Berryman and L. Thigpen, Nonlinear and semilinear dynamic poroelasticity with microstructure, *J. Mech. Phys. Solids* **33**, 97–116 (1985).

J. G. Berryman and H. F. Wang, The elastic coefficients of double-porosity models for fluid transport in jointed rock, *J. Geophys. Res.* **100**, 24611–24627 (1995).

M. A. Biot, Mechancis of deformation and acoustic propagation in porous media, *J. Appl. Phys.* **33**, 1482–1498 (1962).

M. A. Biot and D. G. Willis, The elastic coefficients of the theory of consolidation, *J. Appl. Mech.* **24**, 594–601 (1957).

R. J. S. Brown and J. Korringa, On the dependence of the elastic properties of a porous rock on the compressibility of a pore fluid, *Geophysics* **40**, 608–616 (1975).

F. Gassmann, Über die elastizität poröser medien, *Veirteljahrsschrift der Naturforschenden Gesellschaft in Zürich* **96**, 1–23 (1951).

B. Gurevich and S. L. Lopatnikov, Velocity and attenuation of elastic waves in finely layered porous rocks, *Geophys. J. Int.* **121**, 933–947 (1995).

T. Mukerji and G. Mavko, Pore fluid effects on seismic velocity in anisotropic rocks, *Geophysics* **59**, 233–244 (1994).

A. N. Norris, Low-frequency dispersion and attenuation in partially saturated rocks, *J. Acoust. Soc. Am.* **94**, 359–370 (1993).

G. W. Postma, Wave propagation in a stratified medium, *Geophysics* **20**, 780–806 (1955).

J. R. Rice and M. P. Cleary, Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents, *Rev. Geophys.* **14**, 227–241 (1976).

M. Schoenberg and F. Muir, A calculus for finely layered anisotropic media, *Geophysics* **54**, 581–589 (1989).

A. W. Skempton, The pore-pressure coefficients A and B, *Geotechnique* **4**, 143–147 (1954).

L. Thomsen, Weak anisotropic reflections, in *Offset-Dependent Reflectivity — Theory and Practice of AVO Analysis*, J. P Castagna and M. M. Backus (eds.), Society of Exploration Geophysicists, Tulsa, Oklahoma, 1993, pp. 103–111.

Technical Information Department **• Lawrence Livermore National Laboratory**
University of California • Livermore, California 94551