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Generalized Finite Strains, Generalized Stresses, and a Hybrid Variational Principle for Finite-Element Computer Programs Using Curvilinear Coordinates

W. A. Cook

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Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

NOMENCLATURE

Accelerations	F_I, β^a (for undeformed and reference configurations)
Base vectors	$\mathbf{B}_a$
Cartesian coordinates	Z^I, z^i, Y^a (for undeformed, deformed, and reference configurations)
Constitutive tensors	c_{abde}, κ^{abde}
Curvilinear coordinates	y^a
Densities	P, ζ (for undeformed and reference configurations)
Displacement tensors	w_a, v_a (Cartesian and curvilinear)
Metric tensors	B_{ab}
Normal vector	$\boldsymbol{\nu}$
Strain tensors	ε_{ab}
Stress tensors	τ^{ab}
Surfaces	s, ω (for deformed and reference configurations)
Traction components	t_i, Γ^a (for deformed and reference configurations)
Unit vectors	$\mathbf{i}_a$
Volumes	V, Ω (for undeformed and reference configurations)

GENERALIZED FINITE STRAINS, GENERALIZED STRESSES, AND A HYBRID VARIATIONAL PRINCIPLE FOR FINITE-ELEMENT COMPUTER PROGRAMS USING CURVILINEAR COORDINATES

by

W. A. Cook

ABSTRACT

A generalized hybrid variational principle is presented that includes, as special cases, hybrid variational principles for the Lagrangian and Eulerian formulations. This generalized hybrid principle approximates force equilibrium and constitutive equations. This report is an extension of a similar report entitled "Generalized Finite Strains, Generalized Stresses, and a Hybrid Variational Principle for Finite-Element Computer Programs." The difference between these reports is in the use of curvilinear coordinates. Presented in these reports are two approaches for solving nonlinear stress analysis problems with this generalized hybrid variational principle. Both approaches use the finite-element method: one iterates the nodal displacements, and the other iterates the incremental displacements.

I. INTRODUCTION

This report is an extension of the generalized hybrid variational principle described in Cook (1988), which was an extension of the generalized virtual work principle described in Cook and Genin (1988). This report differs from Cook (1988) in that this one uses curvilinear coordinates instead of Cartesian coordinates for strain and stress.

This report describes the theory used to develop a continuum nonlinear geometry computer program that uses generalized finite strain tensors, generalized stress tensors, and a generalized hybrid variational principle. These strain tensors are generalizations of the mathematical Green or Lagrangian strain tensors that use undeformed or original geometry as their reference configuration and the Almansi or Eulerian strain tensors that use deformed or final geometry as their reference configuration. The parameter Lambda (Λ) specifies the portion of the deformed and undeformed configurations used for the reference configuration (Cook 1986 and Cook and Genin 1988). When Lambda equals one, the reference configuration is the deformed configuration; when Lambda equals zero, the reference configuration is the undeformed configuration; and when Lambda equals one-half, the reference configuration is an average of the deformed and undeformed configurations. Tensor transformations exist to transform between the generalized strain tensors and the Green or Almansi strain tensors.

Generalized stress tensors and the energy conjugate generalized strain tensors use the same reference configuration (Bathe 1982). Consequently, Lambda (like the generalized strain tensors) specifies the reference configuration associated with generalized stress tensors. Relative tensor transformations of weight one exist to transform between the the generalized stress tensors and the second Piola-Kirchhoff or Cauchy stress tensors. The weight parameter for these relative tensors is the ratio of the densities.

The hybrid variational principle for the Lagrangian continuum formulation uses Green strain tensors and second Piola-Kirchhoff stress tensors. The hybrid variational principle for the Eulerian formulation uses Almansi strain tensors and Cauchy stress tensors. Similarly, a generalized hybrid variational principle is defined that uses generalized strain and stress tensors. When Lambda is specified as one, zero, or one-half, the continuum formulation may be Lagrangian, Eulerian, or centered, and Lambda may be any other value between one and zero. For all formulations, the body force and traction integrals are equal. Thus, because the body force is known in the Lagrangian formulation that uses the undeformed configuration, the body force integral in the Lagrangian formulation is used in the final hybrid variational principle. Similarly, because the traction integral is known in the Eulerian formulation (deformed configuration) for follower forces, the traction integral in the Eulerian formulation is used in the final hybrid variational principle.

The continuum model is approximated with finite elements by using Lagrangian interpolation formulas to approximate stress and displacement quantities. The stress quantities are eliminated at the element level, thus allowing the stress quantities to be discontinuous. The resulting nonlinear finite-element equations are solved by iterating linear solutions to converge to the nonlinear solution. Two techniques have been used to do this: one iterates the nodal displacements, and the other iterates the incremental nodal displacements. Each of these techniques has performed well for particular problems.

II. GENERALIZED FORMULATION

This section defines the reference configuration as a linear combination of the undeformed and deformed configurations. The generalized strain tensors for this formulation are specified. A generalized hybrid variational principle, the constitutive equations, and the boundary conditions are then presented.

When Lambda is defined as shown in Eq. (II-1), the reference configuration is a linear combination of the undeformed and deformed configurations, as shown in the following equations:

$$Y^1 = (1-\Lambda)Z^1 + \Lambda z^1 \quad , \qquad \text{(II} - 1\text{)}$$

$$Y^2 = (1-\Lambda)Z^2 + \Lambda z^2 \quad ,$$

and

$$Y^3 = (1-\Lambda)Z^3 + \Lambda z^3 \quad .$$

The generalized strain tensors (Cook 1986) are

$$\varepsilon_{ab} = \frac{1}{2}\left[v_{a;b} + v_{b;a} + (1-2\Lambda)\, v_{c;a} v^{c}_{\ ;b}\right] \quad , \qquad \text{(II} - 2\text{)}$$

where v_a are the covariant components of the displacement vector

$$\mathbf{u} = v_a \mathbf{B}^a \quad . \qquad \text{(II} - 3\text{)}$$

The hybrid variational principle for the generalized formulation is

$$\int_\Omega \tau^{ab}\left[(\delta v_a)_{;b} + (1-\Lambda) v^{c}_{\ ;a}(\delta v_c)_{;b}\right] d\Omega \qquad \text{(II} - 4\text{)}$$

$$+ \int_\Omega \left[v_{a;b} + \left(\frac{1-2\Lambda}{2}\right) v^{c}_{\ ;a} v_{c;b} - c_{abde}\tau^{de}\right] \delta\tau^{ab}\, d\Omega$$

$$- \int_{\Omega} \zeta \beta^a \delta v_a \, d\Omega - \int_{\omega} \Gamma^a \delta v_a \, d\omega = 0 \quad ,$$

where

τ^{ab} are the generalized stress tensors,

c_{abde} are the material tensors,

ζ is the density of the reference configuration,

β^a are the components of the body acceleration vector ($\boldsymbol{\beta} = \beta^a \mathbf{B}_a$),

$\boldsymbol{\nu}$ is the normal vector to the reference configuration boundary ω ($\boldsymbol{\nu} = \nu^a \mathbf{B}_a$),

Γ^a are the components of the traction vector with respect to the normal vector $\boldsymbol{\nu}$,

Ω is the volume of the reference configuration, and

ω is the reference configuration boundary surface.

The generalized constitutive equations are generalized strain tensors as a function of generalized stress tensors or generalized stress tensors as a function of generalized strain tensors, and they are written as

$$\varepsilon_{ab} = c_{abde} \tau^{de} \tag{II – 5}$$

or

$$\tau^{ab} = \kappa^{abde} \varepsilon_{de} \quad ,$$

where c_{abde} and κ^{abde} are material tensors.

The traction boundary conditions are

$$\Gamma^a = \left[\tau^{ab} + (1 - \Lambda) \tau^{cb} v^a_{;c} \right] \nu_b \tag{II – 6}$$

or the displacement boundary conditions are

$$\delta v_a = 0 \quad . \tag{II – 7}$$

The equations presented in this section reduce to the Lagrangian formulation when Lambda equals zero. In this case, the generalized strain tensors become the Green strain tensors, the generalized stress tensors become the second Piola-Kirchhoff stress tensors, and the generalized hybrid variational principle becomes the hybrid variational principle used in the Lagrangian formulation. Similarly, when Lambda equals one, the generalized formulation becomes the Eulerian formulation, the generalized strain tensors become Almansi strain tensors, the generalized stress tensors become the Cauchy stress tensors, and the generalized hybrid variational principle becomes the hybrid variational principle used in the Eulerian formulation. Any formulation in which Lambda is any value between zero and one is a legitimate formulation. Note: When Lambda equals one-half, the strain tensors are simplified, as shown in Eqs. (II-2).

III. COVARIANT DIFFERENTIATION OF DISPLACEMENTS

This section derives the covariant differentiation of displacements v_a as functions of the partial derivatives of the Cartesian displacements w_a.

The displacement transformations from the curvilinear coordinates y^a to the Cartesian coordinates Y^a (reference configuration) are

$$v_a = \frac{\partial Y^c}{\partial y^a} w_c \quad . \qquad \text{(III} - 1\text{)}$$

Covariant differentiation of displacements are

$$v_{a;b} = \frac{\partial v_a}{\partial y^b} - \left\{ {c \atop a\ b} \right\} v_c \quad , \qquad \text{(III} - 2\text{)}$$

where $\left\{ {c \atop a\ b} \right\}$ are the Christoffel symbols of the second kind.

Since Y^a are Cartesian coordinates and the Christoffel symbols of the second kind are zero in these coordinates (Fung 1965:46), we see that

$$\left\{ {c \atop a\ b} \right\} = \frac{\partial^2 Y^d}{\partial y^a \partial y^b} \frac{\partial y^c}{\partial Y^d} \quad . \qquad \text{(III} - 3\text{)}$$

By substituting Eqs. (III-1) and (III-3) into Eqs. (III-2), we get

$$v_{a;b} = \frac{\partial Y^c}{\partial y^a} \frac{\partial w_c}{\partial y^b} \quad . \qquad \text{(III} - 4\text{)}$$

Similarly, the variations of the displacements are

$$(\delta v_a)_{;b} = \frac{\partial Y^c}{\partial y^a} \frac{\partial (\delta w_c)}{\partial y^b} \quad . \qquad \text{(III} - 5\text{)}$$

IV. DISPLACEMENT ITERATION APPROACH

This section presents the form of the generalized hybrid variational principle that is used for the displacement iteration approach. The finite-element method is used to derive the stiffness and force tensors.

It can be shown from Fung (1965) that all of the body force integrals and all of the traction integrals from the virtual work principles in Cook and Genin (1988) are equal. Since the body forces in the virtual work principle used in the Lagrangian formulation and the tractions in the virtual work principle used in the Eulerian formulation are known, the generalized hybrid variational principle, Eq. (II-4), can be written as

$$\int_{\Omega} \tau^{ab} \left[(\delta v_a)_{;b} + \left(\frac{1-2\Lambda}{2} \right) v^{c}_{\ ;a} (\delta v_c)_{;b} \right] d\Omega \qquad \text{(IV - 1)}$$

$$+ \int_{\Omega} \left[v_{a;b} + \left(\frac{1-2\Lambda}{2} \right) v^{c}_{\ ;a} v_{c;b} - c_{abde} \tau^{de} \right] \delta\tau^{ab} \, d\Omega$$

$$+ \frac{1}{2} \int_{\Omega} \tau^{ab} v^{c}_{\ ;a} (\delta v_c)_{;b} \, d\Omega - \int_{V} P F_I \delta w_I \, dV - \int_{s} t_i \delta w_i \, ds = 0 \quad .$$

When we know Ω (volume), t (deformed traction), and c (nonlinear materials) from the last iteration, the integrals from Eq. (IV-1) are as follows:

- the first and second are the stiffness integrals,
- the third modifies the force vector integral, and
- the fourth and fifth are the usual force vector integrals.

Finite-element approximations can be represented as

$$w_a = p_\alpha w_a^\alpha \qquad \text{(IV - 2)}$$

and

$$\tau^{ab} = p^{\alpha}\tau_{\alpha}^{ab} \quad , \tag{IV - 3}$$

where

p_{α} and p^{α} are finite-element shape functions,

w_{a}^{α} are Cartesian nodal point displacements, and

τ_{α}^{ab} are nodal point generalized stresses.

Repeated lower-case Greek letters indicate the summation of all element nodal points.

By substituting Eqs. (IV-2) into Eqs. (III-4), we get

$$v_{a;b} = \frac{\partial Y^{c}}{\partial y^{a}}\frac{\partial p_{\alpha}}{\partial y^{b}}w_{c}^{\alpha} \quad . \tag{IV - 4}$$

The variations of Eqs. (IV-2) and Eqs. (IV-3) are

$$\delta w_{a} = p_{\alpha}\delta w_{a}^{\alpha} \tag{IV - 5}$$

and

$$\delta\tau^{ab} = p^{\alpha}\delta\tau_{\alpha}^{ab} \quad . \tag{IV - 6}$$

By substituting Eqs. (IV-5) into Eqs. (III-5), we get

$$(\delta v_{a})_{;b} = \frac{\partial Y^{c}}{\partial y^{a}}\frac{\partial p_{\alpha}}{\partial y^{b}}\delta w_{c}^{\alpha} \quad . \tag{IV - 7}$$

The substitution of Eqs. (IV-4), (IV-5), (IV-6), and (IV-7) into Eq. (IV-1) gives the generalized hybrid variational principle (where k designates element number) as

$$\sum_k \left(\left\{ \int_{(\Omega_k)^n} \left[\delta_a^d + \left(\frac{1-2\Lambda}{2} \right) \left(v^d_{\;;a} \right)^n \right] p^\beta \frac{\partial Y^c}{\partial y^d} \frac{\partial p_\alpha}{\partial y^b} \, d\Omega \right\} \tau_\beta^{ab} \delta w_c^\alpha \right. \tag{IV − 8}$$

$$+ \left\{ \int_{(\Omega_k)^n} \left[\delta_a^d + \left(\frac{1-2\Lambda}{2} \right) \left(v^d_{\;;a} \right)^n \right] p^\beta \frac{\partial Y^c}{\partial y^d} \frac{\partial p_\alpha}{\partial y^b} \, d\Omega \right\} w_c^\alpha \delta \tau_\beta^{ab}$$

$$- \left[\int_{(\Omega_k)^n} c_{abde} p^\beta p^\alpha \, d\Omega \right] \tau_\alpha^{de} \delta \tau_\beta^{ab}$$

$$+ \frac{1}{2} \left[\int_{(\Omega_k)^n} \left(\tau^{ab} \right)^n \left(v^d_{\;;a} \right)^n \frac{\partial Y^c}{\partial y^d} \frac{\partial p_\alpha}{\partial y^b} \, d\Omega \right] \delta w_c^\alpha$$

$$\left. - \left[\int_{V_k} P F_c p_\alpha \, dV \right] \delta w_c^\alpha - \left[\int_{(\omega_k)^n} (t_c)^n \, p_\alpha \, ds \right] \delta w_c^\alpha \right) = 0 \quad .$$

The superscript n refers to the last increment or iteration.

The $\left(v^a_{\;;b} \right)^n$ are

$$\left(v^a_{\;;b} \right)^n = \left(B^{ac} v_{c;b} \right)^n \quad , \tag{IV − 9}$$

where

$$\left(v_{a;b} \right)^n = \frac{\partial Y^c}{\partial y^a} \frac{\partial \left(w_c \right)^n}{\partial y^b} \quad . \tag{IV − 10}$$

The $\left(\tau^{ab} \right)^n$ are

$$\left(\tau^{ab} \right)^n = \left(\kappa^{abde} \right)^n \left(\varepsilon_{de} \right)^n \quad , \tag{IV − 11}$$

where

$$\left(\varepsilon_{ab} \right)^n = \frac{1}{2} \left[\left(v_{a;b} \right)^n + \left(v_{b;a} \right)^n + (1-2\Lambda) \left(v^c_{\;;a} \right)^n \left(v_{c;b} \right)^n \right] \quad . \tag{IV − 12}$$

We use the following equations to update the geometry

$$\left(Y^1\right)^n = Z^1 + \Lambda\left(w^1\right)^n \quad , \qquad \text{(IV} - 13)$$

$$\left(Y^2\right)^n = Z^2 + \Lambda\left(w^2\right)^n \quad ,$$

$$\left(Y^3\right)^n = Z^3 + \Lambda\left(w^3\right)^n \quad ,$$

$$\left(z^1\right)^n = Z^1 + \left(w^1\right)^n \quad , \qquad \text{(IV} - 14)$$

$$\left(z^2\right)^n = Z^2 + \left(w^2\right)^n \quad ,$$

and

$$\left(z^3\right)^n = Z^3 + \left(w^3\right)^n \quad .$$

Everywhere within Ω, w^1, w^2, and w^3 are Cartesian displacements.

The nonlinear finite-element equations are obtained from Eq. (IV-8). We eliminate the nodal stresses at the element level and iterate nodal displacements for a solution. This ensures continuous displacements but allows the the stresses to be discontinuous.

V. INCREMENTAL DISPLACEMENT ITERATION APPROACH

The generalized hybrid variational principle presented in Section II is modified for incremental displacements. The stiffness and force tensors are then derived using the finite-element method.

The incremental Cartesian displacements are defined as

$$(w_a)^{n+1} = (w_a)^n + \Delta w_a \quad . \tag{V-1}$$

The curvilinear incremental stress tensors are

$$\left(\tau^{ab}\right)^{n+1} = \left(\tau^{ab}\right)^n + \Delta \tau^{ab} \quad . \tag{V-2}$$

The variation of the Cartesian displacements are

$$\delta w_a = \delta(\Delta w_a) \quad . \tag{V-3}$$

The geometry approximations are

$$(\Omega)^{n+1} = (\Omega)^n \tag{V-4}$$

and

$$(s)^{n+1} = (s)^n \quad . \tag{V-5}$$

Thus, the first and second integrals of Eq. (II-4) can be written in an incremental form as

$$\int_\Omega \tau^{ab} \left[(\delta v_a)_{;b} + (1-\Lambda)\, v^c_{\ ;a} \,(\delta v_c)_{;b} \right] d\Omega \tag{V-6}$$

$$+ \int_\Omega \left[v_{a;b} + \left(\frac{1-2\Lambda}{2}\right) v^c_{\ ;a} v_{c;b} - c_{abde}\tau^{de} \right] \delta\tau^{ab}\, d\Omega$$

$$= \int_{(\Omega)^n} \left(\tau^{ab}\right)^n \left[\delta^c_a + (1-\Lambda)\left(v^c_{;a}\right)^n\right] \frac{\partial Y^d}{\partial y^c} \frac{\partial\left[\delta\left(\Delta w_d\right)\right]}{\partial y^b}\, d\Omega$$

$$+ \int_{(\Omega)^n} \Delta\tau^{ab} \left[\delta^c_a + (1-\Lambda)\left(v^c_{;a}\right)^n\right] \frac{\partial Y^d}{\partial y^c} \frac{\partial\left[\delta\left(\Delta w_d\right)\right]}{\partial y^b}\, d\Omega$$

$$+ \int_{(\Omega)^n} \left\{\left[\delta^c_a + (1-2\Lambda)\left(v^c_{;a}\right)^n\right] \frac{\partial Y^d}{\partial y^c} \frac{\partial\left(\Delta w_d\right)}{\partial y^b} - c_{abde}\Delta\tau^{de}\right\} \delta(\Delta\tau^{ab})\, d\Omega \quad .$$

The superscript n refers to the last increment or iteration.

The $\left(v^c_{;a}\right)^n$ are

$$\left(v^a_{;b}\right)^n = \left(B^{ac} v_{c;b}\right)^n \quad , \qquad (V-7)$$

where

$$\left(v_{a;b}\right)^n = \frac{\partial Y^c}{\partial y^a} \frac{\partial\left(w_c\right)^n}{\partial y^b} \quad . \qquad (V-8)$$

The $\left(\tau^{ab}\right)^n$ are

$$\left(\tau^{ab}\right)^n = \left(\kappa^{abde}\right)^n \left(\varepsilon_{de}\right)^n \quad , \qquad (V-9)$$

where

$$\left(\varepsilon_{ab}\right)^n = \frac{1}{2}\left[\left(v_{a;b}\right)^n + \left(v_{b;a}\right)^n + (1-2\Lambda)\left(v^c_{;a}\right)^n \left(v_{c;b}\right)^n\right] \quad . \qquad (V-10)$$

Symmetry does not exist between the incremental displacements and the incremental stress tensors unless the 2Λ is replaced with Λ. This is the same approximation used in Cook and Genin 1988:18-23 and Cook 1988:14-19. With this approximation, Eq. (V-6) becomes

$$\int_{\Omega} \tau^{ab} \left[(\delta v_a)_{;b} + (1-\Lambda)\, v^c_{;a}\, (\delta v_c)_{;b} \right] d\Omega \qquad (V-11)$$

$$+ \int_{\Omega} \left[v_{a;b} + \left(\frac{1-2\Lambda}{2} \right) v^c_{;a} v_{c;b} - c_{abde} \tau^{de} \right] \delta\tau^{ab}\, d\Omega$$

$$= \int_{(\Omega)^n} \left(\tau^{ab}\right)^n \left[\delta^c_a + (1-\Lambda) \left(v^c_{;a}\right)^n \right] \frac{\partial Y^d}{\partial y^c} \frac{\partial \left[\delta\left(\Delta w_d\right)\right]}{\partial y^b}\, d\Omega$$

$$+ \int_{(\Omega)^n} \Delta\tau^{ab} \left[\delta^c_a + (1-\Lambda) \left(v^c_{;a}\right)^n \right] \frac{\partial Y^d}{\partial y^c} \frac{\partial \left[\delta\left(\Delta w_d\right)\right]}{\partial y^b}\, d\Omega$$

$$+ \int_{(\Omega)^n} \left\{ \left[\delta^c_a + (1-\Lambda) \left(v^c_{;a}\right)^n \right] \frac{\partial Y^d}{\partial y^c} \frac{\partial \left(\Delta w_d\right)}{\partial y^b} - c_{abde} \Delta\tau^{de} \right\} \delta(\Delta\tau^{ab})\, d\Omega \quad .$$

The substitution of Eq. (V-11) into Eq. (IV-1) gives the generalized hybrid variational principle for incremental displacements as

$$\int_{(\Omega)^n} \left(\tau^{ab}\right)^n \left[\delta^c_a + (1-\Lambda) \left(v^c_{;a}\right)^n \right] \frac{\partial Y^d}{\partial y^c} \frac{\partial \left[\delta\left(\Delta w_d\right)\right]}{\partial y^b}\, d\Omega \qquad (V-12)$$

$$+ \int_{(\Omega)^n} \Delta\tau^{ab} \left[\delta^c_a + (1-\Lambda) \left(v^c_{;a}\right)^n \right] \frac{\partial Y^d}{\partial y^c} \frac{\partial \left[\delta\left(\Delta w_d\right)\right]}{\partial y^b}\, d\Omega$$

$$+ \int_{(\Omega)^n} \left\{ \left[\delta^c_a + (1-\Lambda) \left(v^c_{;a}\right)^n \right] \frac{\partial Y^d}{\partial y^c} \frac{\partial \left(\Delta w_d\right)}{\partial y^b} - c_{abde} \Delta\tau^{de} \right\} \delta(\Delta\tau^{ab})\, d\Omega$$

$$- \int_V P F_I \delta(\Delta w_I)\, dV - \int_{(s)^n} t_i \delta(\Delta w_i)\, ds = 0 \quad ,$$

When we know Ω (volume), τ (generalized stress tensor), t (deformed traction), and c (nonlinear materials) from the last iteration, the integrals from Eq. (V-12) are as follows:

- the first modifies the force vector integral,
- the second and third are the stiffness tensor integrals, and
- the third and fourth are the usual force vectors.

Finite-element approximations can be represented as

$$\Delta w_a = p_\alpha \Delta w_a^\alpha \qquad (V-13)$$

and

$$\Delta \tau^{ab} = p^\alpha \Delta \tau_\alpha^{ab} \quad , \qquad (V-14)$$

where

p_α and p^α are finite-element shape functions,

Δw_a^α are Cartesian nodal point displacements, and

$\Delta \tau_\alpha^{ab}$ are nodal point incremental generalized stresses.

Repeated lower-case Greek letters indicate the summation of all element nodal points.

Equation (V-13) can be differentiated as

$$\frac{\partial (\Delta w_a)}{\partial y^b} = \frac{\partial p_\alpha}{\partial y^b} \Delta w_a^\alpha \quad . \qquad (V-15)$$

The variations of Eqs. (V-13) and Eqs. (V-14) are

$$\delta(\Delta w_\alpha) = p_\alpha \delta(\Delta w_a^\alpha) \qquad (V-16)$$

and

$$\delta(\tau^{ab}) = p^\alpha \delta(\Delta \tau_\alpha^{ab}) \quad . \qquad (V-17)$$

The differentiation of Eqs. (V- 16) are

$$\frac{\partial\left[\delta(\Delta w_a)\right]}{\partial y^b} = \frac{\partial p_\alpha}{\partial y^b}\delta(\Delta w_a^\alpha) \quad . \tag{V − 18}$$

The substitution of Eqs. (V-13), (V-14), (V-15), (V-16), (V-17), and (V-18) into Eq. (V-12) gives the generalized incremental hybrid variational principle (where k designates element number) as

$$\begin{aligned}
\sum_k \Bigg(& \left\{ \int_{(\Omega_k)^n} \left(\tau^{ab}\right)^n \left[\delta_a^c + (1-\Lambda)\left(v^c_{;a}\right)^n\right] \frac{\partial Y^d}{\partial y^c}\frac{\partial p_\alpha}{\partial y^b}\, d\Omega \right\} \delta(\Delta w_d^\alpha) \\
& + \left\{ \int_{(\Omega_k)^n} \left[\delta_a^c + (1-\Lambda)\left(v^c_{;a}\right)^n\right] \frac{\partial Y^d}{\partial y^c}\frac{\partial p_\alpha}{\partial y^b} p^\beta\, d\Omega \right\} \Delta\tau_\beta^{ab}\delta(\Delta w_d^\alpha) \\
& + \left\{ \int_{(\Omega_k)^n} \left[\delta_a^c + (1-\Lambda)\left(v^c_{;a}\right)^n\right] \frac{\partial Y^d}{\partial y^c}\frac{\partial p_\beta}{\partial y^b} p^\alpha\, d\Omega \right\} \Delta w_d^\beta \delta(\Delta\tau_\alpha^{ab}) \\
& - \left[\int_{(\Omega_k)^n} c_{abde} p^\alpha p^\beta\, d\Omega \right] \Delta\tau_\beta^{de}\delta(\Delta\tau_\alpha^{ab}) \\
& - \left[\int_{V_k} PF_a p_\alpha\, dV \right] \delta(\Delta w_a^\alpha) - \left[\int_{(s_k)^n} (t_a)^n p_\alpha\, ds \right] \delta(\Delta w_a^\alpha) \Bigg) = 0 \quad .
\end{aligned} \tag{V − 19}$$

The geometry is then updated with Eqs. (IV-13) and (IV-14). The nonlinear finite-element equations are obtained from Eq. (V-19). We eliminate the nodal stresses at the element level and iterate nodal displacements for a solution. This ensures continuous displacements but allows the the stresses to be discontinuous.

VI. CONCLUSIONS

The hybrid variational principle presented in this report was checked by solving several problems, including a rectangular body with concentrated loads, body force loads, and traction loads; a cylindrical body with traction loads; and a spherical body with traction loads.

Two methods for solving nonlinear continuum problems are described in this report. Each of these methods works well in accuracy vs computer time for particular problems. Neither method, however, has proved to be consistently better than the other.

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