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Analysis of Error in

Monte Carlo Transport Calculations

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Monte Carlo Transport Calculations

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INTRODUCTION

The Monte Carlo method attempts to get the average behavior of neutrons by simulating a large sample of neutron histories and using the sample mean as an approximation[†] to the average predicted by the Neutron Transport Equation. The Monte Carlo method for neutron transport calculations suffers, in part, because of the inherent statistical errors associated with the method. Without an estimate of these errors in advance of the calculation, it is difficult to decide what estimator and biasing scheme to use, or whether a proposed Monte Carlo calculation will be competitive with a discrete ordinates calculation (a question not treated here).

Perhaps the most compelling reason for predicting statistical errors is cost. If one is contemplating writing a Monte Carlo code (or making major modifications to an existing code) incorporating a new Monte Carlo technique, he would like to know how well the new technique will work - before investing a great deal of time and money writing the code.

If one uses a standard Monte Carlo code he would probably try several techniques and uses the sample variance to select the best technique. This trial and error method works well if one has enough insight into both the transport problem and the Monte Carlo techniques he applies to solve it. However, if one's insight is poor, trial and error will result

[†] Using the law of large numbers

in mostly error. Furthermore, the sample variance, by which one judges the technique, becomes more and more unreliable as the technique departs from optimum. Appendix B has an example of an unreliable sample variance.

Recent work[2,3,4,7,8] on the prediction of statistical errors in Monte Carlo calculations stems from a paper Amster and Djomehri[1] published in 1976, in which a set of integral equations were derived that, when solved, predicted errors in Monte Carlo calculations in non-multiplying media. The present work allows error prediction in non-analog Monte Carlo calculations of multiplying systems, even when supercritical. Non-analog techniques such as biased kernels, particle splitting, and Russian Roulette have been incorporated in this work. Coincidentally (and independently of this work), Sarkar and Prasad[3] have investigated errors in non-analog Monte Carlo calculations of subcritical multiplying systems.

Chapter I. Illustration of Concepts by Example

The Monte Carlo method simulates neutron behavior on a computer. Real neutron transport problems are quite complex and the mathematics required to predict errors in their Monte Carlo simulations is correspondingly complex. In fact, the mathematics often obscures some very simple concepts. This chapter illustrates, for a very simple transport problem, concepts such as analog game, non-analog game, particle weight, score, and importance sampling - using only high-school mathematics. The reader who is familiar with Monte Carlo may skip to Chapter II which treats real transport problems.

An analog (Monte Carlo) game is a faithful simulation of the real transport process; a non-analog game is a simulation in which the simulated neutron does not behave exactly like a real neutron. As a simple example, a two-state transport problem is simulated - first by an analog game in I-A and then by a non-analog game in I-B. Section I-C links the analysis of this simple problem to the analysis of the real transport problems considered in Chapters II and III.

A few definitions are required before addressing the two-state transport problem. In subsequent discussions, particle refers to a computer simulated neutron. A particle's history is a sequence of events, chosen in a random fashion, ending with the loss of the particle. Every event (e.g. scattering) has two numbers associated with it: the probability the event occurs and the event score. A particle's history

score (total score) is the sum of all the event scores the particle generates in its history. Although nature supplies event probabilities, at least for analog games, the calculator must assign appropriate event scores. For example, if one is interested in the mean number of scattering events per particle he might assign each scattering event a "one" and any other event a "zero"; a particle's history score would then be equal to the number of times the particle was scattered.

A) Analog Game

Consider the simple transport problem below where a particle can be in one of two states, labeled 1 and 2. The six possible events and their probabilities and scores are depicted in figure 1 and described in table 1. These six events are independent and mutually exclusive.
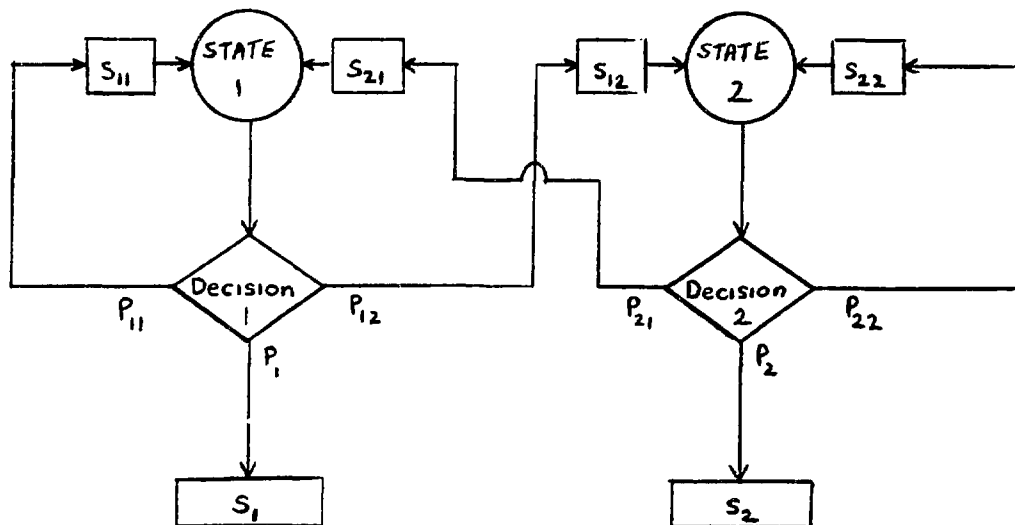
Fig 1. The two-state transport problem.

Table 1

Definitions for the two-state transport problem

Event	Event Probability	Event Score
e_{12} scattering from state 1 to state 2	p_{12}	s_{12}
e_{21} scattering from state 2 to state 1	p_{21}	s_{21}
e_{11} scattering from state 1 to state 1	p_{11}	s_{11}
e_{22} scattering from state 2 to state 2	p_{22}	s_{22}
e_1 absorption from state 1	p_1	s_1
e_2 absorption from state 2	p_2	s_2

Decision 1 will be made by picking a random number, ζ_i, between 0 and

1:

$$\vdash\!\!\!\!-\!\!\!\!-\!\!\!\!+\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!+\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!+$$

$$0 \qquad p_1 \qquad\quad p_1+p_{11} \qquad\quad p_1+p_{11}+p_{12}=1$$

$$\leftarrow p_1 \rightarrow\leftarrow p_{11} \longrightarrow\leftarrow\!\!\!\!-\!\!\!\!-\!\!\!\!-\; p_{12} \longrightarrow$$
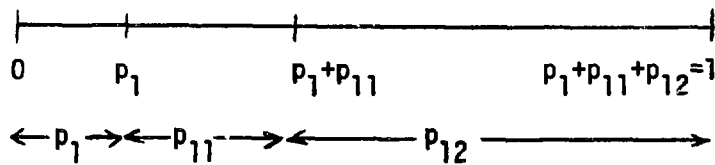
e_1 occurs if $0<\zeta_i\leq p_1$,

e_{11} occurs if $p_1<\zeta_i\leq p_1+p_{11}$, and

e_{12} occurs if $p_1+p_{11}<\zeta_i\leq 1$.

Decision 2 will be made by picking a random number, ζ_i, between 0 and

1:

$$\vdash\!\!\!\!-\!\!\!\!-\!\!\!\!+\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!+\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!+$$

$$0 \qquad p_2 \qquad\quad p_2+p_{22} \qquad\quad p_2+p_{22}+p_{21}=1$$
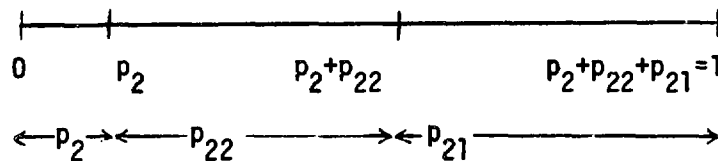
$$\leftarrow p_2 \rightarrow\leftarrow\!\!\!\!-\; p_{22} \longrightarrow\leftarrow p_{21} \longrightarrow$$

e_2 occurs if $0<\zeta_i\leq p_2$,

e_{22} occurs if $p_2<\zeta_i\leq p_2+p_{22}$, and

e_{21} occurs if $p_2+p_{22}<\zeta_i\leq 1$.

As an example, consider a particle starting in state 1. Suppose that the ζ_i are:

$$p_1+p_{11}<\zeta_1\leq 1 \quad \text{implies } e_{12} \text{ occurs,}$$

$$p_2<\zeta_2\leq p_2+p_{22} \quad \text{implies } e_{22} \text{ occurs,}$$

$$p_2+p_{22}\leq\zeta_3<1 \quad \text{implies } e_{21} \text{ occurs, and}$$

$$0<\zeta_4\leq p_1 \qquad \text{implies } e_1 \text{ occurs.}$$

In this case, referring to Table 1, the particle first scatters from state 1 to state 2 (event e_{12}) and scores s_{12}, next scatters from state 2 to state 2 (event e_{22}) and scores s_{22}, next scatters from state 2 to state 1 (event e_{21}) and scores s_{21}, and lastly suffers absorption from state 1 (event e_1) and scores s_1. The history score for this series of events is then $s = s_{12} + s_{22} + s_{21} + s_1$.

This simple system could represent a particle roaming about in an infinite homogeneous two nuclide medium (with only scattering and absorption)

State 1 = particle entering a collision with nuclide type 1

State 2 = particle entering a collision with nuclide type 2

If the scores are taken to be

$$s_{11} = s_{12} = s_{21} = s_{22} = s_2 = 0 \text{ and } s_1 = 1,$$

then a particle absorbed by nuclide 1 would have a history score of 1 and a particle absorbed by nuclide 2 would have a history score of 0. If a large number of particles starting at nuclide j are followed, the mean history score is an estimate of the mean number of particles absorbed by nuclide 1 per particle entering a collision with nuclide j. (This problem is easily solved without Monte Carlo.)

Now return to the general two state problem. It is important to know how many particles must be followed to obtain an acceptable error in the estimate of the mean history score. Let M_1 be the mean score and M_2 be the mean-square score. According to the "central limit theorem,"[1] the means of samples, each of h histories, for h large,

are approximately normally distributed about M_1 with a variance

$$\sigma^2 = \frac{M_2 - M_1^2}{h} .$$ (1)

An estimate of this variance can be obtained by substituting the sample means $\overline{M}_1$ and $\overline{M}_2$ in place of the means M_1 and M_2. There are two problems with using the sample variance:

1) The error estimate is available only after the problem has been programmed and run on a computer. Thus much investment in programming time and computer time might be required simply to determine that the variance is unacceptably high.

2) The sample means $\overline{M}_1$ and $\overline{M}_2$ tend to be unreliable estimates of M_1 and M_2 when the variance is high. Suppose $\overline{M}_2 < M_2$ and $M_1^2 > \overline{M}_1^2$; this results in a sample variance,

$$\overline{\sigma}^2 = \frac{\overline{M}_2 - \overline{M}_1^2}{h} < \frac{M_2 - M_1^2}{h} = \sigma^2 ,$$ (2)

which could be much less than the true variance.

As an example of the second problem, consider the two nuclide problem with the absorption cross section for nuclide 1 many orders of magnitude smaller than any other cross sections. This corresponds to p_1 being orders of magnitude smaller than any of the other probabilities. Thus 100,000 particles might be followed and all result in absorption by nuclide 2, i.e. history score zero. In this case,

$\overline{M}_1 = 0$, sample mean history score,

$\overline{M}_2 = 0$, sample mean-square history score, and

$$\overline{\sigma}^2 = \frac{\overline{M}_2 - \overline{M}_1{}^2}{h} = 0, \text{ sample variance,} \tag{3}$$

whereas, in fact, $M_1 = \epsilon$, where ϵ is some number greater than zero. Thus, $\overline{M}_1$ is infinitely smaller than the true mean M_1 at the same time that the sample variance of zero indicates a very good sample mean.

To avoid problems one and two it is useful to obtain an estimate of the variance before attempting a Monte Carlo solution of the problem. To obtain this estimate three terms must be defined:

$\psi(j,s)$ = probability that a particle in state j will contribute

a cumulative score s in its subsequent history; (4)

$$M_r(j) = \sum_s s^r \psi(j,s) = r^{th} \text{ moment of history score}$$

(where the sum is over all possible

s values); and (5)

$$\delta_{ss'} = \begin{cases} 0 \text{ if } s \neq s', \\ 1 \text{ if } s = s', \end{cases} \text{ where } s \text{ and } s' \text{ are not limited to integers. (6)}$$

From these definitions eqn. (7) may be derived.

$$\psi(i,s) = p_i \delta_{ss_i} + \sum_j p_{ij} \psi(j, s-s_{ij}) \tag{7}$$

Let s_{ne} ($s_{ne} \epsilon \{s_1, s_{11}, s_{12}, s_2, s_{22}, s_{21}\}$) be the score contributed by the next event. The probability that a particle in state i subsequently contributes a cumulative score s is equal to the sum, over all possible

next events, of the probability that the given event occurs multiplied by the probability that all subsequent events contribute the remaining score $s-s_{ne}$, so that the total score is $s_{ne} + (s-s_{ne}) = s$. If the next event is an absorption then the probability that the cumulative score is s is δ_{ss_i}; that is, if the absorption contributes a score s the cumulative score will be s and if the absorption does not contribute a score s the cumulative score will not be s. If the next event is a scattering to state j then there is a score s_{ij} contributed by the scattering, which leaves a score $s-s_{ij}$ to be contributed in subsequent events. The probability that a particle in state j contributes a cumulative score $s-s_{ij}$ in subsequent events is $\psi(j,s-s_{ij})$ by definition (4).

Multiplying eq. (7) by s^r and summing over all possible scores,

$$\sum_s \psi(i,s)s^r = \sum_s \delta_{ss_i} p_i s^r + \sum_s \sum_j p_{ij} \psi(j,s-s_{ij})s^r. \qquad (8)$$

Using eq. (5) and recalling that s_i is a possible value for s,

$$M_r(i) = p_i s_i^r + \sum_s \sum_j p_{ij} \psi(j,s-s_{ij})s^r. \qquad (9)$$

Letting $s' = s-s_{ij}$ or $s = s' + s_{ij}$,

$$\sum_s \sum_j p_{ij} \psi(j,s-s_{ij})s^r = \sum_{s'} \sum_j p_{ij} \psi(j,s')(s'+s_{ij})^r;$$

then using the binomial expansion,

$$= \sum_{s'} \sum_j p_{ij} \psi(j,s') \sum_{n=0}^{r} \binom{r}{n} s'^n s_{ij}^{r-n}$$

$$= \sum_j p_{ij} \sum_{n=0}^{r} \binom{r}{n} s_{ij}^{r-n} \sum_{s'} \psi(j,s')s'^{n},$$

and by using eq. (5)

$$= \sum_j p_{ij} \sum_{n=0}^{r} \binom{r}{n} s_{ij}^{r-n} M_n(j) \tag{10}$$

Substituting eq. (10) into eq. (9) yields the results,

$$M_r(i) = p_i s_i^r + \sum_j p_{ij} \sum_{n=0}^{r} \binom{r}{n} s_{ij}^{r-n} M_n(j). \tag{11}$$

Pulling out the M_r term from the sum,

$$M_r(i) = \sum_j p_{ij} M_r(j) + p_i s_i^r + \sum_j p_{ij} \sum_{n=0}^{r-1} \binom{r}{n} s_{ij}^{r-n} M_n(j) \tag{12}$$

and noting that

$$M_0(j) = \sum_s \psi(j,s) = 1 \tag{13}$$

since $\psi(j,s)$ is a probability function, eq. (12) for r=1 and r=2 may be written:

$$M_1(1) = p_{11} M_1(1) + p_{12} M_1(2) + p_1 s_1 + p_{11} s_{11} + p_{12} s_{12} \tag{14}$$

$$M_1(2) = p_{21} M_1(1) + p_{22} M_1(2) + p_2 s_2 + p_{22} s_{22} + p_{21} s_{21} \tag{15}$$

and

$$M_2(1) = p_{11} M_2(1) + p_{12} M_2(2) + p_1 s_1^2 + p_{11}(s_{11}^2 + 2s_{11}M_1(1))$$

$$+ p_{12}(s_{12}^2 + 2s_{12}M_1(2)) \tag{16}$$

$$M_2(2) = p_{22} M_2(2) + p_{21} M_2(1) + p_2 s_2^2 + p_{22} (s_{22}^2 + 2s_{22} M_1(2))$$

$$+ p_{21} (s_{21}^2 + 2s_{21} M_1(1)) \tag{17}$$

IB Non-Analog Games

The equations derived above are for an analog simulation of the real transport process; the probability that event i occurs in the simulation is the same as the probability that event i occurs in the real transport process. In practice, the transport problem can sometimes be solved more accurately, for a given number of particle histories, if fictitious probabilities are used instead of the real transport probabilities. Any Monte Carlo solution which is not an analog simulation of the real transport process is termed a non-analog Monte Carlo solution.

To understand non-analog Monte Carlo it is necessary to introduce the concept of statistical weight for a particle. Statistical weight, or simply weight, is a score multiplier. Thus, for a given event a particle of weight w scores w times as much as a particle of weight one.

Consider the general two state problem with the particle in state j. There are three chains of events which may occur in the analog game considered previously:

1) the particle is absorbed in state j contributing a cumulative history score $s_a = s_j$

2) the particle scatters to state 1 and contributes a score s_{j1}.

Subsequently, from state 1 the particle contributes a score

$s_b - s_{j1}$ resulting in a cumulative history score from state j of

$$s_{j1} + (s_b - s_{j1}) = s_b$$

3) the particle scatters to state 2 and contributes a score s_{j2}.

Subsequently, from state 2 the particle contributes a score $s_c - s_{j2}$

resulting in a cumulative history score from state j of

$$s_{j2} + (s_c - s_{j2}) = s_c.$$

The mean score for a particle in state j is then

$$M_1(j) = \sum_{i=1}^{3} \text{(prob chain i occurs)*(score due to chain i)} \qquad (18)$$

$$M_1(j) = p_j s_a + p_{j1} s_b + p_{j2} s_c. \qquad (19)$$

Suppose that the probabilities are altered in the following manner,

$$\tilde{p}_j = \frac{1}{\alpha} p_j, \qquad \tilde{p}_{j1} = \frac{1}{\beta} p_{j1}, \text{ and } \qquad \tilde{p}_{j2} = \frac{1}{\gamma} p_{j2} \qquad (20)$$

subject to the constraint that the probability that one of the three

events occurs is still one:

$$\tilde{p}_j + \tilde{p}_{j1} + \tilde{p}_{j2} = 1. \qquad (21)$$

Then substituting eq. (20) into eq. (19),

$$M_1(j) = \alpha \tilde{p}_j s_a + \beta \tilde{p}_{j1} s_b + \gamma \tilde{p}_{j2} s_c. \qquad (22)$$

Recall that for the same chain of events a particle of weight w contributes w times the score that a particle of weight one contributes.

Identifying $\alpha = w_1$, $\beta = w_2$, and $\gamma = w_3$ as particle weights eq. (18) can be generalized to:

$$M_1(j) = \sum_{i=1}^{3} (\text{prob chain i occurs in calc})*(\text{score due to chain i}$$

$$\text{from a particle of weight } w_i). \tag{23}$$

While the mean score, M_1, remains the same with the fictitious probabilities the mean square score in general does not. To put these ideas on a more rigorous footing eqs. (14)-(17) will be rederived for arbitrary branch probabilities and branch weight multipliers. When the branch probabilities are not the same as the real transport probabilities this is called biasing or importance sampling.

IB-1 Biased Kernels

Consider a completely fictitious game in which the probabilities, weights, and scores are completely arbitrary; later it will be shown that with suitable choices for probabilities, weights, and scores, this non-analog game can be made to yield the same mean score, M_1, as the analog game. The following definitions are needed:

$\tilde{\psi}(j,w,s)$ = probability that a particle of weight w in state j

contributes a score s, (24)

w_i = weight multiplier for absorption from state i, (25)

w_{ij} = weight multiplier for scattering from state i to state j, (26)

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$\tilde{p}_i$ = prob of absorption from state i, $\qquad$ (27)

$\tilde{p}_{ij}$ = prob of scattering from state i to state j, $\qquad$ (28)

$\tilde{M}_r(j,w) = \sum\limits_s s^r \tilde{\psi}(j,w,s) = r^{th}$ moment of history score for a

particle of weight w in state j, $\qquad$ (29)

$\tilde{s}_i(w)$ = score for absorption from state i of a particle of

weight w, $\qquad$ (30)

$\tilde{s}_{ij}(w)$ = score for scattering from state i to state j for a

particle of weight w, and $\qquad$ (31)

$\delta_{ss'} = \begin{cases} 0 \text{ if } s' \neq s \\ 1 \text{ if } s = s' \end{cases}$ where s and s' are not necessarily integers. (32)

Next, consider eqn. (33) which is displayed and explained below.

$$\tilde{\psi}(i,w,s) = \tilde{p}_i \; \delta_{s\tilde{s}_i(ww_i)} + \sum_j \tilde{p}_{ij} \; \tilde{\psi}(j, \; ww_{ij}, \; s-\tilde{s}_{ij}(ww_{ij}) \;) \qquad (33)$$

Let s_{ne} be the score contributed by the next event. The probability that a particle of weight w in state i subsequently contributes a cumulative score s is equal to the sum, over all possible next events, of the probability that the given event occurs multiplied by the probability that all subsequent events contribute the remaining score $s-s_{ne}$, so that the total score is $s_{ne} + (s-s_{ne}) = s$.

If the next event is absorption then the particle's weight becomes ww_i and the probability that the cumulative score is s is

$\delta_{s\tilde{s}_i(ww_i)}$ That is, the cumulative score will be s if and only if the absorption contributes a score s.

If the next event is a scattering to state j then the particle's weight becomes ww_{ij} and there is a score $\tilde{s}_{ij}(ww_{ij})$ contributed by the scattering. This leaves a score $s-\tilde{s}_{ij}(ww_{ij})$ to be contributed in subsequent events. The probability that a particle of weight ww_{ij} in state j contributes a cumulative score $s-\tilde{s}_{ij}(ww_{ij})$ in subsequent events is by definition $\psi(j,ww_{ij},s-\tilde{s}_{ij}(ww_{ij})\ )$.

Multiplying eqn. (33) by s^r and summing over all possible values for s yields:

$$\sum_s \tilde{\psi}(i,w,s)s^r = \sum_s \tilde{p}_i\ \delta_{s\tilde{s}_i(ww_i)}s^r + \sum_s \sum_j \tilde{p}_{ij}$$
$$x\ \tilde{\psi}(j,ww_{ij},s-\tilde{s}_{ij}(ww_{ij})\ )s^r. \qquad (34)$$

Using the definition of $\tilde{M}_r$ and recalling that $\tilde{s}_i(ww_i)$ is a possible value for s,

$$\tilde{M}_r(i,w) = \tilde{p}_i\tilde{s}_i^r(ww_i) + \sum_s \sum_j \tilde{p}_{ij}\ \tilde{\psi}(j,ww_{ij},s-\tilde{s}_{ij}(ww_{ij})\ )s^r. \qquad (35)$$

Letting $s' = s - \tilde{s}_{ij}(ww_{ij})$ or $s = s' + \tilde{s}_{ij}(ww_{ij})$ yields $\qquad (36)$

$$\sum_s \sum_j \tilde{p}_{ij}\ \tilde{\psi}(j,ww_{ij},s-\tilde{s}_{ij}(ww_{ij})\ )s^r$$
$$= \sum_{s'} \sum_j \tilde{p}_{ij}\ \tilde{\psi}(j,ww_{ij},s')(s' + \tilde{s}_{ij}(ww_{ij})\ )^r,$$

and using the binomial expansion,

$$= \sum_{s'} \sum_{j} \tilde{p}_{ij} \tilde{\psi}(j, ww_{ij}, s') \sum_{n=0}^{r} \binom{r}{n} s'^{n} [\tilde{s}_{ij}(ww_{ij})]^{r-n}$$

$$= \sum_{j} \tilde{p}_{ij} \sum_{n=0}^{r} \binom{r}{n} [\tilde{s}_{ij}(ww_{ij})]^{r-n} \sum_{s'} \tilde{\psi}(j, ww_{ij}, s') s'^{n}.$$

From the definition of $\tilde{M}_r$,

$$= \sum_{j} \tilde{p}_{ij} \sum_{n=0}^{r} \binom{r}{n} [\tilde{s}_{ij}(ww_{ij})]^{r-n} \tilde{M}_n(j, ww_{ij}) \tag{37}$$

hence,

$$\tilde{M}_r(i, w) = \tilde{p}_i \tilde{s}_i^r(ww_i) + \sum_{j} \tilde{p}_{ij} \sum_{n=0}^{r} \binom{r}{n} [\tilde{s}_{ij}(ww_{ij})]^{r-n} \tilde{M}_n(j, ww_{ij}), \tag{38}$$

and pulling out the M_r term from the sum,

$$\tilde{M}_r(i, w) = \sum_{j} \tilde{p}_{ij} \tilde{M}_r(j, ww_{ij}) + \tilde{p}_i \tilde{s}_i^r(ww_i)$$

$$+ \sum_{j} \tilde{p}_{ij} \sum_{n=0}^{r-1} \binom{r}{n} [\tilde{s}_{ij}(ww_{ij})]^{r-n} \tilde{M}_n(j, ww_{ij}). \tag{39}$$

Since for a given event a particle of weight w scores w times as much as a particle of weight 1,

$$\tilde{s}_i(ww_i) = ww_i \tilde{s}_i(1) \tag{40}$$

$$\tilde{s}_{ij}(ww_{ij}) = ww_{ij} \tilde{s}_i(1). \tag{41}$$

Writing eqn. (39) for $r = 1$ (Noting as in eqn. (13)

$$\tilde{M}_0(j, w) = \sum_{s} \tilde{\psi}(j, w, s) = 1), \tag{42}$$

$$\tilde{M}_1(i, w) = \sum_{j} \tilde{p}_{ij} \tilde{M}_1(j, ww_{ij}) + \tilde{p}_i ww_i \tilde{s}_i(1) + \sum_{j} \tilde{p}_{ij} ww_{ij} \tilde{s}_{ij}(1). \tag{43}$$

A particle's history score is proportional to its weight because:

1) the event probabilities are assumed independent of weight; thus

2) a particle's history (sequence of events) is independent of weight, and

3) each event score is proportional to the particle's weight.

Thus the probability that a particle of weight w contributes a history score s is equal to the probability that a particle of weight one contributes a history score s/w, that is

$$\tilde{\psi}(i,w,s) = \tilde{\psi}(i,1,s/w) \tag{43A}$$

Hence using eqs. (29) and (43A)

$$\tilde{M}_r(i,w) = \sum_s \tilde{\psi}(i,w,s)s^r = w^r \sum_s \tilde{\psi}(i,1,s/w)(s/w)^r = w^r \tilde{M}_r(i,1) \tag{44}$$

Thus eq. (43) becomes

$$\tilde{M}_1(i,1) = \sum_j \tilde{p}_{ij} w_{ij} \tilde{M}_1(j,1) + \tilde{p}_i w_i \tilde{s}_i(1) + \sum_j \tilde{p}_{ij} w_{ij} \tilde{s}_{ij}(1). \tag{45}$$

Suppose the probabilities, weights, and scores are chosen such that (remember they were arbitrary):

$$\tilde{p}_{ij} w_{ij} = p_{ij} \qquad \tilde{p}_i w_i = p_i \qquad \tilde{s}_i(1) = s_i \qquad \tilde{s}_{ij}(1) = s_{ij} \tag{46}$$

then,

$$\tilde{M}_1(i,1) = \sum_j p_{ij} \tilde{M}_1(j,1) + p_i s_i + \sum_j p_{ij} s_{ij} \tag{47}$$

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This is the same eqn. that $M_1(i)$ satisfies [cf. eq. (12)], hence

$$\tilde{M}_1(i,1) = M_1(i). \tag{48}$$

The significance of this result is that a non-analog transport game (with relations (46) holding) can be played with any probabilities $\tilde{p}_{ij} > 0$ ($\tilde{p}_{ij}$ must be greater than zero so that w_{ij} is finite) and the mean score for the non-analog game will be equal to the mean score for the analog game. The second moments for the two games are in general not equal, as will be shown below.

Writing eq. (39) for $r=2$,

$$\tilde{M}_2(i,w) = \sum_j \tilde{p}_{ij}\tilde{M}_2(j,ww_{ij}) + \tilde{p}_i[ww_i\tilde{s}_i(1)]^2$$
$$+ \sum_j \tilde{p}_{ij}\left([ww_{ij}\tilde{s}_{ij}(1)]^2 + 2ww_{ij}\tilde{s}_{ij}(1)ww_{ij}\tilde{M}_1(j,1)\right). \tag{49}$$

From eq. (44) with $r=2$,

$$\tilde{M}_2(i,w) = w^2\tilde{M}_2(i,1) \tag{50}$$

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Letting w=1 and using eq. (50)

$$\tilde{M}_2(i,1) = \sum_j \tilde{p}_{ij} w_{ij}^2 \, \tilde{M}_2(j,1) + \tilde{p}_i [w_i \tilde{s}_i(1)]^2$$

$$+ \sum_j \tilde{p}_{ij} ([w_{ij}\tilde{s}_{ij}(1)]^2 + 2w_{ij}^2 \, \tilde{s}_{ij}\tilde{M}_1(j,1)). \qquad (51)$$

Using the relations (46) which resulted in $\tilde{M}_1(i,1) = M_1(i)$,

$$\tilde{M}_2(i,1) = \sum_j p_{ij} w_{ij} \tilde{M}_2(j,1) + p_i w_i s_i^2$$

$$+ \sum_j p_{ij} (w_{ij} s_{ij}^2 + 2w_{ij} s_{ij}\tilde{M}_1(j)). \qquad (52)$$

This is not the same as eqns. (16) and (17), hence

$$\tilde{M}_2(i,1) \neq M_2(i) \qquad (53)$$

It has been shown that there is a class of non-analog transport games which have the same mean score but different mean square scores and hence different variances. The object of importance sampling (using non-analog probabilities) is to decrease the variance. The central limit theorem (see eqn.(1)) says that if M_2 can be reduced, while M_1 remains the same, the variance will be reduced.

IB-2   Splitting, Russian Roulette, and Weight Cutoff

In general, the more often a state is sampled the better the statistics associated with that state. In a given time only a certain number of particles can be followed; usually some states are under-sampled while other states are oversampled. Splitting refers to replacing a particle of weight w in state j by k particles of weight w/k, thus increasing the number of times state j is sampled while keeping the total weight in state j constant. On the other hand, if a state is being sampled more frequently than necessary, Russian Roulette is played in which a fraction, say $1-\alpha$, of the particles are killed off and the remaining α particles are followed with their weight increased by a factor $1/\alpha$. A Russian Roulette game cuts the number of times a state is sampled while keeping the mean weight followed in that state a constant.

A particle of very low weight is uninteresting since it cannot make much contribution to the score; yet following the particle takes time. To solve this problem a weight cutoff game is played in which a particle whose weight w is less than the weight cutoff w_{co} is either killed, or followed with a new weight w_n. The particle is killed with probability $1-w/w_n$ and followed (with weight w_n) with probability w/w_n (this is a type of Russian Roulette).

IC   Remarks on the Relationship of Chapter I to Chapter III

In the transport problems of Chapter III there is a continuum of states a particle can occupy. The transition from Chapter I to Chapter III is largely a transition from discrete states, probabilities, and scores to continuous states, probabilities, and scores. Thus, in eqn. (33) the discrete probabilities $\tilde{p}_i$ and $\tilde{p}_{ij}$ will be replaced by continuous probabilities and the sum over the discrete states will be replaced by an integral over the continuum of states. The Kronecker delta function (eqn. (6)) for discrete scores will be replaced by the Dirac delta function (eqn. (90)) for continuous scores. These delta functions serve similar purposes. Whereas the Kronecker delta function has the property (when f is a function of a discrete variable)

$$\sum_{s_i} f(s_i)\delta_{s_i s_j} = f(s_j),$$

the Dirac delta function has the property (when f is a function of a continuous variable)

$$\int f(s)\delta(s-s')ds = f(s').$$

There are five other differences between Chapter I and Chapter III. In Chapter III,

1) Particles are followed only for a finite time

2) Russian Roulette and splitting are included as events

3) A weight cutoff game is played as a particle departs an event

4) A particle may multiply (produce secondary particles as in fission)

5) The score for an event is sampled from a probability function.

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Chapter II. <u>Definitions</u>

This chapter contains the definitions of terms used to derive the equations of Chapter III for the expected errors in non-analog Monte Carlo calculations of time dependent particle transport problems. It should be emphasized that these terms are for a non-analog game with particle weights, biased kernels, Russian Roulette, and splitting. It is shown later, in Chapter III, that particular choices of weights and kernels exist that make the mean score the same as in an analog game.

<u>Definitions</u>

$$R = (\vec{r}, \vec{v}) = (\text{particle's position, particle's velocity}) \tag{54}$$

$$P = (\vec{r}, \vec{v}, w) = (\text{particle's position, particle's velocity,}$$
$$\text{particle's weight}) \tag{55}$$

$$P = (R, w) \tag{56}$$

P completely describes the state of a particle in this treatment.

* is the multiplication symbol $A*B = AB$

The probability that a particle contributes a score s in ds in the next event is taken to be:

$p_d(P,P',s)ds$ for a departure from P followed by a collisionless
free flight to P' $\tag{57}$

$p_0(P,P',s)ds$ for a departure from P followed directly by an
absorption at P' $\tag{58}$

$p_1(P,P',s)ds$ for a departure from P followed directly by a
scattering at P' $\tag{59}$

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$p_k(P,P',s)ds$ for a departure from P followed directly by a

collision at P' in which k particles emerge from the

collision   (60)

$p_{sk}(P,P',s)ds$ for a departure from P followed directly by a k for

one splitting event at P'   (61)

$p_\rho(P,P',s)ds$ for a departure from P followed directly by a game of

Russian Roulette at P'   (62)

The following weight multipliers for use in non-analog games are
defined:

$w_t(R,R^+)$ is the factor by which the particle's weight is multiplied

if the particle makes an eventless transit from R to R^+. (63)

$w_\sigma(R')$ is the factor by which the particle's weight is multiplied

if the particle has a collision at R'.   (64)

$w_{\varepsilon_1}(R',R'')$ is the factor by which the particle's weight is multiplied

if the particle enters a scattering event at R' and exits

at R".   (65)

$w_{\varepsilon_k}(R',R_1,\cdots,R_k)$ is the factor by which the k secondary particles

which exit a multiplying event have their weights multiplied.

                                                                                   (66)

w_{co} is the weight cutoff. No particle may exit a collision with

weight less than w_{co} (67)

w_e is the new weight assigned to a particle if the particle survives

the weight cutoff game.   (68)

The following transport kernels are defined:

Transmission Probability

$$T(P,P^+)d\vec{v}^+dw^+ = T(R,R^+)\delta(w^+-w_t w)d\vec{v}^+dw^+$$

$$= T(\vec{r},\vec{r}^+,\vec{v})\delta(\vec{v}^+-\vec{v})\delta(w^+-w_t w)d\vec{v}^+dw^+$$ is the probability that a

particle departing P will arrive at P^+, with velocity in $d\vec{v}^+$

and weight in dw^+, without undergoing any events. Note that

the velocity remains constant and the weight is multiplied by

w_t when the particle arrives at P^+. (69)

Collision Probability

$$\sigma(P^+,P')d\vec{r}^+dP' = \sigma(R^+)\delta(R'-R^+)\delta(w'-w_\sigma w^+)d\vec{r}^+dR'dw'$$ is the probability

that a particle entering $d\vec{r}^+$ about $\vec{r}^+$ will collide in $d\vec{r}^+$ and

change state to P' in dP'. Note that the only coordinate which

changes in this process is the particle's weight, which is

multiplied by w_σ. $\sigma(R^+)$ is the macroscopic total cross

section at R^+. (70)

Splitting Probability

$$S_k(P^+)d\vec{r}^+$$ is the probability that a particle (at $\vec{v}^+$ and w^+) entering

$d\vec{r}^+$ about $\vec{r}^+$ will undergo a k for 1 split. (71)

Russian Roulette Probability

$$R_0(P^+)d\vec{r}^+$$ is the probability that a particle (at $\vec{v}^+$ and w^+) entering

$d\vec{r}^+$ about $\vec{r}^+$ will play Russian Roulette. (72)

Collisionless Free Flight (Drift) Probability

$$D(P,P^+,t)dP^+ = T(P,P^+)\delta(|\vec{r}^+-\vec{r}|-vt)dP^+$$

$$= T(\vec{r},\vec{r}^+,\vec{v})\delta(\vec{v}^+-\vec{v})\delta(w^+-w_t w)\delta(|\vec{r}^+-\vec{r}|-vt)d\vec{r}^+d\vec{v}^+dw^+$$

is the probability that a particle departing P will have a

collisionless free flight (or "drift") for time t and end this

free flight in dP^+ about P^+. Note that the velocity remains

constant in free flight while the position changes by vt

and the particle's weight is multiplied by w_t. (73)

Absorption Probability

$A(P')$ is the probability that a collision at P' results in

absorption. (74)

Scattering Probability

$$E(P',P'')dP'' \equiv \varepsilon_1(P',P'')dP'' = E(R',R'')\delta(w''-w_{\varepsilon_1}w')dR''dw''$$

$$= E(\vec{v}',\vec{v}'')\delta(\vec{r}''-\vec{r}')\delta(w''-w_{\varepsilon_1}w')d\vec{r}''d\vec{v}''dw'' \text{ is the probability}$$

that a particle colliding at P' will scatter into dP'' about

P''. Note that the velocity, but not the position, changes

in a scattering event. The particle's weight is multiplied

by w_{ε_1} in this process. (75)

Multiplication Probability

$$\varepsilon_k(P',P_1,\cdots,P_k)dP_1\cdots dP_k = \varepsilon_k(R',R_1,\cdots,R_k)\delta(w_1-w_{\varepsilon_k}w')\cdots\delta(w_k-w_{\varepsilon_k}w')$$

$$\times dR_1dw_1\cdots dR_kdw_k$$

$$= \varepsilon_k(\vec{v}',\vec{v}_1,\cdots,\vec{v}_k)\delta(\vec{r}_1-\vec{r}')\cdots\delta(\vec{r}_k-\vec{r}')\delta(w_1-w_{\varepsilon_k}w')\cdots\delta(w_k-w_{\varepsilon_k}w')$$

$$\times d\vec{r}_1d\vec{v}_1dw_1\cdots d\vec{r}_kd\vec{v}_kdw_k$$

is the probability that a particle colliding at P' results

in k particles, the jth of which ($j=1,2,\cdots,k$) exits in dP_j

about P_j. These are multiplying events such as (n,2n) and

fission. As with the scattering kernel immediately preceding,

an exiting particle's weight is multiplied (here by w_{ε_k})

while the position is unchanged. (76)

Weight Change Probability for Splitting

$$B_{sk}(P^+,P'')dP''=\delta(\vec{r}''-\vec{r}^+)\delta(\vec{v}''-\vec{v}^+)\delta(w''-w^+/k)d\vec{r}''d\vec{v}''dw''$$

is the probability that a particle undergoing a k for l

split at P^+ results in k particles exiting at P" in dP".

Note that only the particle's weight, which is multiplied

by 1/k, changes in a splitting event.   (77)

Weight Change Probability for Russian Roulette

$$B_0(\alpha,P^+,P'')dP''d\alpha=\delta(\vec{r}''-\vec{r}^+)\delta(\vec{v}''-\vec{v}^+)\{\alpha\delta(w''-w^+/\alpha)+(1-\alpha)\delta(w'')\}$$

$$x\ d\vec{r}''d\vec{v}''dw''d\alpha$$

is the probability that a particle undergoing Russian

Roulette at P^+ (with probability α in $d\alpha$ of survival)

exits in dP" about P". Note that only the particle's

weight changes; the particle is either killed (i.e. w"=0)

or the particle's weight is multiplied by $1/\alpha$.   (78)

Weight Change Probability for Weight Cutoff

$$B(P'',P^C)dP^C=\delta(\vec{r}_c-\vec{r}'')\delta(\vec{v}_c-\vec{v}'')[H(w''-w_{co})\delta(w_c-w'') +$$

$$(1-H(w''-w_{co}))\{\frac{w''}{w_e}\delta(w_c-w_e)+(1-w''/w_e)\delta(w_c)\}]d\vec{r}_c d\vec{v}_c dw_c$$

is the probability that a particle exiting a collision

at P" will have its coordinates changed to P^C in dP^C

before the particle's next flight begins. This is the

weight cutoff kernel - note that only the particle's

weight changes. If the exit weight, w", is greater than

or equal to the cutoff weight, w_{co}, the particle continues

with weight w_c=w". If $w''<w_{co}$ then a game of Russian

Roulette is played in which the particle is either killed,

$w_c = 0$, or the particle is followed with weight $w_c = w_e$.   (79)

Next define the history score probability function and the rth

moment of the history score:

$\psi(P,s,t)ds$ is the probability that a particle at P will contribute

a cumulative score in ds about s in the remaining time t.   (80)

$M_r(P,t) = \int \psi(P,s,t)s^r ds$ is the rth moment of the history score.   (81)

Now define

$A_0(R') = A(P')$ is the probability zero particles exit a collision,

$A_1(R') = \int dR'' E(R',R'')$ is the probability one particle exits a

collision, and

$A_k(R') = \int dR_1 \cdots \int dR_k \varepsilon_k(R',R_1,\cdots,R_k)$ is the probability k

particles exit a collision.   (82)

An obvious requirement is:

$$\sum_{k=0}^{\infty} A_k(R') = 1$$ (83)

The score assigned to an event should be proportional to the particle's

weight; the probability that a particle of weight w' contributes a

score in ds about s is equal to the probability that a particle of

unit weight contributes a score s/w' in ds/w'. Thus

$p(P,R',w',s)ds = p(P,R',1,s/w')ds/w'$ and

$\int p(P,R',w',s)s^r ds = w'^r \int p(P,R',1,s)s^r ds$, or

$\int p(P,R',w_0 w,s)s^r ds = w_0^r \int p(P,R',w,s)s^r ds$.   (84)

This thesis also uses the notations

$$H(x) = \begin{cases} 0 \text{ if } x<0 \\ 1 \text{ if } x\geq 0 \end{cases} \qquad \text{Heaviside function} \qquad (85)$$

$$\delta(\vec{q}) = \begin{cases} 0 \text{ if } \vec{q} \neq 0 \\ \infty \text{ if } \vec{q} = 0 \end{cases} \qquad \text{Dirac delta "function"} \qquad (86)$$

$$\int \delta(\vec{q})d\vec{q}=1 \qquad (87)$$

$$\int_Q f(\vec{q})\delta(\vec{q}-\vec{q}')d\vec{q} = \begin{cases} f(\vec{q}') \text{ if } \vec{q}' \text{ is in the region } Q \\ 0 \text{ if } \vec{q}' \text{ is not in the region } Q \end{cases} \qquad (88)$$

* is the multiplication operator, A*B=AB.

Chapter III.   Derivation of Moment Equations and Choice of Weights

In this chapter equations for the rth moment, $M_r(P,t)$, of the history score are derived. The "central limit theorem"[1] states that the means of samples, each of h histories, are, for large h, approximately normally distributed about M_1 with a variance

$$VAR = (M_2 - M_1^2)/h \qquad (89)$$

Thus M_1 and M_2 determine the expected error in a Monte Carlo transport calculation.

A)   Derivation

The equation for M_r is obtained by deriving a probability conservation law (eq. (91)) and multiplying this law by the rth power of the history score, s^r, and integrating over all possible scores. Let $t^+ = t'$ be the time required for a free flight from P to P^+. (Since a collision process requires no time, t' may be interpreted as the time between departing P and a collision at P'.) Note that $\psi(P,s,t)ds$ is the probability of obtaining a score in ds about s in the time t remaining; a particle departing P is limited in its possible next event points to those states P^+ that can be reached from P in time $t^+ < t$. This is the reason for the $t^+ < t$ limitation on the integrations over P^+ in eq. (91). The first term in eq. (91) (see page 36) has a $t^+ = t$ restriction on the integral over P^+ since this term is due to a free flight for the entire time t; thus the states P^+ are limited to

those states that can be reached in <u>exactly</u> time t.

In order to obtain a history score s in ds a particle that obtains score s' in the next event must obtain a cumulative score in ds about s-s' in all subsequent events. If there are no subsequent events, as in TERM1 and TERM2 of eq. (91), then the entire history score must be contributed at the next event, i.e. s'=s in ds.

There are six possible ways a particle departing P can end its free flight:

1) run the clock out (free flight for the entire time t)
2) absorption at P'
3) play Russian Roulette at P^+
4) split at P^+
5) scatter at P'
6) multiply at P'

Each of the six terms in eq. (91) is a product of probabilities of the form

$$
\left(\begin{array}{c} \text{probability} \\ \text{event occurs} \end{array}\right) * \left(\begin{array}{c} \text{probability event} \\ \text{contributes} \\ \text{score s'} \end{array}\right) * \left(\begin{array}{c} \text{probability that all} \\ \text{subsequent events} \\ \text{contribute the remaining} \\ \text{score in ds about s-s'} \\ \text{in the remaining time t-t'} \end{array}\right)
$$

(90)

Each of these three factors will be listed for each of the six terms of eq. (91). After these factors have been listed a detailed explanation of eq. (91) will be given.

1) Free Flight for remaining time t

$$(D(P,P^+,t)dP^+) \times (p_d(P,P^+,s')ds') \times (\delta(s-s')ds)$$

2) Absorption (free flight to P^+, collision in dP^+, absorption at P')

$$(T(P,P^+)\sigma(P^+,P')dP^+dP'A(P')) \times (p_0(P,P's')ds') \times (\delta(s-s')ds)$$

3) Russian Roulette (free flight to P^+, Russian Roulette game at P^+, exit Russian Roulette game at P'')

$$(T(P,P^+)R_0(P^+)dP^+B_0(\alpha,P^+,P'')d\alpha dP'') \times (p_\rho(P,P^+,s')ds')$$

$$\times (\psi(P'',s-s',t-t')ds)$$

4) Splitting (free flight to P^+, split k for 1 at P^+, exit splitting at P'')

$$(T(P,P^+)S_k(P^+)dP^+B_{sk}(P^+,P'')dP'') \times (p_{sk}(P,P^+,s')ds')$$

$$\times (\psi(P'',s_1,t-t^+)ds_1 \cdots \psi(P'',s_{k-1},t-t^+)ds_{k-1}$$

$$\times \psi(P'',s-(s'+s_1+\cdots+s_{k-1}),t-t^+)ds)$$

5) Scattering (free flight to P^+, collision in dP^+, exit collision in dP'', Russian Roulette game at P'', exit Russian Roulette game in dP^C)

$$(T(P,P^+)\sigma(P^+,P')dP^+dP'E(P',P'')dP''B(P'',P^C) \times (p_1(P,P',s')ds')$$

$$\times (\psi(P^C,s-s',t-t')ds)$$

6) Multiplying (free flight to P^+, collision in dP^+, k particles exit collision in $dP_1,\cdots,dP_k$, Russian Roulette games at $P_1,\cdots,P_k$, exit Russian Roulette games in $dP_1^C,\cdots,dP_k^C$)

$$(T(P,P^+)\sigma(P^+,P')dP^+dP'\epsilon_k(P',P_1,\cdots,P_k)dP_1\cdots dP_kB(P_1,P_1^C)dP_1^C\cdots$$

$$\times B(P_k,P_k^C)dP_k^C) \times (p_k(P,P',s')ds') \times (\psi(P_1^C,s_1,t-t')ds_1\cdots$$

$$\times \psi(P_{k-1}^C,s_{k-1},t-t')ds_{k-1}\psi(P_k^C,s-(s'+s_1+\cdots+s_{k-1}),t-t')ds)$$

The score accumulation probability $\psi(P,s,t)ds$ is separated into the six terms given on the right hand side of eq. (91). These six terms are score accumulation probabilities for particles that end their first flights from P by:

1) a collisionless free flight for the remaining time t   (TERM1)
2) being absorbed at P'   (TERM2)
3) playing Russian Roulette at P^+ (TERM3)
4) splitting at P^+ (TERM4)
5) scattering at P'   (TERM5)
6) multiplying at P'   (TERM6)

In TERM1 $D(P,P^+,t)dP^+$ of the particles departing P arrive in dP^+ about P^+ in exactly time t without undergoing any events and $P_d(P,P^+,s')ds'$ of the $D(P,P^+,t)dP^+$ particles contribute a score s' in ds'; and $\delta(s'-s)ds$ of the $D(P,P^+,t)dP^+ P_d(P,P',s')ds'$ particles contribute a score in ds about s. The history score, s, for a particle departing P which undergoes no events in the entire time t consists only of the single score s' due to the free flight of the particle; thus the history score will be in ds about s if and only if s' is in ds about s, hence the delta function.

In TERM2 $T(P,P^+)$ of the particles departing P make a free flight to P^+. $\sigma(P^+,P')dP^+dP'$ of the $T(P,P^+)$ particles collide in dP^+ and have their state changed (only the weight coordinate changes) to P' in dP'. $A(P')$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'$ particles are absorbed at P'. $P_0(P,P',s')ds'$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'A(P')$ particles contribute a score s' in ds'. The history score, s, for a particle that is absorbed in its first event consists only of the single score s'

due to the absorption; thus the history score will be in ds about s if and only if s' is in ds about s. Hence $T(P,P^+)\sigma(P^+,P')dP^+dP'A(P')$ $\times \delta(s-s')ds$ of the particles departing P contribute a score in ds about s by being absorbed in the next event.

In TERM3 $T(P,P^+)$ of the particles departing P free flight to P^+. $R_0(P^+)dP^+$ of the $T(P,P^+)$ particles play Russian Roulette in dP^+. $B_0(\alpha,P^+,P'')d\alpha dP''$ of the $T(P,P^+)R_0(P^+)dP^+$ particles exit the Russian Roulette game in dP''. $p_\rho(P,P^+,s')ds'$ of the $T(P,P^+)R_0(P^+)dP^+B_0(\alpha,P^+,P'')$ $\times d\alpha dP''$ particles contribute a score s' in ds' at this event. $\psi(P'',s-s',t-t^+)ds$ of the $T(P,P^+)R_0(P^+)dP^+B_0(\alpha,P^+,P'')d\alpha dP''p_\rho(P,P^+,s')ds'$ particles subsequently contribute cumulative scores in ds about s-s' to make the history score in ds about $s'+(s-s')=s$. Thus $T(P,P^+)R_0(P^+)$ $\times dP^+B_0(\alpha,P^+,P'')d\alpha dP''p_\rho(P,P^+,s')ds'\psi(P'',s-s',t-t^+)ds$ of the particles departing P contribute a score in ds about s by having the next event be a Russian Roulette game.

In TERM4 $T(P,P^+)$ of the particles departing P free flight to P^+. $S_k(P^+)dP^+$ of the $T(P,P^+)$ particles undergo a k for 1 split in dP^+. $B_{sk}(P^+,P'')dP''$ collections of k particles exit the $T(P,P^+)S_k(P^+)dP^+$ splitting events (into dP''). $p_{sk}(P,P^+,s')ds'$ of the $T(P,P^+)S_k(P^+)dP^+$ $\times B_{sk}(P^+,P'')dP''$ splitting events contribute a score s' in ds'.

$$\int \psi(P'',s_1,t-t^+)ds_1 \cdots \int \psi(P'',s_{k-1},t-t^+)ds_{k-1}$$
$$\times \psi(P'',s-(s'+s_1+\cdots+s_{k-1}),t-t^+)ds$$

of the collections of k particles subsequently contribute the remaining

score, $s_1+s_2+\cdots+s_{k-1}+[s-(s'+s_1+\cdots+s_{k-1})]] = s-s'$, to make the history

score in ds about $s'+s_1+\cdots+s_{k-1}+[s-(s'+s_1+\cdots+s_{k-1})]=s$. Restating

this last idea, each of the $T(P,P^+)S_k(P^+)dP^+B_{sk}(P^+,P'')dP''p_{sk}(P,P^+,s')ds'$

collections of k particles exiting P" has already contributed a score s'

of the desired history score s; thus the k particles exiting P" must

collectively contribute a cumulative score in ds about s-s' in the

remaining time $t-t^+$. The probability that k particles departing P"

collectively contribute a cumulative score in ds about s-s' in time $t-t^+$

is

$$\int ds_1 \cdots \int ds_{k-1} \psi(P'',s_1,t-t^+)\cdots\psi(P'',s_{k-1},t-t^+)$$
$$\times\psi(P'',s-(s'+s_1+\cdots+s_{k-1}),t-t^+)ds$$

That is, if particles 1 through k-1 contribute scores s_1 through s_{k-1},

then the kth particle must contribute a score in ds about

$s-(s'+s_1+\cdots+s_{k-1})$ so that the score s', due to splitting, plus the

scores due to the k particles departing dP" results in a total score

in ds about s. The integrations over s_1 to s_{k-1} give all possible

ways that s_1 through s_k can sum to s-s'.

In TERMS $T(P,P^+)$ of the particles departing P free flight to P^+.

$\sigma(P^+,P')dP^+dP'$ of the $T(P,P^+)$ particles collide in dP^+ and change

state (only the weight coordinate changes) to P' in dP'.

$E(P',P'')dP''$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'$ particles exit the collision

in dP". $B(P'',P^C)dP^C$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'E(P',P'')dP''$

particles playing the weight cutoff game at P" exit the game in dP^C.

$p_1(P,P',s')ds'$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'E(P',P'')dP''B(P'',P^C)dP^C$

particles contribute a score s' (in ds') in the scattering event.

$\psi(P^C,s-s',t-t')ds$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'E(P',P'')dP''B(P'',P^C)dP^C$

x $p_1(P,P',s')ds'$ particles departing P^C subsequently contribute the

remaining score in ds about s-s' in the remaining time t-t' so that

the total history score is s'+(s-s')=s in ds.

In TERM6 $T(P,P^+)$ of the particles departing P free flight to P^+.

$\sigma(P^+,P')dP^+dP'$ of the $T(P,P^+)$ particles collide in dP^+ and change

state (only the weight coordinate changes) to P' in dP'.

$\varepsilon_k(P',P_1,\cdots,P_k)dP_1\cdots dP_k$ collections of k particles exit

(the $T(P,P^+)\sigma(P^+,P')dP^+dP'$ collisions in dP^+) in $dP_1,\cdots,dP_k$.

$B(P_1,P_1^C)dP_1^C\cdots B(P_k,P_k^C)dP_k^C$ of the $T(P,P^+)\sigma(P^+,P')dP^+dP'\varepsilon_k(P',P_1,\cdots,P_k)$

x $dP_1\cdots dP_k$ collections of k particles playing the weight cutoff

game at $P_1,\cdots,P_k$ exit the game in $dP_1^C,\cdots,dP_k^C$. $p_k(P,P',s')ds'$ of

the $T(P,P^+)\sigma(P^+,P')dP^+dP'\varepsilon_k(P',P_1,\cdots,P_k)dP_1\cdots dP_k$

x $B(P_1,P_1^C)dP_1^C\cdots B(P_k,P_k^C)dP_k^C$ collections of k particles departing

$P_1^C,\cdots,P_k^C$ contribute a score s' (in ds') in the multiplying event.

$$\int\psi(P_1^C,s_1,t-t')ds_1\cdots\int\psi(P_{k-1}^C,s_{k-1},t-t')ds_{k-1}$$

$$x\ \psi(P_k^C,s-(s'+s_1+\cdots+s_{k-1}),t-t')ds$$

of the collections of k particles contribute the remaining score

$s_1+s_2+\cdots+s_{k-1}+[s-(s'+s_1+\cdots+s_{k-1})]=s-s'$ in the remaining time t-t',

to make the history score in ds about $s'+s_1+\cdots+s_{k-1}+[(s-(s'+s_1+\cdots+s_{k-1}))]=s$.

Restating this last idea, each of the $T(P,P^+)\sigma(P^+,P')dP^+dP'p_k(P,P',s')ds'$

$x \ \varepsilon_k(P',P_1,\cdots,P_k)dP_1\cdots dP_k B(P_1,P_1^C)dP_1^C\cdots B(P_k,P_k^C)dP_k^C$ collections of k

particles exiting $P_1^C,\cdots,P_k^C$ has already contributed a score s' of the

desired history score s; thus the k particles exiting $P_1^C,\cdots,P_k^C$ must

collectively contribute a cumulative score in ds about s-s' in the

remaining time t-t'. The probability that k particles departing

$P_1^C,\cdots,P_k^C$ collectively contribute a cumulative score in ds about s-s'

in time t-t' is

$$\int \psi(P_1^C,s_1,t-t')ds_1 \cdots \int \psi(P_{k-1}^C,s_{k-1},t-t')ds_{k-1}$$

$$x \ \psi(P_k^C,s-(s'+s_1+\cdots+s_{k-1}),t-t')ds$$

That is, if particles 1 through k-1 contribute scores s_1 through s_{k-1},

then the kth particle must contribute a score in ds about

$s-(s'+s_1+\cdots+s_{k-1})$ so that the score s', due to multiplication, plus

the scores due to the k particles exiting $P_1^C,\cdots,P_k^C$ results in a total

score in ds about s. The integrations over s_1 to s_{k-1} give all possible

ways that s_1 through s_k can sum to s-s'.

Adding TERM1 through TERM6 results in the final equation for the

probability of obtaining score s in the time t remaining

$$\psi(P,s,t)ds = {}_{t^+=t}\!\!\int dP^+ D(P,P^+,t)\int ds' P_d(P,P^+,s')\delta(s-s')ds$$

$$+ \ {}_{t^+<t}\!\!\int dP^+ T(P,P^+)\int dP'\sigma(P^+,P')\int ds' P_0(P,P',s')A(P')\delta(s-s')ds$$

$$+ \ {}_{t^+<t}\!\!\int dP^+ T(P,P^+)R_0(P^+)\int ds' P_\rho(P,P^+,s')\int dP''\int_0^1 d\alpha B_0(\alpha,P^+,P'')$$

$$x \ \psi(P'',s-s',t-t^+)ds \ +\sum_{k=2}^{\infty} \ {}_{t^+<t}\!\!\int dP^+ T(P,P^+)S_k(P^+)\int ds' P_{sk}(P,P^+,s')$$

$$x \int dP'' B_{sk}(P^+,P'')\int ds_1 \cdots \int ds_{k-1}\psi(P'',s_1,t-t^+)\cdots\psi(P'',s_{k-1},t-t^+)$$

$$x \; \psi(P'',s-(s'+s_1+\cdots+s_{k-1}),t-t^+)ds + {}_{t^+<t}\!\int dP^+ T(P,P^+) \int dP' \sigma(P^+,P')$$

$$x \int ds' p_1(P,P',s') \int dP'' E(P',P'') \int dP^C B(P'',P^C) \psi(P^C,s-s',t-t')ds$$

$$+ \sum_{k=2}^{\infty} {}_{t^+<t}\!\int dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int ds' p_k(P,P',s')$$

$$x \int dP_1 \cdots \int dP_k \epsilon_k(P',P_1,\cdots,P_k) \int dP_1^C B(P_1,P_1^C) \cdots \int dP_k^C B(P_k,P_k^C) \int ds_1 \cdots \int ds_{k-1}$$

$$x \; \psi(P_1^C,s_1,t-t') \cdots \psi(P_{k-1}^C,s_{k-1},t-t') \psi(P_k^C,s-(s'+s_1+\cdots+s_{k-1}),t-t')ds \quad (91)$$

This equation is of the form

$$\text{TERM0} = \sum_{I=1}^{6} \text{TERMI}$$

Multiplying eq. (91) by s^r and integrating over s results in the

desired equation for M_r. The integrations are done term by term below.

$$\text{TERM0} = \int ds \psi(P,s,t) s^r = M_r(P,t) \tag{92}$$

$$\text{TERM1} = {}_{t^+=t}\!\int dP^+ D(P,P^+,t) \int ds p_d(P,P^+,s) s^r \tag{93}$$

$$\text{TERM2} = {}_{t^+<t}\!\int dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int ds p_0(P,P',s) A(P') s^r \tag{94}$$

$$\text{TERM3} = {}_{t^+<t}\!\int dP^+ T(P,P^+) R_0(P^+) \int ds' p_\rho(P,P^+,s') \int dP'' \int_0^1 d\alpha B_0(\alpha,P^+,P'')$$

$$x \int ds \psi(P'',s-s',t-t^+) s^r \tag{95}$$

Using eq. (A5)

$$\text{TERM3} = {}_{t^+<t}\!\int dP^+ T(P,P^+) R_0(P^+) \int ds' p_\rho(P,P^+,s') \int dP'' \int_0^1 d\alpha B_0(\alpha,P^+,P'')$$

$$x \sum_{n=0}^{r} \binom{r}{n} s'^{r-n} M_n(P'',t-t^+) \tag{96}$$

$$\text{TERM4} = \sum_{k=2}^{\infty} \int_{t^+ < t} dP^+ T(P,P^+) S_k(P^+) \int ds' p_{sk}(P,P^+,s') \int dP'' B_{sk}(P^+,P'')$$

$$\times \int ds_1 \cdots \int ds_{k-1} \psi(P'',s_1,t-t^+) \cdots \psi(P'',s_{k-1},t-t^+) \int ds\ s^r$$

$$\times \psi(P'',s-(s'+s_1+\cdots+s_{k-1}),t-t^+) \tag{97}$$

Using eq. (A5) on $\int ds$

$$\text{TERM4} = \sum_{k=2}^{\infty} \int_{t^+ < t} dP^+ T(P,P^+) S_k(P^+) \int ds' p_{sk}(P,P^+,s') \int dP'' B_{sk}(P^+,P'')$$

$$\times \int ds_1 \cdots \int ds_{k-1} \psi(P'',s_1,t-t^+) \cdots \psi(P'',s_{k-1},t-t^+)$$

$$\times \sum_{n_1=0}^{r} \binom{r}{n_1} (s'+s_1+\cdots+s_{k-1})^{r-n_1} M_{n_1}(P'',t-t^+) \tag{98}$$

Using eq. (A5) on $\int ds_{k-1}$

$$\text{TERM4} = \sum_{k=2}^{\infty} \int_{t^+ < t} dP^+ T(P,P^+) S_k(P^+) \int ds' p_{sk}(P,P^+,s') \int dP'' B_{sk}(P^+,P'')$$

$$\times \int ds_1 \cdots \int ds_{k-2} \psi(P'',s_1,t-t^+) \cdots \psi(P'',s_{k-2},t-t^+) \sum_{n_1=0}^{r} \binom{r}{n_1} M_{n_1}(P'',t-t^+)$$

$$\times \sum_{n_2=0}^{r-n_1} \binom{r-n_1}{n_2} M_{n_2}(P'',t-t^+)(s'+s_1+\cdots+s_{k-2})^{r-n_1-n_2} \tag{99}$$

Using eq. (A5) repeatedly on $\int ds_{k-2} \cdots \int ds_1$

$$\text{TERM4} = \sum_{k=2}^{\infty} \int_{t^+ < t} dP^+ T(P,P^+) S_k(P^+) \int ds' p_{sk}(P,P^+,s') \int dP'' B_{sk}(P^+,P'')$$

$$\times \sum_{n_1=0}^{r} \binom{r}{n_1} M_{n_1}(P'',t-t^+) \cdots \sum_{n_k=0}^{r-n_1-\cdots-n_{k-1}} \binom{r-n_1-\cdots-n_{k-1}}{n_k} M_{n_k}(P'',t-t^+)$$

$$\times s'^{r-n_1-\cdots-n_k} \tag{100}$$

$$\text{TERM5} = \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP'\sigma(P^+,P') \int ds' p_1(P,P',s') \int dP''E(P',P'')$$

$$x\int dP^C B(P'',P^C) \int ds\psi(P^C,s-s',t-t')s^r \qquad (101)$$

Using eq. (A5)

$$\text{TERM5} = \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP'\sigma(P^+,P') \int ds' p_1(P,P',s') \int dP''E(P',P'')$$

$$x\int dP^C B(P'',P^C) \sum_{n=0}^{r} \binom{r}{n} M_n(P^C,t-t')s'^{r-n} \qquad (102)$$

$$\text{TERM6} = \sum_{k=2}^{\infty} \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP'\sigma(P^+,P') \int ds' p_k(P,P',s')$$

$$x\int dP_1\cdots\int dP_k \varepsilon_k(P',P_1,\cdots,P_k) \int dP_1^C B(P_1,P_1^C)\cdots\int dP_k^C B(P_k,P_k^C)$$

$$x\int ds_1\cdots\int ds_{k-1}\psi(P_1^C,s_1,t-t')\cdots\psi(P_{k-1}^C,s_{k-1},t-t')\int ds$$

$$x\ \psi(P_k^C,s-(s'+s_1+\cdots+s_{k-1}),t-t')s^r \qquad (103)$$

Using eq. (A5) repeatedly as in TERM4

$$\text{TERM6} = \sum_{k=2}^{\infty} \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP'\sigma(P^+,P') \int ds' p_k(P,P',s') \int dP_1\cdots\int dP_k$$

$$x\ \varepsilon_k(P',P_1,\cdots,P_k) \int dP_1^C B(P_1,P_1^C)\cdots\int dP_k^C B(P_k,P_k^C)$$

$$x\ \sum_{n_1=0}^{r}\binom{r}{n_1} M_{n_1}(P_k^C,t-t')\sum_{n_2=0}^{r-n_1}\binom{r-n_1}{n_2} M_{n_2}(P_{k-1}^C,t-t')\cdots$$

$$x\ \sum_{n_k=0}^{r-n_1-\cdots-n_{k-1}}\binom{r-n_1-\cdots-n_{k-1}}{n_k} M_{n_k}(P_1^C,t-t')s'^{r-n_1-\cdots-n_k} \qquad (104)$$

Changing the dummy variable s' to the dummy variable s and adding

$$\text{TERM0} = \sum_{I=1}^{6} \text{TERMI}$$

gives the equation for the rth moment

$$M_r(P,t) = \int_{t'=t}^{+} dP^+ D(P,P^+,t) \int ds p_d(P,P^+,s) s^r$$

$$+ \int_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int ds p_0(P,P',s) A(P') s^r$$

$$+ \int_{t^+<t} dP^+ T(P,P^+) R_0(P^+) \int ds p_\rho(P,P^+,s) \int dP'' \int_0^1 d\alpha \ell_0(\alpha,P^+,P'')$$

$$\times \sum_{n=0}^r \binom{r}{n} M_n(P'',t-t^+) s^{r-n} + \sum_{k=2}^\infty \int_{t^+<t} dP^+ T(P,P^+) S_k(P^+) \int ds p_{sk}(P,P^+,s)$$

$$\times \int dP'' B_{sk}(P^+,P'') \sum_{n_1=0}^r \binom{r}{n_1} M_{n_1}(P'',t-t^+) \cdots \sum_{n_k=0}^{r-n_1-\cdots-n_{k-1}} \binom{r-n_1-\cdots-n_{k-1}}{n_k}$$

$$\times M_{n_k}(P'',t-t^+) s^{r-n_1-\cdots-n_k} + \int_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int ds p_1(P,P',s)$$

$$\times \int dP'' E(P',P'') \int dP^C B(P'',P^C) \sum_{n=0}^r \binom{r}{n} M_n(P^C,t-t') s^{r-n} + \sum_{k=2}^\infty \int_{t^+<t} dP^+$$

$$\times T(P,P^+) \int dP' \sigma(P^+,P') \int ds p_k(P,P',s) \int dP_1 \cdots \int dP_k \varepsilon_k(P',P_1,\cdots,P_k)$$

$$\times \int dP_1^C B(P_1,P_1^C) \cdots \int dP_k^C B(P_k,P_k^C) \sum_{n_1=0}^r \binom{r}{n_1} M_{n_1}(P_k^C,t-t') \cdots$$

$$\times \sum_{n_k=0}^{r-n_1-\cdots-n_{k-1}} \binom{r-n_1-\cdots-n_{k-1}}{n_k} M_{n_k}(P_1^C,t-t') s^{r-n_1-\cdots-n_k} \tag{105}$$

Recall $p(P,P',s)$ and $\psi(P,s,t)$ are probability functions so that

$$\int p(P,P',s) s^0 ds = 1 \tag{106}$$

$$M_0(P,t) = \int \psi(P,s,t) s^0 ds = 1 \tag{107}$$

Note in TERM4 and TERM6 that any one of the n_j $(j=1,\ldots,k)$ could be equal to r and hence there are k ways to get M_r from these terms. Collecting the M_r terms

$$M_r(P,t) = \int\limits_{t^+<t} dP^+ T(P,P^+) R_0(P^+) \int dP'' \int_0^1 d\alpha B_0(\alpha,P^+,P'') M_r(P'',t-t^+)$$

$$+ \sum_{k=2}^{\infty} \int\limits_{t^+<t} dP^+ T(P,P^+) S_k(P^+) \int dP'' B_{sk}(P^+,P'') k M_r(P'',t-t^+)$$

$$+ \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int dP'' E(P',P'') \int dP^C B(P'',P^C) M_r(P^C,t-t')$$

$$+ \sum_{k=2}^{\infty} \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int dP_1 \cdots \int dP_k \epsilon_k(P',P_1,\cdots,P_k)$$

$$\times \int dP_1^C B(P_1,P_1^C) \cdots \int dP_k^C B(P_k,P_k^C) \sum_{j=1}^k M_r(P_j^C,t-t') + Q_r(P,t) \qquad (108)$$

where

$$Q_r(P,t) = \int\limits_{t^+=t} dP^+ D(P,P^+,t) \int ds p_d(P,P^+,s) s^r + \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P')$$

$$\times \int ds p_0(P,P',s) A(P') s^r + \int\limits_{t^+<t} dP^+ T(P,P^+) R_0(P^+) \int ds p_\rho(P,P^+,s)$$

$$\times \int dP'' \int_0^1 d\alpha B_0(\alpha,P^+,P'') \sum_{n=0}^{r-1} \binom{r}{n} M_n(P'',t-t') s^{r-n} + \sum_{k=2}^{\infty} \int\limits_{t^+<t} dP^+ T(P,P^+) S_k(P^+)$$

$$\times \int ds p_{sk}(P,P^+,s) \int dP'' B_{sk}(P^+,P'') \sum_{n_1=0}^r \binom{r}{n_1} M_{n_1}(P'',t-t^+) \cdots$$

$$\times \sum_{n_k=0}^{r-n_1-\cdots-n_{k-1}} \binom{r-n_1-\cdots-n_{k-1}}{n_k} M_{n_k}(P'',t-t^+) s^{r-n_1-\cdots-n_k} \prod_{i=1}^k (1-\delta_{n_i r})$$

$$+ \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P') \int ds p_1(P,P',s) \int dP'' E(P',P'') \int dP^C B(P'',P^C)$$

$$\times \sum_{n=0}^{r-1} \binom{r}{n} M_n(P^C,t-t') s^{r-n} + \sum_{k=2}^{\infty} \int\limits_{t^+<t} dP^+ T(P,P^+) \int dP' \sigma(P^+,P')$$

$$\times \int ds p_k(P,P',s) \int dP_1 \cdots \int dP_k \epsilon_k(P',P_1,\cdots,P_k) \int dP_1^C B(P_1,P_1^C) \cdots \int dP_k^C B(P_k,P_k^C)$$

$$\times \sum_{n_1=0}^{r} \binom{r}{n_1} M_{n_1}(P_k^c, t-t') \sum_{n_2=0}^{r-n_1} \binom{r-n_1}{n_2} M_{n_2}(P_{k-1}^c, t-t') \cdots$$

$$\times \sum_{n_k=0}^{r-n_1-\cdots-n_{k-1}} \binom{r-n_1-\cdots-n_{k-1}}{n_k} M_{n_k}(P_1^c, t-t') s^{r-n_1-\cdots-n_k} \prod_{i=1}^{k}(1-\delta_{n_i r}) \quad (109)$$

The first two moments are usually the only ones desired. The source term Q_1 for the M_1 equation and the source term Q_2 for the M_2 equation are given below. From eq. (109)

$$Q_1(P,t) = \int_{t'=t}^{t^+} dP^+ D(P,P^+,t) \int ds p_d(P,P^+,s) s + \int_{t'<t}^{t^+} dP^+ T(P,P^+) \int dP' \sigma(P^+,P')$$

$$\times \int ds p_0(P,P',s) A(P') s + \sum_{k=2}^{\infty} \int_{t'<t}^{t^+} dP^+ T(P,P^+) S_k(P^+) \int ds p_{sk}(P,P^+,s) s$$

$$+ \int_{t'<t}^{t^+} dP^+ T(P,P^+) R_0(P^+) \int ds p_\rho(P,P^+,s) s + \int_{t'<t}^{t^+} dP^+ T(P,P^+) \int dP' \sigma(P^+,P')$$

$$\times \int ds p_1(P,P',s) \int dP'' E(P',P'') s + \sum_{k=2}^{\infty} \int_{t'<t}^{t^+} dP^+ T(P,P^+) \int dP' \sigma(P^+,P')$$

$$\times \int ds p_k(P,P',s) \int dP_1 \cdots \int dP_k \varepsilon_k(P',P_1,\cdots,P_k) s \quad (110)$$

For r=2 the splitting and multiplication terms in eq. (109) have a number of sums which must be evaluated. First consider the sum in the multiplication term in eq. (109) for r=2

$$X_m = \sum_{n_1=0}^{2} \binom{2}{n_1} M_{n_1}(P_k^c, t-t') \sum_{n_2=0}^{2-n_1} \binom{2-n_1}{n_2} M_{n_2}(P_{k-1}^c, t-t') \cdots$$

$$\times \sum_{n_k=0}^{2-n_1-\cdots-n_{k-1}} \binom{2-n_1-\cdots-n_{k-1}}{n_k} M_{n_k}(P_1^c, t-t') s^{r-n_1-\cdots-n_k} \prod_{i=1}^{k}(1-\delta_{n_i 2}) \quad (111)$$

Note by the summation limits that

$$0 \leq n_1 + n_2 + \cdots + n_k \leq 2 \tag{111A}$$

and also that the $(1-\delta_{n_i 2})$ factor requires $n_i \neq 2$ so that

$$0 \leq n_i \leq 1 \tag{111B}$$

for $1 \leq i \leq k$. The terms in the sum (111) are separated below into $k+1$ mutually exclusive groups of terms which are evaluated and then summed to produce a simpler expression for χ_m. The jth group $(j = 1,2,\cdots,k)$ of terms consists of all terms for which $n_1 = n_2 = \cdots = n_{j-1} = 0$ and $n_j = 1$. The $k+1^{\underline{st}}$ group of terms consists of the single term for which $n_1 = n_2 = \cdots = n_k = 0$. These groups are evaluated in the following paragraph.

If $n_1 = 1$ then by eq. (111A) either one or zero other $n_i = 1$ $(i=2,3,\cdots,k)$, yielding

$$\binom{2}{1} M_1(P_k^C, t-t') \left[\sum_{j=1}^{k-1} M_1(P_j^C, t-t') + s \right] \tag{111C}$$

If $n_1 = 0$ and $n_2 = 1$ then either one or zero other $n_i = 1$ $(i=3,4,\cdots,k)$

$$\binom{2}{1} M_1(P_{k-1}^C, t-t') \left[\sum_{j=1}^{k-2} M_1(P_j^C, t-t') + s \right] \tag{112}$$

If $n_1 = n_2 = 0$ and $n_3 = 1$ then either one or zero other $n_i = 1$ $(i=4,5,\cdots,k)$

$$\binom{2}{1} M_1(P_{k-2}^C, t-t') \left[\sum_{j=1}^{k-3} M_1(P_j^C, t-t') + s \right] \tag{113}$$

$$\vdots$$

If $n_1 = n_2 = \cdots = n_{k-2} = 0$ and $n_{k-1} = 1$ then either one or zero other $n_i = 1$ $(i=k)$

$$\binom{2}{1} M_1(P_2^C, t-t') \left[\sum_{j=1}^{1} M_1(P_j^C, t-t') + s \right] \qquad (114)$$

If $n_1 = \cdots = n_{k-1} = 0$ and $n_k = 1$

$$\binom{2}{1} M_1(P_1^C, t-t') s \qquad (115)$$

If $n_1 = \cdots = n_k = 0$

$$s^2 \qquad (116)$$

Hence adding eqs. (111C)-(116)

$$\chi_m = s^2 + 2s \sum_{\ell=1}^{k} M_1(P_\ell^C, t-t') + 2 \sum_{\ell=2}^{k} M_1(P_\ell^C, t-t')$$

$$\times \sum_{j=1}^{\ell-1} M_1(P_j^C, t-t') \qquad (117)$$

Note that the term in eq. (109) involving the k particles emerging from a splitting event has the same form as the term involving k particles emerging from a collision except that in a splitting event all particles exit at P". Thus using eq. (117) with $P_j^C = P''$

$$\chi_s = s^2 + 2 \, skM_1(P'', t-t^+) + 2M_1^2(P'', t-t^+) \sum_{\ell=2}^{k} \sum_{j=1}^{\ell-1} 1 \qquad (118)$$

$$\sum_{\ell=2}^{k} \sum_{j=1}^{\ell-1} 1 = \sum_{\ell=2}^{k} (\ell-1) = 1+2+\cdots+k-1 = \frac{k(k-1)}{2} \qquad (119)$$

$$X_s = s^2 + 2skM_1(P'',t-t^+) + k(k-1)M_1^2(P'',t-t^+) \tag{120}$$

From eqs. (109),(117), and (120)

$$Q_2(P,t) = \int_{t^+=t} dP^+ D(P,P^+,t) \int ds p_d(P,P^+,s)s^2 + \int_{t^+<t} dP^+ T(P,P^+)$$

$$x\int dP'\sigma(P^+,P')\int ds p_0(P,P',s)A(P')s^2 + \int_{t^+<t} dP^+ T(P,P^+)R_0(P^+)$$

$$x\int ds p_\rho(P,P^+,s)\int dP'' \int_0^1 d\alpha B_0(\alpha,P^+,P'') \sum_{n=0}^1 \binom{2}{n}M_n(P'',t-t')s^{2-n}$$

$$+ \sum_{k=2}^\infty \int_{t^+<t} dP^+ T(P,P^+)S_k(P^+)\int ds p_{sk}(P,P^+,s)\int\!\!\int dP''B_{sk}(P^+,P'')\Big\{s^2 +$$

$$2skM_1(P'',t-t^+) + k(k-1)M_1^2(P'',t-t^+)\Big\} + \int_{t^+<t} dP^+ T(P,P^+)$$

$$x\int dP'\sigma(P^+,P')\int ds p_1(P,P',s)\int dP''E(P',P'')\int dP^C B(P'',P^C) \sum_{n=0}^1 \binom{2}{n}s^{2-n}$$

$$x\ M_n(P^C,t-t') + \sum_{k=2}^\infty \int_{t^+<t} dP^+ T(P,P^+)\int dP'\sigma(P^+,P')\int ds p_k(P,P',s)$$

$$x\int dP_1\cdots\int dP_k \varepsilon_k(P',P_1,\cdots,P_k)\int dP_1^C B(P_1,P_1^C)\cdots\int dP_k^C B(P_k,P_k^C)$$

$$x\Big\{s^2 + 2s\sum_{\ell=1}^k M_1(P_\ell^C,t-t') + 2\sum_{\ell=2}^k M_1(P_\ell^C,t-t')\sum_{j=1}^{\ell-1} M_1(P_j^C,t-t')\Big\} \tag{121}$$

B) Choice of Non-Analog Kernels and Weight Multipliers.

By using non-analog kernels one tries to obtain the same M_1 as in the analog game but with a smaller variance. If there are no Russian Roulette or splitting events, i.e.

$$R_0(P^+) = S_k(P^+) = 0$$

then it is easy, if somewhat tedious, to show how to pick kernels and weight multipliers which result in a game with the same M_1 as in the analog game. Substituting the definitions of Chapter II into eq. (108) results in

$$
\begin{aligned}
M_r(R,w,t) &= \int_{t^+ < t} dR^+ dw^+ T(R,R^+)\delta(w^+ - w_t w)\int dR' dw' \sigma(R') \\
&\times \delta(R'-R^+)\delta(w'-w_\sigma w^+)\int dR'' dw'' E(R',R'')\delta(w''-w_{\varepsilon_1}w')\int dR^C dw_C \\
&\times \delta(R^C - R'')\left\{ H(w''-w_{co})\delta(w_c - w'') + \left(1 - H(w''-w_{co})\right)\right. \\
&\times \left. \left[\frac{w''}{w_e}\delta(w_c-w_e) + \left(1-\frac{w''}{w_e}\right)\delta(w_c)\right]\right\} M_r(P^C, t-t') \\
&+ \sum_{k=2}^{\infty} \int_{t^+ < t} dR^+ dw^+ T(R,R^+)\delta(w^+-w_t w)\int dR' dw' \sigma(R')\delta(R'-R^+) \\
&\times \delta(w'-w_\sigma w^+)\int dR_1 dw_1 \cdots \int dR_k dw_k \varepsilon_k(R',R_1,\cdots,R_k)\delta(w_1-w_{\varepsilon_k}w')\cdots \\
&\times \delta(w_k - w_{\varepsilon_k}w') \prod_{i=1}^{k}\int dR_i^C dw_{c_i}\,\delta(R_i^C-R_i)\left\{ H(w_i-w_{co})\delta(w_{c_i}-w_i)\right. \\
&+ \left. \left(1-H(w_i-w_{co})\right)\left[\frac{w_i}{w_e}\delta(w_{c_i}-w_e) + \left(1-\frac{w_i}{w_e}\right)\delta(w_{c_i})\right]\right\} \\
&\times \sum_{j=1}^{k} M_r(R_j^C, w_{c_j}, t-t') + Q_r(P,t)
\end{aligned}
\tag{122}
$$

Performing the integrations over δ functions and noting:

1) $w_i = w_{\epsilon_k} w'$ for all i, thus either $w_i \geq w_{co}$ or $w_i < w_{co}$ for all i $\quad$ (123)

2) $M_r(R.0,t) = 0$ since a particle of weight zero can contribute

   nothing to the score, hence $M_r(R,w\delta_{n,1},t) = \delta_{n,1} M_r(R,w,t)$ $\quad$ (124)

3) t^+ changes to t' when the integration of $\delta(R'-R^+)$ makes

   $R' = R^+$ $\quad$ (124A)

one arrives at

$$M_r(R,w,t) = \int_{t'<t} dR'T(R,R')\sigma(R')\int dR''E(R',R'')\left\{ H(ww_t w_\sigma w_{\epsilon_1} - w_{co}) \right.$$

$$\times M_r(R'',ww_t w_\sigma w_{\epsilon_1},t-t') + \left(1-H(ww_t w_\sigma w_{\epsilon_1} - w_{co})\right)\frac{ww_t w_\sigma w_{\epsilon_1}}{w_e}$$

$$\times M_r(R'',w_e,t-t') \right\} + \sum_{k=2}^{\infty} \int_{t'<t} dR'T(R,R')\sigma(R')\int dR_1 \cdots \int dR_k$$

$$\times \epsilon_k(R',R_1,\cdots,R_k)\left\{ H(ww_t w_\sigma w_{\epsilon_k} - w_{co})\sum_{j=1}^{k} M_r(R_j,ww_t w_\sigma w_{\epsilon_k},t-t') \right.$$

$$+ \left(1-H(ww_t w_\sigma w_{\epsilon_k} - w_{co})\right)\sum_{b_1=0}^{1}\cdots\sum_{b_k=0}^{1}\left(\frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\right)^{b_1+\cdots+b_k}\left(1-\frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\right)^{k-b_1-\cdots-b_k}$$

$$\times \sum_{j=1}^{k} M_r(R_j,w_e,t-t')\delta_{b_j,1} \right\} + Q_r(R,w,t) \qquad (124B)$$

The sums over the b_ℓ's give all possible combinations for the

survivals or deaths of the k particles playing the weight cutoff

game. Survival of the ℓth particle corresponds to $b_\ell = 1$ and death

corresponds to $b_\ell = 0$. The $\delta_{b_j,1}$ in eq. (124B) allows only surviving

particles to contribute.

Using eqs. (A16) and (A18) one arrives at

$$
M_r(R,w,t) = \int\limits_{t'<t} dR'\,T(R,R')\sigma(R')\int dR''E(R',R'') \left\{ H(ww_t{}^w\sigma^w\varepsilon_1 - w_{co}) \right.
$$

$$
\times\, M_r(R'',ww_t{}^w\sigma^w\varepsilon_1,t-t') + \left(1-H(ww_t{}^w\sigma^w\varepsilon_1 - w_{co})\right) \frac{ww_t{}^w\sigma^w\varepsilon_1}{w_e}
$$

$$
\times\, M_r(R'',w_e,t-t') \Big\} + \sum_{k=2}^{\infty}\int\limits_{t'<t} dR'\,T(R,R')\sigma(R')\int dR_1\cdots\int dR_k
$$

$$
\times\, \varepsilon_k(R',R_1,\ldots,R_k)\Big\{ H(ww_t{}^w\sigma^w\varepsilon_k - w_{co})\sum_{j=1}^{k} M_r(R_j,ww_t{}^w\sigma^w\varepsilon_k,t-t')
$$

$$
+ \frac{ww_t{}^w\sigma^w\varepsilon_k}{w_e}\left(1-H(ww_t{}^w\sigma^w\varepsilon_k - w_{co})\right)\sum_{j=1}^{k} M_r(R_j,w_e,t-t') \Big\}
$$

$$
+\, Q_r(R,w,t) \tag{125}
$$

Eq. (125) is of the form

$$
M_r(R,w,t) = KM_r(R,w,t) + Q_r(R,w,t) \tag{125A}
$$

where the form of the operator K can be inferred from eq. (125).

A Neumann sequence may be generated using the iterative procedure (index i)

$$
M_r^0(R,w,t) = 0 \tag{125B}
$$

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and

$$M_r^{i+1}(R,w,t) = KM_r^i(R,w,t) + Q_r(R,w,t).$$ (125C)

With suitable restrictions on K (see Appendix C) the Neumann

sequence converges to a unique solution. From here on, it will be

assumed K is such that there exists a unique solution to the M_r equation.

Now attempt to separate the weight dependence of M_r. Replace w by

aw in eq. (125) and then substitute

$$M_r(R,aw,t) = a^r F_r(R,w,t) = a^r w^r F(R,1,t)$$ (125D)

into eq. (125), divide the result by a^r, and note that for $a>0$

$H(ax-b)$ equals $H(x-b/a)$, this yields

$$F_r(R,w,t) = \int_{t'<t} dR'T(R,R')\sigma(R')\int dR''E(R',R'') \left\{ H(ww_t w_\sigma w_{\epsilon_1} - w_{co}/a) \right.$$

$$\times F_r(R'',ww_t w_\sigma w_{\epsilon_1},t-t') + \left(1-H(ww_t w_\sigma w_{\epsilon_1} - w_{co}/a)\right) \frac{ww_t w_\sigma w_{\epsilon_1}}{w_e/a}$$

$$\times F_r(R'',\frac{w_e}{a},t-t') \left.\right\} + \sum_{k=2}^{\infty} \int_{t'<t} dR'T(R,R')\sigma(R')\int dR_1 \cdots \int dR_k$$

$$\times \epsilon_k(R',R_1,\cdots,R_k) \left\{ H(ww_t w_\sigma w_{\epsilon_1} - w_{co}/a) \sum_{j=1}^{k} F_r(R_j,ww_t w_\sigma w_{\epsilon_k},t-t') \right.$$

$$+ \frac{ww_t w_\sigma w_{\epsilon_k}}{w_e/a} \left(1-H(ww_t w_\sigma w_{c_1} - w_{co}/a)\right) \sum_{j=1}^{k} F_r(R_j,\frac{w_e}{a},t-t') \left.\right\}$$

$$+ Q_r(R,aw,t)a^{-r}$$ (125E)

If w_e and w_{co} are proportional to the particle weight, that is:

$$w_e(aw) = aw_e(w)$$ (125F)

$$w_{co}(aw) = aw_{co}(w)$$ (125G)

and if Q_r is proportional to the $r\underline{th}$ power of the particle weight, that is

$$Q_r(R,aw,t) = a^r Q_r(R,w,t) \tag{125H}$$

then eq. (125B) for F_r is independent of "a" and identical to eq. (125) for M_r. Thus if eqs. (125F) - (125H) hold the separation works and F_r equals M_r, which implies

$$M_r(R,w,t) = w^r M_r(R,1,t). \tag{125I}$$

(Conditions (125F) and (125G) make the particle's history independent of its weight as in Chapter 1)

When r is equal to one, conditions (125F) and (125G) are not required. To see this, use eqs. (125D) and (125H) to set every weight argument in eq. (125E) to w and then use $H(x) + (1-H(x)) = 1$ to obtain

$$F_1(R,w,t) = \int_{t'<t} dR'T(R,R')\sigma(R')\!\!\int dR''E(R',R'')w_t w_\sigma w_{\epsilon_1} F_1(R,w,t-t')$$

$$+ \sum_{k=2}^{\infty} \int_{t'<t} dR'T(R,R')\sigma(R')\!\!\int dR_1 \cdots \int dR_k \epsilon_k(R',R_1,\cdots,R_k)w_t w_\sigma w_{\epsilon_k}$$

$$\times \sum_{j=1}^{k} F_1(R_j,w,t-t') + Q_1(R,w,t). \tag{125J}$$

Note that F_1 is independent of w_e and w_{co} so eq. (125H) is the only requirement for

$$M_1(R,w,t) = w M_1(R,1,t) \tag{125K}$$

From eq. (110) and the definitions in Chapter II

$$Q_1(R,w,t) = \underset{t^+=t}{+\int} dR^+ dw^+ T(R,R^+) \delta(w^+ - ww_t) \delta(|\vec{r}^{+} - \vec{r}| - vt)$$

$$\times \int dsp_d(P,R^+,w^+,s)s + \underset{t^+<t}{+\int} dR^+ dw^+ T(R,R^+) \delta(w^+ - ww_t)$$

$$\times \int dR' dw' \sigma(R') \delta(R'-R^+) \delta(w'-w_\sigma w^+) \int dsp_0(P,R',w',s) A(R')s$$

$$+ \underset{t^+<t}{+\int} dR^+ dw^+ T(R,R^+) \delta(w^+-ww_t) \int dR' dw' \sigma(R') \delta(R'-R^+) \delta(w'-w^+ w_\sigma)$$

$$\times \int dsp_1(P,R',w',s)s \int dR'' dw'' E(R',R'') \delta(w''-w_{\varepsilon_1} w')$$

$$+ \sum_{k=2}^{\infty} \underset{t^+<t}{+\int} dR^+ dw^+ T(R,R^+) \delta(w^+-ww_t) \int dR' dw' \sigma(R') \delta(R'-R^+)$$

$$\times \delta(w'-w_\sigma w^+) \int dsp_k(P,R',w',s)s \int dR_1 dw_1 \cdots \int dR_k dw_k$$

$$\times \varepsilon_k(R',R_1,\cdots,R_k) \delta(w_1 - w_{\varepsilon_k} w') \cdots \delta(w_k - w_{\varepsilon_k} w') \tag{126}$$

Performing the integrations over the delta functions and using

(123) and (124A)

$$Q_1(R,w,t) = \underset{t^+=t}{+\int} dR^+ T(R,R^+) \delta(|\vec{r}^{+}-\vec{r}|-vt) \int dsp_d(P,R^+,ww_t,s)s$$

$$+ \underset{t'<t}{\int} dR' T(R,R') \sigma(R') A(R') \int dsp_0(P,R',ww_t w_\sigma,s)s$$

$$+ \underset{t'<t}{\int} dR' T(R,R') \sigma(R') \int dsp_1(P,R',ww_t w_\sigma,s)s \int dR'' E(R',R'')$$

$$+ \sum_{k=2}^{\infty} \underset{t'<t}{\int} dR' T(R,R') \sigma(R') \int dsp_k(P,R',ww_t w_\sigma,s)s \int dR_1 \cdots \int dR_k \varepsilon_k(R',R_1,\cdots,R_k) \tag{127}$$

By eqs. (84) and (127) Q_1 must be proportional to w. Thus, by eqs. (125H)

and (125K)

$$M_1(R,ww_0,t) = wM_1(R,w_0,t) \tag{128}$$

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From eq. (121) and the definitions of Chapter II

$$Q_2(R,w,t) = \int_{t^+ \leq t} dR^+ dw^+ T(R,R^+)\delta(w^+ - ww_t)\delta(|\vec{r}^+ - \vec{r}| - vt)$$

$$\times \int ds p_d(P,R^+,w^+,s)s^2 + \int_{t^+ < t} dR^+ dw^+ T(R,R^+)\delta(w^+ - ww_t)$$

$$\times \int dR' dw' \sigma(R')\delta(R' - R^+)\delta(w' - w^+ w_\sigma)\int ds p_0(P,R',w',s)s^2 A(R')$$

$$+ \int_{t^+ < t} dR^+ dw^+ T(R,R^+)\delta(w^+ - ww_t)\int dR' dw' \sigma(R')\delta(R' - R^+)\delta(w' - w^+ w_\sigma)$$

$$\times \int ds p_1(P,R',w',s)\int dR'' dw'' E(R',R'')\delta(w'' - w_{\epsilon_1} w')\int dR^c dw_c \delta(R^c - R'')$$

$$\times \left\{ H(w'' - w_{co})\delta(w_c - w'') + \left(1 - H(w'' - w_{co})\right)\left[\frac{w''}{w_e}\delta(w_c - w_e)\right.\right.$$

$$\left.\left. + (1 - \frac{w''}{w_e})\delta(w_c)\right]\right\}\sum_{n=0}^{1}\binom{2}{n} M_n(R^c,w_c,t-t')s^{2-n}$$

$$+ \sum_{k=2}^{\infty} \int_{t^+ < t} dR^+ dw^+ T(R,R^+)\delta(w^+ - ww_t)\int dR' dw' \sigma(R')\delta(R' - R^+)$$

$$\times \delta(w' - w^+ w_\sigma)\int ds p_k(P,R',w',s)\int dR_1 dw_1 \cdots \int dR_k dw_k \epsilon_k(R',R_1,\cdots,R_k)$$

$$\times \delta(w_1 - w_{\epsilon_k} w')\cdots\delta(w_k - w_{\epsilon_k} w')\prod_{i=1}^{k}\int dR_i^c dw_{c_i}\delta(R_i^c - R_i)$$

$$\times \left\{ H(w_i - w_{co})\delta(w_{c_i} - w_i) + \left(1 - H(w_i - w_{co})\right)\left[\frac{w_i}{w_e}\delta(w_{c_i} - w_e)\right.\right.$$

$$\left.\left. + \left(1 - \frac{w_i}{w_e}\right)\delta(w_{c_i})\right]\right\}\left\{s^2 + 2s\sum_{\ell=1}^{k} M_1(R_\ell^c,w_{c_\ell},t-t')\right.$$

$$\left. + 2\sum_{\ell=2}^{k} M_1(R_\ell^c,w_{c_\ell},t-t')\sum_{j=1}^{\ell-1} M_1(R_j^c,w_{c_j},t-t')\right\} \tag{129}$$

Integrating over the delta functions and noting (123)-(124A)

$$Q_2(R,w,t) = \int_{t^+=t} dR^+ T(R,R^+)\delta(|\vec{r}^+-\vec{r}|-vt)\int ds\, p_d(P,R^+,ww_t,s)s^2$$

$$+ \int_{t'<t} dR'T(R,R')\sigma(R')A(R')\int ds\, p_0(P,R',ww_t w_\sigma,s)s^2$$

$$+ \int_{t'<t} dR'T(R,R')\sigma(R')\int ds\, p_1(P,R',ww_t w_\sigma,s)\int dR''E(R',R'')$$

$$\times \left\{ H(ww_t w_\sigma w_{\epsilon_1}-w_{co}) \sum_{n=0}^{1} \binom{2}{n} M_n(R'',ww_t w_\sigma w_{\epsilon_1},t-t')s^{2-n} \right.$$

$$+ \left. \left(1-H(ww_t w_\sigma w_{\epsilon_1}-w_{co})\right)\left(s^2+2s\frac{ww_t w_\sigma w_{\epsilon_1}}{w_e}M_1(R'',w_e,t-t')\right)\right\}$$

$$+ \sum_{k=2}^{\infty} \int_{t'<t} dR'T(R,R')\sigma(R')\int ds\, p_k(P,R',ww_t w_\sigma,s)\int dR_1\cdots\int dR_k$$

$$\times \epsilon_k(R',R_1,\cdots,R_k)\left[H(ww_t w_\sigma w_{\epsilon_k}-w_{co})\left\{ s^2+2s\sum_{\ell=1}^{k}M_1(R_\ell,ww_t w_\sigma w_{\epsilon_k},t-t') \right.\right.$$

$$+ 2\sum_{\ell=2}^{k} M_1(R_\ell,ww_t w_\sigma w_{\epsilon_k},t-t') \sum_{j=1}^{\ell-1} M_1(R_j,ww_t w_\sigma w_{\epsilon_k},t-t')\Bigg\}$$

$$+ \left(1-H(ww_t w_\sigma w_{\epsilon_k}-w_{co})\right) \sum_{b_1=0}^{1}\cdots\sum_{b_k=0}^{1}\left(\frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\right)^{b_1+\cdots+b_k}$$

$$\times \left(1-\frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\right)^{k-b_1-\cdots-b_k} \Bigg\} s^2 + 2s\sum_{\ell=1}^{k}M_1(R_\ell,w_e,t-t')\delta_{b_\ell,1}$$

$$+ 2\sum_{\ell=2}^{k} M_1(R_\ell,w_e,t-t')\delta_{b_\ell,1} \sum_{j=1}^{\ell-1} M_1(R_j,w_e,t-t')\delta_{b_j,1}\Bigg\}\Bigg] \qquad (130)$$

Combining the scattering and multiplication terms and using eqs. (A12)-(A21)

$$Q_\zeta'(R,w,t) = \int_{\substack{+\\t^+=t}} dR^+ T(R,R^+)\delta(|\vec{r}^+-\vec{r}|-vt)\int ds \, p_d(P,R^+,ww_t,s)s^2$$

$$+ \int_{t'<t} dR' T(R,R')\sigma(R')A(R')\int ds \, p_0(P,R',ww_t w_\sigma,s)s^2$$

$$+ \sum_{k=1}^{\infty} \int_{t'<t} dR' T(R,R')\sigma(R')\int ds \, p_k(P,R',ww_t w_\sigma,s)\int dR_1 \cdots dR_k$$

$$\times \, \epsilon_k(R',R_1,\cdots,R_k)\left[H(ww_t w_\sigma w_{\epsilon_k}-w_{co})\left\{s^2 + 2s\sum_{\ell=1}^{k}M_1(R_\ell,ww_t w_\sigma w_{\epsilon_k},t-t')\right.\right.$$

$$\left. + 2\sum_{\ell=2}^{k}M_1(R_\ell,ww_t w_\sigma w_{\epsilon_k},t-t')\sum_{j=1}^{\ell-1}M_1(R_j,ww_t w_\sigma w_{\epsilon_k},t-t')\right\}$$

$$+ \left(1-H(ww_t w_\sigma w_{\epsilon_k}-w_{co})\right)\left\{s^2 + 2s\frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\sum_{\ell=1}^{k}M_1(R_\ell,w_e,t-t')\right.$$

$$\left.\left. + 2\left(\frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\right)^2\sum_{\ell=2}^{k}M_1(R_\ell,w_e,t-t')\sum_{j=1}^{\ell-1}M_1(R_j,w_e,t-t')\right\}\right] \qquad (130A)$$

Using eq. (128), and noting $H(x) + (1-H(x)) = 1$

$$Q_2(R,w,t) = \int_{t^+=t} dR^+ T(R,R^+)\delta(|\vec{r}^+ - \vec{r}| - vt)\int ds\, p_d(P,R^+,ww_t,s)s^2$$

$$+ \int_{t'<t} dR'T(R,R')\sigma(R')A(R')\int ds\, p_0(P,R',ww_t w_\sigma,s)s^2$$

$$+ \sum_{k=1}^{\infty} \int_{t'<t} dR'T(R,R')\sigma(R')\int ds\, p_k(P,R',ww_t w_\sigma,s)\int dR_1 \cdots \int dR_k$$

$$\times\ \epsilon_k(R',R_1,\cdots,R_k)\ \Big\{ s^2 + 2s \sum_{\ell=1}^{k} M_1(R_j,ww_t w_\sigma w_{\epsilon_k},t-t')$$

$$+ 2\sum_{\ell=2}^{k} M_1(R_\ell,ww_t w_\sigma w_{\epsilon_k},t-t') \sum_{j=1}^{\ell-1} M_1(R_j,ww_t w_\sigma w_{\epsilon_k},t-t')\Big\} \tag{130B}$$

Note that Q_1 and Q_2 (eqs. (127) and (130B)) are independent of the weight cutoff game. By eqs. (84), (128), and (130B) Q_2 must be proportional to w^2. Thus if the exit weight and cutoff weight are chosen as in eqs. (125F)-(125G) then eq. (125I) with r=2 implies

$$M_2(R,ww_0,t) = w^2 M_2(R,w_0,t) \tag{131}$$

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Consider eq. (125) for M_1. Let * indicate analog functions.

For an analog game eq. (127) with $w_t = w_\sigma = w_{\epsilon_1} = w_{\epsilon_k} = 1$ and $w_{co} = 0$ becomes

$$Q_1^*(R,w,t) = \int_{t^+=t} dR^+ T^*(R,R^+)\delta(|\vec{r}^+ - \vec{r}| - vt)\int dsp_d^*(P,R^+,w,s)s$$

$$+ \int_{t'<t} dR' T^*(R,R')\sigma^*(R')A^*(R')\int dsp_0^*(P,R',w,s)s$$

$$+ \int_{t'<t} dR' T^*(R,R')\sigma^*(R')\int dsp_1^*(P,R',w,s)s\int dR'' E^*(R',R'')$$

$$+ \sum_{k=2}^{\infty} \int_{t'<t} dR' T^*(R,R')\sigma^*(R')\int dsp_k^*(P,R',w,s)s\int dR_1 \cdots \int dR_k \epsilon_k^*(R',R_1,\cdots,R_k) \quad (132)$$

Now consider a non-analog game with weight multipliers:

$$\left. \begin{aligned} w_t &= T^*(R,R')/T(R,R') \\[4pt] w_\sigma &= \sigma^*(R')/\sigma(R') \\[4pt] w_{\epsilon_1} &= E^*(R',R'')/E(R',R'') \\[4pt] w_{\epsilon_k} &= \epsilon_k^*(R',R_1,\cdots,R_k)/\epsilon_k(R',R_1,\cdots,R_k) \end{aligned} \right\} \qquad (133)$$

Using eqs. (133),(127), and (84) the equation for Q_1 becomes:

$$Q_1(R,w,t) = \int_{t^+=t} dR^+ T(R,R^+)\delta(|\vec{r}^+ - \vec{r}| - vt) \frac{T^*(R,R^+)}{T(R,R^+)}\int dsp_d(P,R^+,w,s)s$$

$$+ \int_{t'<t} dR' T(R,R')\sigma(R') \frac{T^*(R,R')}{T(R,R')} \frac{\sigma^*(R')}{\sigma(R')} A(R')\int dsp_0(P,R',w,s)s$$

$$+ \int_{t'<t} dR' T(R,R')\sigma(R') \frac{T^*(R,R')}{T(R,R')} \frac{\sigma^*(R')}{\sigma(R')}\int dsp_1(P,R',w,s)s\int dR'' E(R',R'')$$

$$+ \sum_{k=2}^{\infty} \int_{t'<t} dR' T(R,R')\sigma(R') \frac{T^*(R,R')}{T(R,R')} \frac{\sigma^*(R')}{\sigma(R')}\int dsp_k(P,R',w,s)s$$

$$\times \int dR_1 \cdots \int dR_k \epsilon_k(R',R_1,\cdots,R_k) \quad (134)$$

Let

$$A_0(R') = A(R')$$

$$A_1(R') = \int dR'' E(R', R'')$$

$$A_k(R') = \int dR_1 \cdots \int dR_k \varepsilon_k(R', R_1, \cdots, R_k) \quad k \geq 2$$

(135)

If

$$P_d(P, R^+, w, s) = p_d^*(P, R^+, w, s), \text{ and} \tag{136}$$

$$\sum_{k=0}^{\infty} A_k(R') p_k(P, R', w, s) = \sum_{k=0}^{\infty} A_k^*(R') p_k^*(P, R', w, s), \text{ then} \tag{137}$$

$$Q_1(R, w, t) = Q_1^*(R, w, t). \tag{138}$$

For an analog game eq. (125) with $w_t = w_\sigma = w_{\varepsilon_1} = w_{\varepsilon_k} = 1$ and $w_{co} = 0$ becomes

$$M_1^*(R, w, t) = \int_{t' < t} dR' T^*(R, R') \sigma^*(R') \int dR'' E^*(R', R'') M_1^*(R'', w, t-t')$$

$$+ \sum_{k=2}^{\infty} \int_{t' < t} dR' T^*(R, R') \sigma^*(R') \int dR_1 \cdots \int dR_k \varepsilon_k(R', R_1, \cdots, R_k) \sum_{j=1}^{k} M_1^*(R_j, w, t-t')$$

$$+ Q_1^*(R, w, t) \tag{139}$$

For the non-analog case eq. (125) becomes (using eqs. (128) and (133))

$$M_1(R, w, t) = \int_{t' < t} dR' T(R, R') \sigma(R') \int dR'' E(R', R'') \left\{ H(w w_t w_\sigma w_{\varepsilon_1} - w_{co}) \right.$$

$$\times \frac{T^*(R, R')}{T(R, R')} \frac{\sigma^*(R')}{\sigma(R')} \frac{E^*(R', R'')}{E(R', R'')} M_1(R'', w, t-t') + \left(1 - H(w w_t w_\sigma w_{\varepsilon_1} - w_{co}) \right)$$

$$\times \frac{T^*(R, R')}{T(R, R')} \frac{\sigma^*(R')}{\sigma(R')} \frac{E^*(R', R'')}{E(R', R'')} M_1(R'', w, t-t') \right\} + \sum_{k=2}^{\infty} \int_{t' < t} dR' T(R, R') \sigma(R')$$

$$\times \int dR_1 \cdots \int dR_k \varepsilon_k(R', R_1, \cdots, R_k) \left\{ H(w w_t w_\sigma w_{\varepsilon_k} - w_{co}) \frac{T^*(R, R')}{T(R, R')} \frac{\sigma^*(R')}{\sigma(R')} \right.$$

$$x \; \frac{\epsilon_k^*(R',R_1,\cdots,R_k)}{\epsilon_k(R',R_1,\cdots,R_k)} \sum_{j=1}^{k} M_1(R_j,w,t-t') + \left(1-H(ww_t w_\sigma w_{\epsilon_k} - w_{co})\right) \frac{T^*(R,R')}{T(R,R')}$$

$$x \; \frac{\sigma^*(R')}{\sigma(R')} \frac{\epsilon_k^*(R',R_1,\cdots,R_k)}{\epsilon_k(R',R_1,\cdots,R_k)} \sum_{j=1}^{k} M_1(R_j,w,t-t') \Big\} + Q_1(R,w,t) \qquad (140)$$

Using $H(x) + (1-H(x)) = 1$ eq. (140) becomes

$$M_1(R,w,t) = \int_{t'<t} dR' T^*(R,R')\sigma^*(R') \int dR'' E^*(R',R'') M_1(R'',w,t-t')$$

$$+ \sum_{k=2}^{\infty} \int_{t'<t} dR' T^*(R,R')\sigma^*(R') \int dR_1 \cdots \int dR_k \epsilon_k^*(R',R_1,\cdots,R_k) \sum_{j=1}^{k} M_1(R_j,w,t-t')$$

$$+ Q_1(R,w,t) \qquad (141)$$

Since $Q_1 = Q_1^*$ this is exactly the same equation that M_1^* satisfies;
hence if eqs. (133), (136), and (137) are satisfied,

$$M_1(R,w,t) = M_1^*(R,w,t). \qquad (142)$$

This shows it is possible to play a non-analog game whose mean score
is the same as the mean score for the analog game. In general
$M_2 \neq M_2^*$, which implies the variances are different; several examples
of this will be given in Chapter IV.

Chapter IV. Examples

In IV-A two different estimators for flux in a region are compared.
Using the equations of Chapter III it is shown (for an arbitrary medium)
that both estimators give the same mean but different variances. The
equations of Chapter III are then specialized in IV-B to one-speed
infinite medium problems, and an analytic solution method is provided
to solve these specialized equations exactly. The equations of IV-B
are then used in IV-C to predict the variances in four different tech-
niques for obtaining the expected number of collisions. Finally, in
IV-D the sample variances in Monte Carlo calculations are compared with
the predicted variances for each of the four techniques in IV-C.

A. Collision and Track Length Estimators Compared

Consider the problem of obtaining the integral of the flux over a
given region Γ in $(\vec{r},\vec{v})$ space. Two estimators can be used to solve
this problem; the collision estimator scores $1/\sigma$ every time a collision
occurs in Γ and the track length estimator scores the distance (track
length) a neutron travels through Γ between events. It is shown below,
first on physical grounds and then using eqs. (125) and (127), that
these estimators result in the same expected value for the flux but
different variances.

Collision Estimator

The number of collisions per unit volume at energy E is equal to
the flux $\phi(\vec{r},E)$ multiplied by the macroscopic cross section $\sigma(\vec{r},E)$, or

$$dN(\vec{r},E) = \sigma(\vec{r},E)\phi(\vec{r},E)dV \tag{143}$$

$$\phi(\vec{r},E)dV = dN(\vec{r},E)/\sigma(\vec{r},E) \tag{144}$$

Each collision in dV contributes $1/\sigma$ to the flux in the volume dV. Thus an estimate of the integral (over space, energy, and time) of the flux in region Γ may be obtained by scoring $1/\sigma(\vec{r},E)$ at each collision occurring in the region in the given time interval.

Track Length Estimator

The significance of this estimator follows from the property that flux at a given energy equals speed times particle number density.[6] Thus, the integral of flux over a given region at a given time equals speed times the number of neutrons then in that region. For each neutron, the integral of speed times the number of neutrons within the region (i.e. one or zero) over the time interval equals the sum of one times the track length within the region plus zero times the track length outside the region. Summing over all neutrons shows that the expected mean track length equals the flux integrated over the region and the time interval.

————————

In accordance with these conclusions, the equations of Chapter III will be shown here to imply that the M_1's, but not the M_2's, of the two estimators are equal. It is sufficient to prove the Q_1's, but not the Q_2's, are equal, for $T(P,P^+)$, $\sigma(P^+,P')$, $B(P',P'')$, $E(P',P'')$, and $\epsilon_k(P',P_1,\cdots,P_k)$ in eq. (108) do not depend on the scoring distributions

p(P,P',s). An analog game will be considered here but one can use the results of eq. (142) to generalize to a non-analog game.

Let x' be the coordinate along $\vec{\Omega}$, the direction of the neutron's flight. Let

$$\vec{r}'-\vec{r} = (x',y',z') \tag{145}$$

so that the neutron's position is described in a Cartesian coordinate system where the X' axis lies along $\vec{\Omega}$. The transmission probability that a particle travels a distance x' through a medium (of macroscopic total cross section σ) is

$$T(R,x') = T(\vec{r},\vec{r}',\vec{v}) = \begin{cases} \exp\left[-\int_0^{x'} \sigma(\vec{r}+\vec{\Omega}h,\vec{v})dh\right] & \text{if } x' \geq 0 \\ 0 & \text{if } x' < 0 \end{cases} \tag{146}$$

That is

$$T(R,R') = \begin{cases} \delta(\vec{v}-\vec{v}')\delta(y')\delta(z')T(R,x') & \text{if } x' \geq 0 \\ 0 & \text{if } x' < 0 \end{cases} \tag{147}$$

Differentiating eq. (146)

$$\frac{dT(R,x')}{dx'} = \begin{cases} -\sigma(\vec{r}+\vec{\Omega}x',\vec{v})T(R,x') & \text{if } x' \geq 0 \\ 0 & \text{if } x' < 0 \end{cases} \tag{148}$$

Now consider a spatially convex region Γ of $(\vec{r},\vec{v})$ space (it will be shown at the end of this section that the results are valid even if Γ is not spatially convex). Let the possible path length inside Γ extend from x_1 to x_2 and let Γ extend over all velocities.
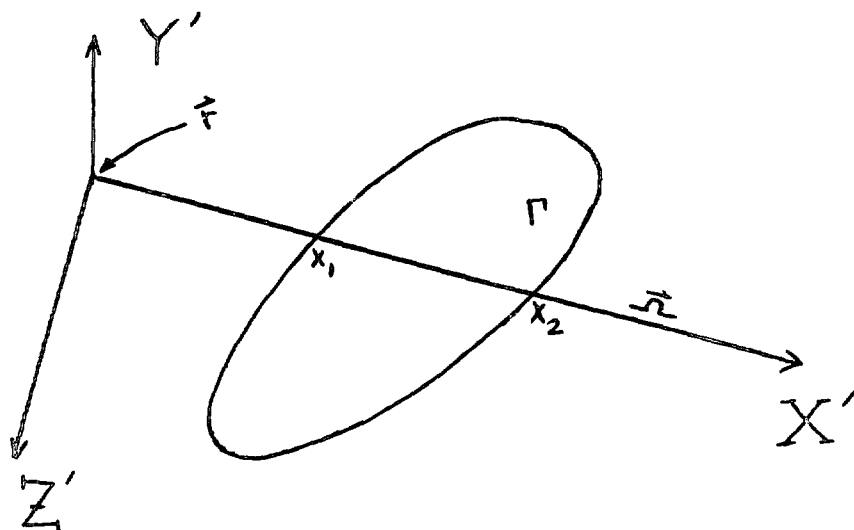
If vt is less than the distance to Γ, then the neutron cannot reach Γ in time t and $x_1 = x_2 = 0$. If $\vec{r}$ is inside Γ then $x_1 = 0$, and if $\vec{r} + \vec{v}t$ is inside Γ then $x_2 = vt$. If the neutron's flight path does not intersect Γ then $x_1 = x_2 = 0$ and no score is possible.

Q's for Collision Estimator

For the collision estimator each collision inside Γ contributes a score $1/\sigma$ and a free flight for the entire time remaining contributes zero. Thus

$$p_k(P,P',s) = \begin{cases} \delta(s-1/\sigma) & R' \text{ in } \Gamma \\ \\ \delta(s) & R' \text{ not in } \Gamma \end{cases} \quad k = 0,1,2\cdots \tag{149}$$

$$p_d(P,P',s) = \delta(s) \tag{150}$$

Let

$$\sigma(x',\vec{v}) = \sigma(\vec{r}+\vec{\Omega}x',\vec{v}) \tag{151}$$

Using equations (146)-(151) with all weights unity, eqs. (127) and (130) become

$$Q_1(R,1,t) = \int_{x_1}^{x_2} dx'T(R,x') \tag{152}$$

$$Q_2(R,1,t) = \int_{x_1}^{x_2} dx'T(R,x')\left[\frac{1}{\sigma(x',\vec{v})} + 2\int dR''E(R',R'')M_1(R'',1,t-t')\right.$$

$$+ \sum_{k=2}^{\infty}\int dR_1\cdots\int dR_k \epsilon_k(R',R_1,\cdots,R_k)\left\{2\sum_{\ell=1}^{k}M_1(R_\ell,1,t-t') + 2\sigma(x',\vec{v})\right.$$

$$\sum_{\ell=2}^{k} M_1(R_\ell,1,t-t')\sum_{j=1}^{\ell-1} M_1(R_j,1,t-t')\left.\right\}\left.\right] \tag{153}$$

Q's for Track Length Estimator

For the track length estimator each track the neutron leaves in Γ contributes a score equal to the length of the track. Thus

$$p_k(P,P',s) = p_d(P,P',s) = p(P,P',s) \text{ for } k = 0,1,2\cdots, \text{ where} \tag{154}$$

$$p(P,P',s) = \begin{cases} \delta(s) & \text{if } x' < x_1 \\ \delta(s-(x'-x_1)) & \text{if } x_1 \le x' \le x_2 \\ \delta(s-(x_2-x_1)) & \text{if } x_2 < x' \end{cases} \tag{155}$$

Using eqs. (146), (147), (148), (154), and (155) with all weights
unity eq. (127) becomes

$$Q_1(R,1,t) = (x_2-x_1)T(R,vt) + \int_{x_1}^{x_2} dx'T(R,x')\sigma(x',\vec{v})(x'-x_1)$$

$$+ \int_{x_2}^{vt} dx'T(R,x')\sigma(x',\vec{v})(x_2-x_1). \tag{156}$$

Substituting eq. (148) into eq. (156) yields

$$Q_1(R,1,t) = (x_2-x_1)T(R,vt) + \int_{x_1}^{x_2}(x'-x_1)\left(\frac{-dT(R,x')}{dx'}\right)dx'$$

$$- (x_2-x_1)\int_{T(R,x_2)}^{T(R,vt)} dT. \tag{157}$$

Integrating the first integral by parts,

$$Q_1(R,1,t) = \int_{x_1}^{x_2} T(R,x')dx' \tag{158}$$

As expected, the Q_1 for the track length estimator equals the Q_1 for
the collision estimator. Substituting eqs. (146) and (147) into eq.
(130), noting that w_{co} is zero for an analog game, and using eqs. (82) -
(83) to collect terms in s^2:

$$Q_2(R,1,t) = T(R,vt)(x_2-x_1)^2 + \int_{x_1}^{x_2} dx' T(R,x')\sigma(x',\vec{v})(x'-x_1)^2$$

$$+ \int_{x_2}^{vt} dx' T(R,x')\sigma(x',\vec{v})(x_2-x_1)^2 + \int_{x_1}^{x_2} dx' T(R,x')\sigma(x',\vec{v})$$

$$\times \left[2(x'-x_1)\int dR'' E(R',R'')M_1(R'',1,t-t') + \int dR_1 \cdots \int dR_k \epsilon_k(R',R_1,\cdots,R_k) \right.$$

$$\times \left\{ 2(x'-x_1) \sum_{\ell=1}^{k} M_1(R_\ell,1,t-t') + 2\sum_{\ell=2}^{k} M_1(R_\ell,1,t-t') \right.$$

$$\left. \left. \times \sum_{j=1}^{\ell-1} M_1(R_j,1,t-t') \right\} \right] + \int_{x_2}^{vt} dx' T(R,x')\sigma(x',\vec{v}) \left[2(x_2-x_1) \right.$$

$$\times \int dR'' E(R',R'')M_1(R'',1,t-t') + \int dR_1 \cdots \int dR_k \epsilon_k(R',R_1,\cdots,R_k)$$

$$\times \left\{ 2(x_2-x_1)\sum_{\ell=1}^{k} M_1(R_\ell,1,t-t') + 2\sum_{\ell=2}^{k} M_1(R_\ell,1,t-t') \right.$$

$$\left. \left. \times \sum_{j=1}^{\ell-1} M_1(R_j,1,t-t') \right\} \right] \tag{159}$$

Q_2 for the collision estimator and Q_2 for the track length estimator
are not generally equal, as can be seen by comparing the case in which
σ approaches zero in the medium

$$Q_2 \rightarrow (x_2-x_1)^2 \quad \text{for the track length estimator} \tag{160}$$

$$Q_2 \rightarrow \infty \qquad \text{for the collision estimator} \tag{161}$$

Thus for optically thin systems the M_2 for the track length estimator will be much less than the M_2 for the collision estimator, resulting in the track length estimator having a much smaller variance than the collision estimator.

The above analysis can be applied to non-convex regions as well as to convex regions. Any bounded region which arises in practice can be approximated, to any accuracy desired, by a finite number of convex parts (see Appendix D). For either estimator the Q_r for any region equals the sum of the Q_r's for its parts.
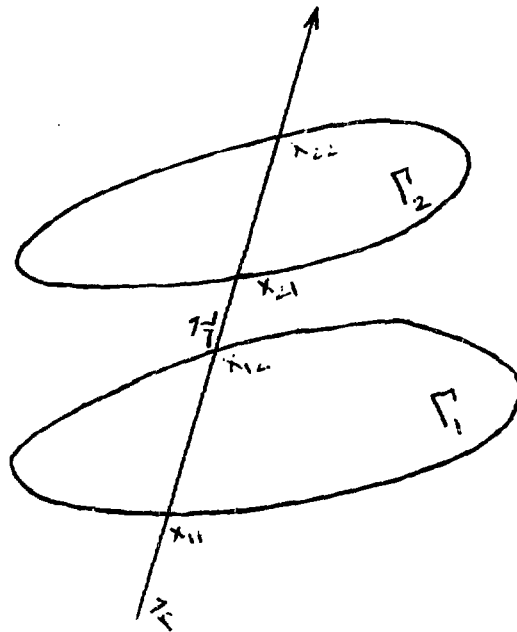
Since the equation for M_r is linear, the M_r for any region equals the sum of the M_r's for its parts. (The M_1 fed into the equation for Q_2 in any subregion should be the M_1 for the entire region Γ and not the M_1 for the subregion.)

B. One-Speed Infinite Medium Equations

Consider a one speed problem in an infinite homogeneous medium (with macroscopic total cross section σ^*) where an event score depends only on the relative coordinate, $\vec{r}'-\vec{r}$, between the next event point and the departure point. Let

$$A_k^*(R')=A_k^* \quad (k=0,1,2,\cdots) \text{ independent of } R' \tag{162}$$

so that all cross sections are constant.

Suppose one attempts to solve this problem using a non-analog Monte Carlo Method[†] in which

$$\sigma(R')=\sigma \quad \text{independent of } R' \tag{163}$$

$$A_k(R')=A_k \quad (k=0,1,2,\cdots) \text{ independent of } R' \tag{164}$$

With these choices, the fictitious Monte Carlo medium looks infinite and homogeneous just as the real medium does. Since the Monte Carlo medium is infinite and homogeneous $M_r(\vec{r},\vec{\Omega},w,t)$ should be independent of $\vec{r}$ and $\vec{\Omega}$. That is, any location looks the same as any other location since there are no reference points, or

$$M_r(\vec{r},\vec{\Omega},w,t)=M_r(w,t). \tag{165}$$

[†] for simplicity weight cutoff is not considered (i.e. $w_{co}=0$).

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The cross sections are all constant so the probability of an eventless transmission from $\vec{r}$ to $\vec{r}'$ is

$$e^{-\sigma|\vec{r}'-\vec{r}|} \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad \begin{array}{l} \text{if } \vec{r}'-\vec{r} \text{ is along the neutron's flight path} \\ \\ \text{otherwise} \end{array} \tag{166}$$

Let

$$\vec{r}'-\vec{r}=(x',y',z') \tag{167}$$

If x' is taken along $\vec{\Omega}$, the direction of the neutron's flight, then the transmission probability kernel becomes:

$$T(P,P')=e^{-\sigma x'} \delta(\vec{\Omega}-\vec{\Omega}')\delta(y')\delta(z')\delta(w'-w_t w) \tag{168}$$

where

$$\sigma(R')=\sigma \tag{169}$$

The free flight kernel becomes

$$D(P,P',t)=e^{-\sigma x'} \delta(\vec{\Omega}-\vec{\Omega}')\delta(y')\delta(z')\delta(w'-w_t w)\delta(x'-vt) \tag{170}$$

Substituting eqs. (82), (84), (86), (168), (169), and (170) into eqs. (125), (127), and (130) results in eqs. (171), (172), and (173).

$$M_r(w,t) = \sum_{k=1}^{\infty}\int_0^{vt} dx' \sigma e^{-\sigma x'} kA_k M_r(ww_t w_\sigma w_{\epsilon_k},t-t')+Q_r(w,t) \tag{171}$$

$$Q_1 = \int dx' e^{-\sigma x'} \delta(x'-vt) \int ds p_d(x',ww_t,s)s$$

$$+ \sum_{k=0}^{\infty} \int_0^{vt} dx' \sigma e^{-\sigma x'} \int ds p_k(x',ww_t w_\sigma,s)A_k s \tag{172}$$

$$Q_2 = \int dx' \delta(x'-vt) \int ds p_d(x',ww_t,s) e^{-\sigma x'} s^2$$

$$+ \sum_{k=0}^{\infty} \int_0^{vt} dx' \sigma e^{-\sigma x'} \int ds p_k(x',ww_t w_\sigma,s)A_k \tag{173}$$

$$\times \left\{ s^2 + 2skM_1(ww_t w_\sigma w_{\varepsilon_k},t-t') + k(k-1)M_1^2(ww_t w_\sigma w_{\varepsilon_k},t-t') \right\}$$

Let

$$y = \sigma vt \tag{174}$$

$$y' = \sigma vt' = \sigma x' \tag{175}$$

Changing variables t and t' to y and y' results in

$$M_r(w,y) = \sum_{k=1}^{\infty} kA_k \int_0^y dy' e^{-y'} M_r(ww_t w_\sigma w_{\varepsilon_k},y-y') + Q_r \tag{176}$$

$$Q_1 = e^{-y} \int ds p_d(y,ww_t,s)s$$

$$+ \sum_{k=0}^{\infty} \int_0^y dy' e^{-y'} \int ds p_k(y',ww_t w_\sigma,s)A_k s \tag{177}$$

$$Q_2 = e^{-y} \int ds p_d(y,ww_t,s)s^2$$

$$+ \sum_{k=0}^{\infty} \int_0^y dy' e^{-y'} \int ds p_k(y',ww_t w_\sigma,s)A_k \left\{ s^2 \right.$$

$$\left. + 2skM_1(ww_t w_\sigma w_{\varepsilon_k},y-y') + k(k-1)M_1^2(ww_t w_\sigma w_{\varepsilon_k},y-y') \right\} \tag{178}$$

From eqs. (176) and (A6)

$$M_r(w,y) = \sum_{k=1}^{\infty} kA_k T[M_r(ww_t w_\sigma w_{\epsilon_k},y)] + Q_r \qquad (179)$$

Expanding M_r and Q_r in power series

$$M_r(w,y) = w^r \sum_{n=0}^{\infty} a_n y^n \qquad (180)$$

$$Q_r(w,y) = w^r \sum_{n=0}^{\infty} b_n y^n \qquad (180A)$$

Substituting eqs. (180) and (180A) into eq. (179):

$$w^r \sum_{n=0}^{\infty} a_n y^n = w^r \sum_{k=1}^{\infty} kA_k (w_t w_\sigma w_{\epsilon_k})^r T\left[\sum_{n=0}^{\infty} a_n y^n\right] + w^r \sum_{n=0}^{\infty} b_n y^n \qquad (181)$$

Let

$$\alpha = \sum_{k=1}^{\infty} kA_k (w_t w_\sigma w_{\epsilon_k})^r \qquad (182)$$

then using eq. (A10)

$$\sum_{n=0}^{\infty} a_n y^n = \alpha \sum_{n=0}^{\infty} (-1)^{n+1} n! a_n \sum_{j=n+1}^{\infty} (-1)^j \frac{y^j}{j!} + \sum_{n=0}^{\infty} b_n y^n \qquad (183)$$

Collecting like powers

$$a_n = \frac{(-1)^{n+1} \alpha}{n!} \sum_{m=1}^{n-1} (-1)^m m! a_m + b_n \qquad (184)$$

Hence as long as Q_r can be expanded in a power series the solution to eq. (125) is given by eq. (184). Thus solving for the expected errors, in a problem where the moments are dependent only on time and weight, has been reduced to solving for the a_n, given the b_n.

This power series method fails, of course, when Q_r is not analytic in y. Q_r need not be analytic. Suppose

$$p_k(y',ww_tw_o,s) = \delta(s) \quad \text{and}$$

$$p_d(y,ww_t,s) = \delta(s-ww_tH(y-y_o)),$$

then eq. (177) becomes

$$Q_1 = ww_tH(y-y_o)e^{-y}.$$

Thus Q_1 is independent of space (recall y is a <u>relative</u> coordinate) and angle, but not analytic in y.

The variances of four Monte Carlo techniques for obtaining the mean number of collisions a particle (and its secondaries) makes in time t are compared below using the equations of section B.

Consider an infinite homogeneous medium with the following properties:

$$A_0^* = 1/3; \quad \text{a third of the collisions result in absorption} \tag{185}$$

$$A_1^* = 1/3; \quad \text{a third of the collisions result in scattering} \tag{186}$$

$$A_2^* = 1/3; \quad \text{a third of the collisions result in two exiting particles} \tag{187}$$

$$A_m^* = 0 \text{ for } m \geq 3; \quad \text{no collision results in more than two exiting}$$
$$\text{particles} \tag{188}$$

$$\sigma^* = \text{macroscopic cross section} \tag{188A}$$

$$\alpha^* = \sum_{k=1}^{\infty} kA_k^* = 1 \tag{189}$$

The following four estimators for collisions are compared:

1) analog collision estimator
2) track length estimator
3) survival biased collision estimator
4) survival biased track length estimator

All kernels, score distributions, and weight multipliers are chosen in accordance with eqs. (133), (136), and (137) so that eq. (142) holds when biased kernels are used, thus ensuring estimators 1) and 3) have the same M_1 and estimators 2) and 4) have the same M_1. Estimators 1)

and 2) are the same as the estimators described in IV-A except that the scores in 1) and 2) are a constant σ times the scores in IV-A. Thus the M_1's for the estimators 1) and 2) will be σ times the M_1's for the corresponding estimators in IV-A. The M_1's in IV-A are equal; hence the M_1's for estimators 1) and 2) are equal. Thus estimators 1), 2), 3) and 4) can be expected to have the same means but different variances.

One possible point of confusion should be addressed before obtaining the variances for the four estimators. Quantities such as T^*, σ^*, A^*, E^*, and ε_k^* which describe the medium must be independent of biasing schemes and choice of estimator. However, the score distributions $p^*(P,P',s)$ (used with unbiased kernels) do depend on the choice of estimator. Thus the $p^*(P,P',s)$ for estimators 1) and 3) are different from the $p^*(P,P',s)$ for estimators 2) and 4).

Game 1   Analog Collision Estimator

Unbiased kernels are used and the particle's weight is scored at every collision. For an analog game a particle's weight never changes.

$$A_0(R') = A_0^*(R') \qquad A_0 = A_0^* = 1/3 \tag{190 A,B}$$

$$T(R,R^+) = T^*(R,R+) \qquad w_t = 1 \tag{191 A,B}$$

$$\sigma(R') = \sigma^*(R') \qquad w_\sigma = 1 \tag{192 A,B}$$

$$E(R',R'') = E^*(R',R'') \qquad w_{\varepsilon_1} = 1 \quad A_1 = A_1^* = 1/3 \tag{193 A,B,C}$$

$$\varepsilon_2(R',R_1,R_2) = \varepsilon_2^*(R',R_1,R_2) \qquad w_{\varepsilon_2} = 1 \quad A_2 = A_2^* = 1/3 \tag{194 A,B,C}$$

$$D(R,R^+,t) = D^*(R,R^+,t) \tag{195}$$

$$P_k(P,P',s) = p_k^*(P,P',s) = \delta(s-w') \quad k = 0,1,2 \tag{196}$$

$$P_d(P,P',s) = p_d^*(P,P',s) = \delta(s) \tag{197}$$

Substituting relations (185)-(197) into eq. (177)

$$Q_1 = \int_0^y e^{-y'} w \left\{ A_0 + A_1 + A_2 \right\} dy' = w \int_0^y e^{-y'} dy' \tag{198}$$

From eq. (182) and (191)-(194)

$$\alpha = \sum_{k=1}^{\infty} k A_k w_t w_\sigma w_{\varepsilon_k} = 1 \tag{199}$$

Using eqs. (A6) and (A10) eq. (198) becomes

$$Q_1 = wT[1] = w \sum_{j=1}^{\infty} (-1)^{j+1} \frac{y^j}{j!} \tag{200}$$

therefore from eq. (180A)

$$b_n = \frac{(-1)^{n+1}}{n!} \qquad n \geq 1 \tag{201}$$

From eqs. (184) and (201)

$$a_n = \frac{(-1)^{n+1}}{n!} \left(\sum_{m=1}^{n-1} (-1)^m m! \, a_m + 1 \right) \tag{202}$$

Solving eq. (202) for the first few a_n:

$$a_1 = 1$$

$$a_2 = 0$$

$$a_3 = 0$$

Inductive Proof that $a_n = 0$ for $n \geq 2$

Suppose $a_n = 0$ then from eq. (202)

$$\sum_{m=1}^{n-1} (-1)^m m! \, a_m + 1 = 0$$

From eq. (202)

$$a_{n+1} = \frac{(-1)^{n+2}}{(n+1)!} \left((-1)^n n! \, a_n + \sum_{m=1}^{n-1} (-1)^m m! \, a_m + 1 \right)$$

$$a_{n+1} = \frac{n!}{(n+1)!} \, a_n = 0$$

Hence $a_n = 0$ implies $a_{n+1} = 0$; but $a_2 = 0$ hence $a_n = 0$ for $n \geq 2$.

Thus by eq. (180)

$$M_1(w,y) = wy = wovt \tag{203}$$

Using eqs. (190)-(197), (203), and (A6), eq. (178) becomes

$$Q_2 = \left[(A_0 + A_1 + A_2)T[1] + (2A_1 + 4A_2)T[y] + 2A_2 T[y^2] \right] w^2 \tag{204}$$

Using eq. (A10)

$$w^{-2}Q_2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{y^n}{n!} + 2 \sum_{n=2}^{\infty} (-1)^n \frac{y^n}{n!} + \frac{2}{3} \cdot 2 \sum_{n=3}^{\infty} (-1)^{n+1} \frac{y^n}{n!} \qquad (205)$$

$$w^{-2}Q_2 = y + \frac{1}{2} y^2 + \frac{1}{3} \sum_{n=3}^{\infty} (-1)^{n+1} \frac{y^n}{n!} \qquad (206)$$

By eq. (180A)

$$b_1 = 1$$

$$b_2 = 1/2$$

$$b_n = \frac{1}{3} \frac{(-1)^{n+1}}{n!} \quad \text{for } n \geq 3 \qquad (207)$$

From eqs. (184), (199), and (207)

$$a_1 = 1$$

$$a_2 = 1$$

$$a_n = \frac{(-1)^{n+1}}{n!} \left[\sum_{m=1}^{n-1} (-1)^m m! \, a_m + \frac{1}{3} \right] \qquad n \geq 3 \qquad (208)$$

Using eq. (208)

$$a_3 = \frac{1}{3!} \frac{4}{3}$$

$$a_4 = 0$$

Proof (by induction) that $a_n = 0$ for $n \geq 4$

Suppose $a_n = 0$; then from eq. (208)

$$\sum_{m=1}^{n-1} (-1)^m m! \, a_m + \frac{1}{3} = 0 \qquad (208A)$$

Writing eq. (208) for $n + 1$

$$a_{n+1} = \frac{(-1)^n}{(n+1)!} \left((-1)^n n! \, a_n + \sum_{m=1}^{n-1} (-1)^m m! \, a_m + \frac{1}{3} \right) \qquad (208B)$$

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Using $a_n = 0$ and eq. (208A) in eq. (208B) results in $a_{n+1} = 0$. Thus $a_4 = 0$ implies $a_n = 0$ for $n \geq 4$.

From eq. (180)

$$M_2(w,y) = w^2(y + y^2 + \frac{2}{9} y^3) \tag{208C}$$

Using eqs. (203) and (208C)

$$VAR = M_2(w,y) - M_1^2(w,y) = w^2(y + \frac{2}{9} y^3) \tag{209}$$

Game 2 Track Length Estimator

Unbiased kernels are used and the event score generated by a particle of unit weight is σ times the distance travelled between events.

$$A_0(R') = A_0^*(R') \qquad A_0 = A_0^* \tag{210}$$

$$T(R,R^+) = T^*(R,R^+) \qquad w_t = 1 \tag{211}$$

$$\sigma(R') = \sigma^*(R') \qquad w_\sigma = 1 \tag{212}$$

$$E(R',R'') = E^*(R',R'') \qquad w_{\epsilon_1} = 1 \qquad A_1 = A_1^* \tag{213}$$

$$\epsilon_2(R',R_1,R_2) = \epsilon_2^*(R',R_1,R_2) \qquad w_{\epsilon_2} = 1 \qquad A_2 = A_2^* \tag{214}$$

$$D(R,R^+,t) = D^*(R,R^+,t) \tag{215}$$

$$p_k(P,P',s) = p_k^*(P,P',s) = \delta(s-w'\sigma x') = \delta(s-w'y') \quad k = 0,1,2 \tag{216}$$

$$p_d(P,P',s) = p_d^*(P,P',s) = \delta(s-w'\sigma x') = \delta(s-w'y') \tag{216A}$$

From eqs. (177) and (216)

$$Q_1 = w \left[ye^{-y} + \int_0^y dy' e^{-y'} \left\{ A_0 y' + A_1 y' + A_2 y' \right\} \right] \tag{217}$$

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Using eq. (A7) and (A8)

$$Q_1/w = ye^{-y} + V[y] = ye^{-y} + e^{-y} \sum_{j=2}^{\infty} \frac{y^j}{j!} \qquad (218)$$

$$Q_1/w = e^{-y} \sum_{j=1}^{\infty} \frac{y^j}{j!} = e^{-y}(e^y - 1) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{y^j}{j!} \qquad (219)$$

and from eq. (182) with $r = 1$

$$\alpha = \sum_{k=1}^{\infty} k A_k w_{\varepsilon_k} w_o w_t = 1 \qquad (220)$$

Q_1 (and hence b_n) and α are the same as in game 1. Thus following the same steps as in game 1 yields

$$M_1(w,y) = wy \qquad (221)$$

From eqs. (170) and (221)

$$Q_2 = w^2 y^2 e^{-y} + w^2 \int_0^y dy' e^{-y} \left\{ A_o y'^2 + A_1 y'^2 + A_2 y'^2 \right.$$

$$\left. + 2A_1 y'(y-y') + 4A_2 y'(y-y') + 2A_2(y-y')^2 \right\} \qquad (222)$$

Using eqs. (A6) - (A10)

$$w^{-2} Q_2 = y^2 e^{-y} - V[y^2] + 2yV[y] + \frac{2}{3} T[y^2] \qquad (223)$$

$$w^{-2} Q_2 = \frac{2}{3}(e^{-y} - 1) + \frac{2}{3}y^2 + \frac{2}{3}y \qquad (224)$$

$$w^{-2} Q_2 = y^2 + \frac{2}{3} \sum_{n=3}^{\infty} (-1)^n \frac{y^n}{n!} \qquad (225)$$

$$b_1 = 0 \qquad b_2 = 1 \quad \text{and for } n \geq 3 \quad b_n = \frac{2}{3}(-1)^n \frac{1}{n!} \qquad (226)$$

From eqs. (182) with $r = 2$

$$\alpha = \sum_{k=1}^{\infty} kA_k w_{\epsilon_k}^2 w_t^2 w_\sigma^2 = 1 \tag{227}$$

Using eq. (184) $a_1 = 0$, $a_2 = 1$, and for $n \geq 3$

$$a_n = \frac{(-1)^{n+1}}{n!} \left[\sum_{m=1}^{n-1} (-1)^m m! \ a_m - \frac{2}{3} \right] \tag{228}$$

Thus $a_3 = \frac{1}{3!} \frac{4}{3}$, $a_4 = 0$, $a_5 = 0$, and $a_6 = 0$.

<u>Proof</u> (by induction) that $a_n = 0$ for $n \geq 4$

Suppose $a_n = 0$; then from eq. (228)

$$\sum_{m=1}^{n-1} (-1)^m m! \ a_m - \frac{2}{3} = 0 \tag{229}$$

Again from eq. (228)

$$a_{n+1} = \frac{(-1)^{n+2}}{(n+1)!} \left[(-1)^n n! \ a_n + \sum_{m=1}^{n-1} (-1)^m m! \ a_m - \frac{2}{3} \right] \tag{230}$$

Thus using eq. (229) in eq. (230) it follows that $a_n = 0$ implies $a_{n+1} = 0$. But $a_4 = 0$; hence for $n \geq 4$, $a_n = 0$.

Thus from eq. (180)

$$M_2(w,y) = w^2(y^2 + \frac{2}{9} y^3), \text{ and} \tag{231}$$

$$VAR = M_2(w,y) - M_1^2(w,y) = w^2 \frac{2}{9} y^3 \tag{231A}$$

Game 3 Survival Biased Collision Estimator

The exit kernels are biased so that absorption cannot occur. This is known as absorption by weight reduction, or survival biasing. A particle of weight w generates a score of w every time the particle collides.

$$A_0(R') = 0 \text{ no absorption, called "survival biased"} \tag{232}$$

$$\left.\begin{array}{l} T(R,R^+) = T^*(R,R^+) \quad w_t = 1 \\[4pt] \sigma(R') = \sigma^*(R') \quad\quad w_\sigma = 1 \end{array}\right\} \quad \begin{array}{l}\text{next collision location} \\ \text{not biased}\end{array} \tag{233}$$
<div style="text-align:right">(234)</div>

$$E(R',R'') = \tfrac{3}{2}E^*(R',R'') \quad w_{\epsilon_1} = \tfrac{2}{3} \quad A_1 = \tfrac{3}{2}A^*_1 = \tfrac{1}{2} \tag{235}$$

$$\epsilon_2(R',R_1,R_2) = \tfrac{3}{2}\epsilon_2^*(R',R_1,R_2) \quad w_{\epsilon_2} = \tfrac{2}{3} \quad A_2 = \tfrac{3}{2}A^*_2 = \tfrac{1}{2} \tag{236}$$

$$p_k(P,P',s) = p_k^*(P,P',s) = \delta(s-w') \quad k = 0,1,2 \tag{237}$$

$$p_d(P,P',s) = p_d^*(P,P',s) = \delta(s) \tag{237A}$$

Substituting eqs. (232) – (237A) into eq. (177) yields

$$Q_1 = w\int_0^y dy' e^{-y'} \tag{238}$$

Using eqs. (A6) and (A10)

$$Q_1/w = T[1] = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{y^n}{n!} \tag{239}$$

Substituting eqs. (233) – (236) into eq. (182) with r=1 yields

$$\alpha = \frac{1}{2}\cdot\frac{2}{3} + 2\cdot\frac{1}{2}\cdot\frac{2}{3} = 1 \tag{240}$$

Q_1 and α are the same as they were in game 1; thus following the same steps as in game 1 yields

$$M_1(w,y) = wy \tag{241}$$

Substituting eqs. (232) - (237A) and (241) into eq. (178) yields

$$Q_2 = w^2 \int_0^y dy' e^{-y'} \left[\frac{1}{2} \left(1 + \frac{4}{3}(y-y') + 1 + \frac{8}{3}(y-y') + \frac{8}{9}(y-y')^2 \right) \right] \tag{242}$$

Using eqs. (A6) and (A10)

$$w^{-2}Q_2 = T[1] + 2T[y] + \frac{4}{9}T[y^2] \tag{243}$$

$$w^{-2}Q_2 = y + \frac{1}{2}y^2 + \frac{1}{9} \sum_{n=3}^{\infty} (-1)^n \frac{y^n}{n!} \tag{244}$$

From eq. (180A)

$$b_1 = 1 \quad b_2 = \frac{1}{2} \quad \text{and for } n \geq 3 \quad b_n = \frac{1}{9} \frac{(-1)^n}{n!} \tag{245}$$

Substituting eqs. (233) - (236) into eq. (182) yields

$$\alpha = \frac{2}{3} \tag{246}$$

Inserting eqs. (245) - (246) into eq. (184) yields

$$a_1 = 1 \quad a_2 = \frac{5}{6} \quad a_3 = \frac{9}{3!3^3} \tag{247}$$

$$a_n = \frac{(-1)^{n+1}}{n!} \left(\frac{2}{3} \sum_{m=1}^{n-1} (-1)^m m! a_m - \frac{1}{9} \right) \quad \text{for } n \geq 3 \tag{248}$$

<u>Proof</u> (by induction) that $a_n = (-1)^{n+1} \frac{9}{n!3^n}$ for $n \geq 3$

Suppose

$$a_n = (-1)^{n+1} \frac{9}{n! \, 3^n} \tag{249}$$

Writing eq. (248) for n+1

$$a_{n+1} = \frac{(-1)^n}{(n+1)!} \left(\frac{2}{3} \sum_{m=1}^{n-1} (-1)^m m! \, a_m + \frac{2}{3}(-1)^n n! \, a_n - \frac{1}{9} \right) \tag{250}$$

Dividing eq. (248) by -3(n+1)

$$\frac{-a_n}{3(n+1)} = \frac{(-1)^n}{3(n+1)!} \left(\frac{2}{3} \sum_{m=1}^{n-1} (-1)^m m! \, a_m - \frac{1}{9} \right) \tag{251}$$

Subtracting eq. (251) from eq. (250)

$$a_{n+1} + \frac{a_n}{3(n+1)} = \frac{(-1)^n}{3(n+1)!} \left(\frac{4}{3} \sum_{m=1}^{n-1} (-1)^m m! \, a_m + (-1)^n 2 n! \, a_n - \frac{2}{9} \right) \tag{252}$$

Substituting a_n from eq. (248) into eq. (252)

$$a_{n+1} + \frac{a_n}{3(n+1)} = 0 \tag{253}$$

Substituting eq. (249) into eq. (253)

$$a_{n+1} = \frac{(-1)^{n+2} \, 9}{(n+1)! \, 3^{n+1}} \tag{254}$$

Eq. (249) is true for n=3; thus by eq. (254) the assertion is proven.

Thus by eq. (180)

$$M_2(w,y) = w^2 \left[y + \frac{5}{6} y^2 + \sum_{n=3}^{\infty} (-1)^{n+1} \frac{9}{n! \, 3^n} y^n \right] \tag{255}$$

$$VAR = M_2(w,y) - M_1^2(w,y) \tag{256}$$

Using eqs. (241) and (255)

$$VAR = w^2 \left[y - \frac{1}{6} y^2 + \sum_{n=3}^{\infty} (-1)^{n+1} \frac{9}{n! \, 3^n} y^n \right] \tag{257}$$

Game 4   Survival Biased Track Length Estimator

The exit kernels are biased so that absorption cannot occur and the event score generated by a particle of unit weight is σ times the distance travelled between events.

$$A_0(R') = 0 \quad \text{no absorption, "survival biased"} \tag{258}$$

$$\left. \begin{array}{l} T(R,R^+) = T*(R,R^+) \qquad w_t = 1 \\[2mm] \sigma(R') = \sigma*(R') \qquad w_\sigma = 1 \end{array} \right\} \begin{array}{l} \text{next collision location} \\ \text{not biased} \end{array} \tag{259}$$
$$\tag{260}$$

$$E(R',R'') = \tfrac{3}{2}E*(R',R'') \qquad w_{\varepsilon_1} = \tfrac{2}{3} \qquad A_1 = \tfrac{3}{2}A_1^* = \tfrac{1}{2} \tag{261}$$

$$\varepsilon_2(R',R_1,R_2) = \tfrac{3}{2}\varepsilon_2^*(R',R_1,R_2) \qquad w_{\varepsilon_2} = \tfrac{2}{3} \qquad A_2 = \tfrac{3}{2}A_2^* = \tfrac{1}{2} \tag{262}$$

$$p_d(P,P',s) = p_k(P,P',s) = p_d^*(P,P',s) = p_k^*(P,P',s)$$

$$= \delta(s - w'\sigma x') = \delta(s - w'y') \qquad k = 0,1,2 \tag{263}$$

Substituting eqs. (258) - (263) into eq. (177)

$$Q_1 = wye^{-y} + w\int_0^y dy' e^{-y'} y' \tag{264}$$

$$Q_1/w = ye^{-y} + e^{-y}(-y-1) + 1 = 1 - e^{-y} \tag{265}$$

$$Q_1/w = \sum_{n=1}^{\infty}(-1)^n \frac{y^n}{n!} \tag{266}$$

Substituting eqs. (259) - (262) into eq. (182) for r=1

$$\alpha = 1 \tag{267}$$

Q_1 and α are the same as in game 1; thus following the same steps as in game 1 yields

$$M_1(w,y) = wy \tag{268}$$

Substituting eqs. (258) - (263) and (268) into eq. (178)

$$Q_2 = w^2 y^2 e^{-y} + w^2 \int_0^y dy' e^{-y'} A_1 \left\{ y'^2 + 2y' w_{\varepsilon_1} (y-y') \right\}$$

$$+ w^2 \int_0^y dy' e^{-y'} A_2 \left\{ y'^2 + 4y' w_{\varepsilon_2} (y-y') + 2w_{\varepsilon_2}^2 (y-y')^2 \right\} \tag{269}$$

Using eqs. (259) - (262) and (A6) - (A7)

$$Q_2/w^2 = y^2 e^{-y} - V[y^2] + 2yV[y] + \frac{4}{9} T[y^2] \tag{270}$$

Usings Eqs. (A8.1) and (A11)

$$Q_2/w^2 = \frac{10}{9} e^{-y} - \frac{10}{9} + \frac{10}{9} y + \frac{4}{9} y^2 \tag{271}$$

$$Q_2/w^2 = y^2 + \frac{10}{9} \sum_{n=3}^{\infty} (-1)^n \frac{y^n}{n!} \tag{272}$$

From eqs. (180A) and (272)

$$b_1 = 0 \quad b_2 = 1 \quad b_n = (-1)^n \frac{10}{9n!} , \quad n \geq 3 \tag{273}$$

Substituting eqs. (259) - (262) into eq. (182) for r=2

$$\alpha = \frac{2}{3} \tag{274}$$

Using eqs. (273) and (274) in eq. (184)

$$a_1 = 0 \qquad a_2 = 1 \qquad a_3 = \frac{6}{3!3^3} \tag{275}$$

$$a_n = \frac{(-1)^{n+1}}{n!} \left(\frac{2}{3} \sum_{m=1}^{n-1} (-1)^m m! \; a_m - \frac{10}{9} \right) \qquad n \geq 3 \tag{276}$$

Proof (by induction) that $a_n = (-1)^{n+1} \dfrac{6}{n!3^n}$ for $n \geq 3$

Suppose

$$a_n = (-1)^{n+1} \frac{6}{n!3^n} \tag{276A}$$

Writing eq. (276) for $n+1$

$$a_{n+1} = \frac{(-1)^n}{(n+1)!} \left(\frac{2}{3} \sum_{m=1}^{n-1} (-1)^m m! a_m + (-1)^n \; \frac{2}{3} n! \; a_n - \frac{10}{9} \right) \tag{277}$$

Dividing eq. (276) by $-3(n+1)$

$$\frac{-a_n}{3(n+1)} = \frac{(-1)^n}{3(n+1)!} \left(\frac{2}{3} \sum_{m=1}^{n-1} (-1)^m m! \; a_m - \frac{10}{9} \right) \tag{278}$$

Subtracting eq. (278) from eq. (277)

$$a_{n+1} + \frac{a_n}{3(n+1)}$$
$$= \frac{(-1)^n}{3(n+1)!} \left(\frac{4}{3} \sum_{m=1}^{n-1} (-1)^m m! \; a_m - \frac{2 \cdot 10}{9} + (-1)^n \; 2n! a_n \right) \tag{279}$$

Using eq. (276) in the right hand side of eq. (279)

$$a_{n+1} + \frac{a_n}{3(n+1)} = 0 \tag{280}$$

Substituting eq. (276A) into eq. (280)

$$a_{n+1} = (-1)^{n+2} \frac{6}{(n+1)!3^{n+1}} \tag{280A}$$

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Now eq. (276) is true for n=3, so by induction it is true for $n \geq 3$. QED.

From eq. (180)

$$M_2(w,y) = w^2 y^2 + w^2 \sum_{n=3}^{\infty} (-1)^{n+1} \frac{6}{n!3^n} y^n \tag{281}$$

Using eqs. (268) and (281)

$$VAR = M_2(w,y) - M_1^2(w,y) = w^2 \sum_{n=3}^{\infty} (-1)^{n+1} \frac{6}{n!3^n} y^n \tag{282}$$

Comparisons of Four Estimators for Collisions

#1 $VAR = y + \frac{2}{9}y^3$

#2 $VAR = \frac{2}{9}y^3$

#3 $VAR = y - \frac{1}{6}y^2 + \sum_{n=3}^{\infty}(-1)^{n+1}\frac{9}{3^n n!}y^n$

#4 $VAR = \sum_{n=3}^{\infty}(-1)^{n+1}\frac{6}{3^n n!}y^n$

#1 $M_1 = y$

#2 $M_1 = y$

#3 $M_1 = y$

#4 $M_1 = y$

#1 Analog

#2 Analog Track Length

Survival Biased Collision #3

Survival Biased Track Length #4

VAR

$y = \sigma vt$
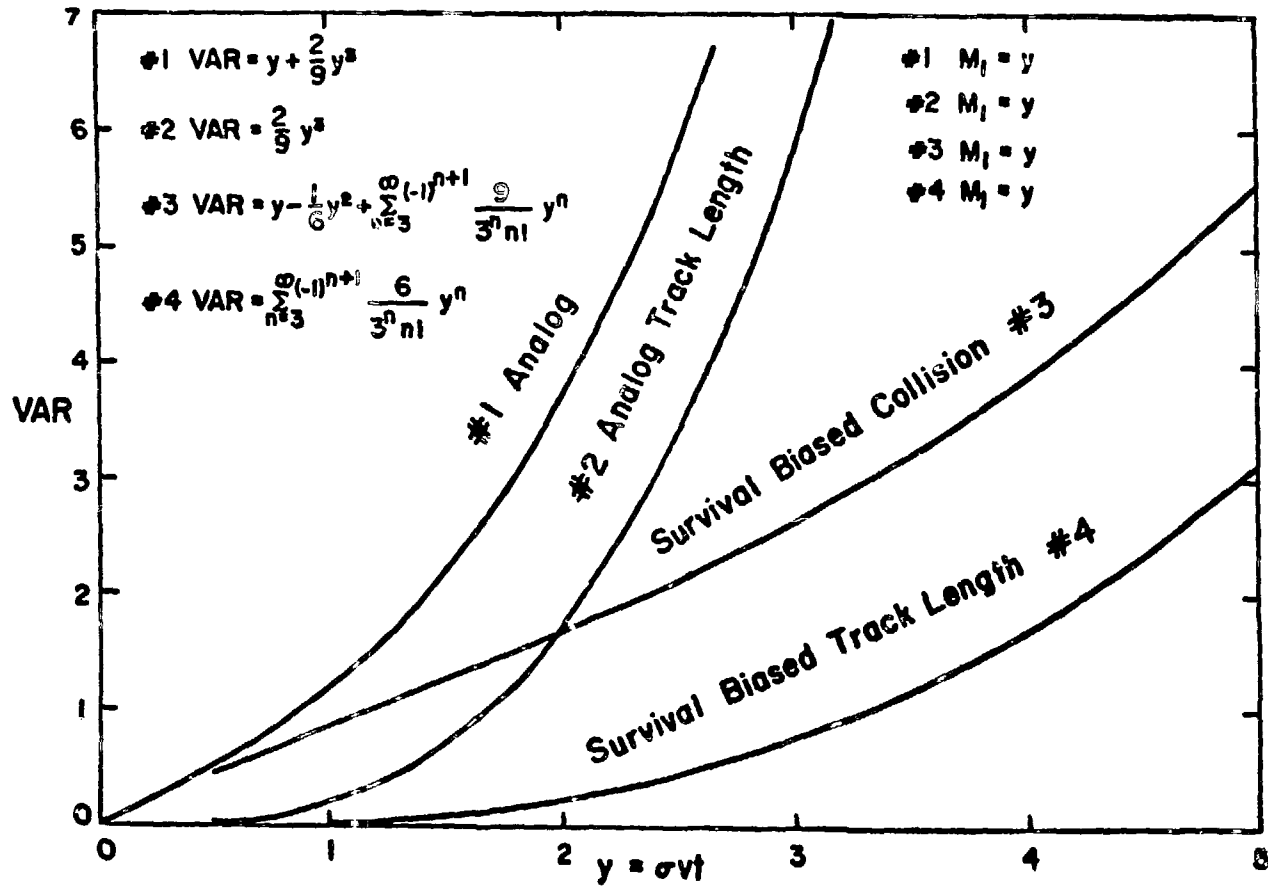
D. Numerical Comparison of Four Estimators for Collisions

Monte Carlo calculations were performed (using the MCNP code at the Los Alamos Scientific Laboratory) for each of the four games in IV-C. Forty thousand neutron histories were followed in each game. The sample means and variances are compared below with the theoretical means and variances predicted in IV-C.

$\overline{M}_1$ = sample mean score

$\overline{M}_2$ = sample mean-square score

$\overline{VAR} = \overline{M}_2 - \overline{M}_1^2$ = sample variance

$\overline{S}_m = (\overline{VAR}/40000)^{1/2}$ = standard deviation of the sample mean

M_1 = mean score

M_2 = mean-square score

$VAR = M_2 - M_1^2$ = variance

$S_m = (VAR/40000)^{1/2}$ = standard deviation of the mean

Game 1   Analog Collision Estimator

	y=1	2	3	4	5
$\overline{M}_1$	.99786	1.9914	3.0000	3.9954	4.9828
$\overline{M}_2$	2.2039	7.6530	17.9861	33.9382	57.1933
$\overline{VAR}$	1.2082	3.6875	8.9857	17.9748	32.3650
$\overline{S}_m$	.00550	.00960	.01499	.02120	.02845
M_1	1.0000	2.0000	3.0000	4.0000	5.0000
M_2	2.2222	7.7778	18.0000	31.2222	57.7778
VAR	1.2222	3.7778	9.0000	18.2222	32.7778
S_m	.00553	.00972	.01500	.02134	.02863

Game 2 Track Length Estimator

	$y=1$	2	3	4	5
$\overline{M}_1$	.99935	2.0009	3.0027	4.0002	4.9816
$\overline{M}_2$	1.2208	5.7783	15.0690	30.1939	52.3384
$\overline{VAR}$	.22207	1.7745	6.0524	14.1815	27.5215
$\overline{S}_m$	.00236	.00666	.01230	.01883	.02623
M_1	1.0000	2.0000	3.0000	4.0000	5.0000
M_2	1.2222	5.7778	15.0000	30.2222	52.7778
VAR	.22222	1.7778	6.0000	14.2222	27.7778
S_m	.00236	.00667	.01224	.01886	.02635

Game 3   Survival Biased Collision Estimator

	$y=1$	2	3	4	5
$\overline{M}_1$	.99735	2.0001	2.9976	3.9895	4.9831
$\overline{M}_2$	1.8763	5.7110	11.6566	19.8487	30.4348
$\overline{VAR}$	.88156	1.7089	2.6711	3.9331	5.6034
$\overline{S}_m$	.00469	.00654	.00817	.00992	.01184
M_1	1.0000	2.0000	3.0000	4.0000	5.0000
M_2	1.8846	5.7126	11.6891	19.9610	30.6335
VAR	.88455	1.7126	2.6891	3.9610	5.6335
S_m	.00470	.00654	.00820	.00995	.01187

Game 4   Survival Biased Track Length Estimator

	y=1	2	3	4	5
$\bar{M}_1$	.99928	1.9989	2.9962	3.9951	4.9862
$\bar{M}_2$	1.0328	4.2499	9.7701	17.7210	28.0534
$\overline{VAR}$	.03420	.25409	.79303	1.7605	3.1910
$\bar{S}_m$	.00092	.00252	.00445	.00663	.00893
M_1	1.0000	2.0000	3.0000	4.0000	5.0000
M_2	1.0341	4.2528	9.7927	17.7518	28.2001
VAR	.03415	.25283	.79272	1.7518	3.2001
S_m	.00092	.00251	.00445	.00662	.00894

Chapter V. Conclusion

The equations for M_1 and M_2 are useful for at least two reasons.
First, it is possible to use the equations to prove certain well known
or conjectured results[2,3,4,7,8] (e.g. see IV-A of this work). Second,
the equations are useful in predicting errors in specific Monte Carlo
transport calculations (e.g. see IV-C). Often a proposed variance
reduction technique will reduce the number of histories required, but
increase the computation time per history. The equations for M_1 and
M_2 allow prediction of how much a specific technique reduces the number
of histories required, to be weighed against a change in the time
required per history.

The moment equations derived here are linear integral equations,
and thus can be solved by several techniques. One possibility is
simply to truncate the Neumann series of eq. (C7). Another possibility
is to convert the moment equations to their integro-differential forms
as Amster and Djomehri[1] do, and then use a P_N or an S_N solution technique.
(Sarkar and Prasad[3] have used a P_1 approximation to the integro-differ-
ential forms of the moments equations to study the exponential transform.)

This work generalizes the theory developed by Amster and Djomehri[1]
to include:

1) Error prediction in Monte Carlo transport calculations of time-
    dependent multiplying systems, even when supercritical.

2) Score distributions which depend on the preceding event point as well as the current event point. This allows prediction of errors in calculations using a track length estimator.

3) Russian Roulette and splitting events. These are standard variance reducing techniques.

APPENDIX A - Useful Results

Result #1

Consider

$$\beta = \int \psi(P, s-s^*, t) s^r ds \tag{A1}$$

Let $\hat{s} = s-s^*$, that is $s = \hat{s}+s^*$ (A2)

$$\beta = \int \psi(P, s, t)(\hat{s}+s^*)^r d\hat{s} \tag{A3}$$

Using the binomial theorem

$$\beta = \int d\hat{s} \sum_{n=0}^{r} \binom{r}{n} s^{*r-n} \hat{s}^n \psi(P, \hat{s}, t)$$

$$= \sum_{n=0}^{r} \binom{r}{n} s^{*r-n} \int d\hat{s} \psi(P, \hat{s}, t) \hat{s}^n \tag{A4}$$

Using eq. (81)

$$\beta = \sum_{n=0}^{r} \binom{r}{n} s^{*r-n} M_n(P, t) = \int ds \psi(P, s-s^*, t) s^r \tag{A5}$$

Result #2

Consider the operators

$$T[R(y)] = \int_0^y e^{-y'} R(y-y')dy' \tag{A6}$$

$$V[R(y)] = \int_0^y e^{-y'} R(y')dy' \tag{A7}$$

From integral tables[5]

$$V[y^m] = m!e^{-y} \sum_{j=m+1}^{\infty} \frac{y^j}{j!} \qquad \text{for } m \geq 0 \tag{A8}$$

$$V[y^m] = m! \left(1-e^{-y} \sum_{j=0}^{m} \frac{y^j}{j!} \right) \qquad \text{for } m \geq 0 \tag{A8.1}$$

Let $x=y-y'$ in eq. (A6)

$$T[R(y)] = e^{-y} \int_0^y e^x R(x)dx \tag{A9}$$

From integral tables[5]

$$T[y^m] = (-1)^{m+1} m! \sum_{j=m+1}^{\infty} (-1)^j \frac{y^j}{j!} \qquad \text{for } m \geq 0 \tag{A10}$$

$$T[y^m] = (-1)^{m+1} m! \left(e^{-y} - \sum_{j=0}^{m} (-1)^j \frac{y^j}{j!} \right) \qquad \text{for } m \geq 0 \tag{A11}$$

Result #3

Consider the sum

$$S = \sum_{b_1=0}^{1} \cdots \sum_{b_k=0}^{1} x^{b_1+\cdots+b_k} (1-x)^{k-b_1-\cdots-b_k} . \tag{A12}$$

This sum can be rewritten

$$S = \sum_{b_1=0}^{1} x^{b_1} (1-x)^{1-b_1} \cdots \sum_{b_k=0}^{1} x^{b_k} (1-x)^{1-b_k} , \tag{A13}$$

but

$$\sum_{n=0}^{1} x^n (1-x)^{1-n} = 1 , \tag{A14}$$

thus

$$S = 1 . \tag{A15}$$

Consider the sum

$$S = \sum_{b_1=0}^{1} \cdots \sum_{b_k=0}^{1} x^{b_1+\cdots+b_k} (1-x)^{k-b_1-\cdots-b_k} \sum_{j} F(j) \delta_{b_j,1} \tag{A16}$$

Rearranging the sum over the b's as in eq. (A13) and using eq. (A14)

$$S = \sum_{j} F(j) \sum_{b_j=0}^{1} x^{b_j} (1-x)^{1-b_j} \delta_{b_j,1} \tag{A17}$$

or

$$S = x \sum_j F(j) \tag{A18}$$

Consider the sum

$$S = \sum_{b_1=0}^{1} \cdots \sum_{b_k=0}^{1} x^{b_1 + \cdots + b_k} (1-x)^{k-b_1-\cdots-b_k} \sum_j F(j)\delta_{b_j,1}$$

$$x \sum_\ell G(\ell)\delta_{b_\ell,1} \tag{A19}$$

Rearranging the sum over the b's as in eq. (A13) and using eq. (A14)

$$S = \sum_j F(j) \sum_{b_j=0}^{1} x^{b_j} (1-x)^{1-b_j} \delta_{b_j,1} \sum_\ell G(\ell) \tag{A20}$$

$$x \sum_{b_\ell=0}^{1} x^{b_\ell} (1-x)^{1-b_\ell} \delta_{b_\ell,1} , \text{ or}$$

$$S = x^2 \sum_j F(j) \sum_\ell G(\ell) \tag{A21}$$

Appendix B

Simple Example of an Unreliable Sample Variance

Consider particles trying to penetrate a thick shield. If the shield is any good, very few of the particles will penetrate the shield. Thus, it is possible to simulate a huge number of particles without any of the particles penetrating the shield. If none of the simulated particles penetrate the shield, the sample mean and the sample variance will both be zero. However, even for the best shields the probability of penetration is some number $\epsilon > 0$. The sample mean is thus in error, at the same time the sample variance is indicating zero error.

APPENDIX C

CONVERGENCE OF THE NEUMANN SERIES

FOR EQUATION (125)

Let x' be the coordinate along $\vec{\Omega}$, the direction of the neutron's flight. Let

$$\vec{r}' - \vec{r} = (x', y', z')$$

so that the neutron's position is described in a Cartesian coordinate system whose origin is at $\vec{r}$ and whose X' axis lies along $\vec{\Omega}$. Let

$$x' = vt'$$

$$\vec{r}' = \vec{r} + \vec{\Omega}x' = \vec{r} + \vec{v}t'$$

$$T(R,R') = \delta(\vec{v} - \vec{v}')\delta(y')\delta(z')T(R,\vec{r}',\vec{v}')$$

$$\sigma(R') = \sigma(\vec{r}',\vec{v}')$$

$$\epsilon_k(R',R_1,\cdots,R_k) = \epsilon_k(\vec{r}',\vec{v}',R_1,\cdots,R_k)$$

Thus eq. (125) may be rewritten:

$$M_r(R,w,t) = \int_0^t v\,dt'\,T(R,\vec{r}+\vec{v}t')\sigma(\vec{r}+\vec{v}t',\vec{v})$$

$$\times \sum_{k=1}^{\infty} \int dR_1 \cdots \int dR_k \epsilon_k(\vec{r}+\vec{v}t',\vec{v},R_1,\cdots,R_k)\Big\{H(ww_tw_\sigma w_{\epsilon_k}-w_{co})$$

$$\times \sum_{j=1}^{k} M_r(R_j,ww_tw_\sigma w_{\epsilon_k},t-t') + \frac{ww_tw_\sigma w_{\epsilon_k}}{w_e}\left(1-H(ww_tw_\sigma w_{\epsilon_k}-w_{co})\right)$$

$$\times \sum_{j=1}^{k} M_r(R_j,w_e,t-t')\Big\} + Q_r(R,w,t) \tag{C1}$$

Define the operator

$$\chi(t')G(R,w,t-t') = vT(R,\vec{r}+\vec{v}t')\sigma(\vec{r}+\vec{v}t',\vec{v})$$

$$\times \sum_{k=1}^{\infty}\int dR_1 \cdots \int dR_k \epsilon_k(\vec{r}+\vec{v}t',\vec{v},R_1,\cdots,R_k)\left\{ H(ww_t w_\sigma w_{\epsilon_k} - w_{co}) \right.$$

$$\times \sum_{j=1}^{k} G(R_j, ww_t w_\sigma w_{\epsilon_k}, t-t') + \frac{ww_t w_\sigma w_{\epsilon_k}}{w_e}\left(1 - H(ww_t w_\sigma w_{\epsilon_k} - w_{co})\right)$$

$$\times \sum_{j=1}^{k} G(R_j, w_e, t-t') \Bigg\} \tag{C2}$$

Thus

$$M_r(R,w,t) = \int_0^t dt_1 \chi(t_1) M_r(R,w,t-t_1) + Q_r(R,w,t) \tag{C3}$$

or replacing t by $t-t'$ and t_1 by t''

$$M_r(R,w,t-t') = \int_0^{t-t'} dt'' \chi(t'') M_r(R,w,t-t'-t'') + Q_r(R,w,t-t') \tag{C4}$$

Now use eqs. (C3)-(C4) to generate a Neumann series.

$$M_r(R,w,t) = \int_0^t dt_1 \chi(t_1)\left\{\int_0^{t-t_1} dt_2 \chi(t_2) M_r(R,w,t-t_1-t_2) \right.$$

$$+ Q_r(R,w,t-t_1) \Bigg\} + Q_r(R,w,t) \tag{C5}$$

Let $T_n = t_1+t_2+\cdots+t_n$ and substitute eq. (C4) into eq. (C5) for the next term in the series

$$M_r(R,w,t) = Q_r(R,w,t) + \int_0^t dt_1 \chi(t_1) Q_r(R,w,t-T_1)$$

$$+ \int_0^t dt_1 \chi(t_1)\int_0^{t-T_1} dt_2 \chi(t_2)\left\{\int_0^{t-T_2} dt_3 \chi(t_3) M_r(R,w,t-T_3)\right.$$

$$+ Q_r(R,w,t-T_2)\Bigg\} \tag{C6}$$

Repetitively substituting for $M_r(R,w,t-T_n)$ yields:

$$\dot{M}_r(R,w,t) = Q_r(R,w,t) + \sum_{q=1}^{\infty} \int_0^t dt_1 \chi(t_1) \int_0^{t-T_1} dt_2 \chi(t_2) \cdots$$

$$\times \int_0^{t-T_{q-1}} dt_q \chi(t_q) Q_r(R,w,t-T_q), \text{ the Neumann series.} \tag{C7}$$

Physically T_n is the time elapsed between a particle's birth ("zeroth event") and n^{th} event, t_n is the elapsed time between a particle's $n-1^{st}$ and n^{th} events, and finally, $t-T_n$ is the history time remaining after a particle's n^{th} event.

In the following theorem, unless otherwise specified, eqs. (C8)-(C15) hold for all possible arguments of the bounded function. All maxima are taken over the full range of all variables. For instance,

$$max(A,B,C) = max(A(X), B(Y), C(Z))$$
$$\text{all } X,Y,Z$$

where X, Y, Z may be n-tuples of vectors.

Theorem C. The Neumann series (C7) converges if there exists a uniform bound, B>1, such that:

1) The particle starting weight is bounded.

$$w < B \tag{C8}$$

2) All weight multipliers are bounded.

$$w_t < B, \ w_\sigma < B, \text{ and } w_{\varepsilon_k} < B \tag{C9}$$

3) The exit weight from a Russian Roulette game is bounded.

$$w_e < B \text{ and } w_{co}(R') < w_e(R', R_1, \cdots, R_k) \tag{C10}$$

4) The speed is bounded.

$$v<B \tag{C11}$$

5) The macroscopic total cross section is bounded, thus

$$T(R,\vec{r}+\vec{v}t')\sigma(\vec{r}+\vec{v}t',\vec{v})<B \tag{C12}$$

6) The number of particles departing an event is bounded.

$$k<B \tag{C13}$$

7) The source function $Q_r(R,w_0,t_0)$ is bounded

$$Q_r(R,w_0,t_0)<B \quad \text{for} \quad 0 \leq w_0 \leq \max(ww_tw_\sigma w_{\varepsilon_k},w_e,w) \tag{C14}$$

$$0 \leq t_0 \leq t$$

Before proving this theorem, let us discuss conditions 1) - 7). Conditions 1), 2), and the first part of 3) ensure that the particle weights are always finite. The second part of 3) simply says that in Russian Roulette, the exit weight is larger than the cutoff. The speed of light bounds any particle velocity, so condition 4) holds. Condition 5) is satisfied for any real problems. (For delta function cross sections condition 5) could be altered to

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} T(R,\vec{r}+\vec{v}t')\sigma(\vec{r}+\vec{v}t',\vec{v})vdt'<2\varepsilon Bv,$$

so that the expected number of next collisions in any time interval is bounded.) Condition 6) should always hold in practice, although it is somewhat stronger than the condition actually required:

$$\sum_{k=1}^{\infty} kA_k<B.$$

That is, the mean number of particles departing an event is bounded.

The source term $Q_r(R,w,t)$ must be bounded because it is the first term of the Neumann series. (The limitation on w_0 in eq. (C14) is required since we desire to bound both $Q_r(R,ww_t w_\sigma w_{\varepsilon_k}, t-t')$ and $Q_r(R,w_e,t-t')$ in order to say $\chi(t')Q_r(R,w,t-t')$ is bounded.) Q_r will be bounded if conditions 1) - 6) are met, M_ℓ is bounded for $\ell<r$, and the moments of the event score are bounded, i.e.,

$$\int ds\, p(P,P',s)s^\ell < S_{max}, \text{ for } \ell \leq r.$$

For example, see eqs. (127) and (130).

Thus we proceed in the following manner. Q_1 is bounded and thus by theorem C, M_1 is bounded. But M_1 bounded implies Q_2 is bounded, and thus by theorem C, M_2 is bounded. But M_1 and M_2 bounded imply Q_3 is bounded, and so forth.

Proof of Theorem C

Consider a bounded function

$$G(R,w_0,t_0) < G_m \quad \text{for} \quad \begin{array}{c} 0 \leq w_0 \leq \max(ww_t w_\sigma w_{\varepsilon_k}, w_e) \\ 0 \leq t_0 \leq t \\ G_m > 1 \end{array} \quad (C15)$$

Note that $H(ww_t w_\sigma w_{\varepsilon_k} - w_{co}) \leq 1$ and that, since by assumption 3) $w_{co} \leq w_e$,

$$\frac{ww_t w_\sigma w_{\varepsilon_k}}{w_e}\left(1 - H(ww_t w_\sigma w_{\varepsilon_k} - w_{co})\right) \leq 1$$

Using these facts and eq. (C2)

$$\chi(t')G(R,w,t-t') < vT(R,\vec{r}+\vec{v}t')\sigma(\vec{r}+\vec{v}t',\vec{v})$$

$$\times \sum_{k=1}^{\infty} \int dR_1 \cdots \int dR_k \varepsilon_k(\vec{r}+\vec{v}t',\vec{v},R_1,\cdots,R_k)2kG_m,$$

(C16)

where the 2k comes from the two sums on j of eq. (C2). By eqs. (82), (C11), and (C12)

$$\chi(t')G(R,w,t-t') < 2B^2 G_m \sum_{k=1}^{\infty} kA_k(\vec{r}+\vec{v}t',\vec{v}).$$

At this point we insist

$$\sum_{k=1}^{\infty} kA_k(\vec{r}+\vec{v}t',\vec{v}) < B_1, \text{ where } B_1 > 1,$$

which will certainly be true if k is bounded, in view of eq. (83).
Thus

$$\chi(t')G(R,w,t-t') < 2B^2 B_1 G_m \triangleq C$$

(C17)

Now let's look at the qth term of the Neumann series of eq. (C7).

$$N_q = \int_0^t dt_1 \chi(t_1) \int_0^{t-T_1} dt_2 \chi(t_2) \cdots \int_0^{t-T_{q-1}} dt_q \chi(t_q) Q_r(R,w,t-T_q)$$

Since the functions Q_r and 1 both meet conditions (C15) then by (C17)

$$N_q < C^q \int_0^t dt_1 \int_0^{t-T_1} dt_2 \cdots \int_0^{t-T_{q-1}} dt_q \triangleq C^q \tau$$

Recall that $T_n = t_1 + \cdots + t_n$ and let

$y_n = t - T_n = t - T_{n-1} - t_n$, then

$$\tau = \int_0^t dy_1 \int_0^{y_1} dy_2 \int_0^{y_2} dy_3 \cdots \int_0^{y_{q-1}} dy_q$$

$$\tau = \int_0^t dy_1 \cdots \int_0^{y_{q-3}} dy_{q-2} \int_0^{y_{q-2}} y_{q-1} dy_{q-1}$$

$$\tau = \int_0^t dy_1 \cdots \int_0^{y_{q-4}} dy_{q-3} \int_0^{y_{q-3}} \frac{1}{2!} \, y_{q-2}^2 \, dy_{q-2}$$

$$\tau = \int_0^t \frac{1}{(q-1)!} \, y_1^{q-1} \, dy_1 = \frac{t^q}{q!}$$

Hence the qth term of the Neumann series is

$$N_q < C^q t^q / q!$$

Thus the Neumann series is less than the series for exp (Ct) and

hence Neumann series (C7) converges. This completes the proof of

theorem C.

APPENDIX D

Approximation of a Bounded n-Dimensional Region

by Convex Subregions

Theorem   Any bounded n-dimensional set possessing a volume as defined
by the Peano-Jordan content, can be approximated arbitrarily closely
(in volume) by a finite number of closed convex, n-dimensional
intervals which are separate.

This theorem can be demonstrated using well-known results from the
theory of volume and measure in mathematics. Here the term closed
interval is defined as the Cartesian product of n one-dimensional
closed intervals -- namely, a closed interval I is defined as the set
of points $P = (x_1, x_2, \cdots, x_n)$ in n-space satisfying the inequalities

$$a_i \leq x_i \leq b_i \qquad 1 \leq i \leq n.$$

By separate it is meant that any two of the closed intervals have only
boundary points in common.

The theorem follows immediately from the development of the
theory of content.[9] A bounded set has content if the inner content is
equal to the outer content, and each of these latter two quantities is
approximated to any desired degree of accuracy by the volume of a finite
sum of separate intervals. (The volume of an n-dimensional interval I
is given by the product $\prod_{i=1}^{n}(b_i - a_i)$, as expected.) Since an interval

is convex, the result follows. Clearly the approximation is not unique - in fact, there are infinitely many ways to make the approximation.

The above result states that whenever a bounded set in n-space has a volume as defined by the content, then the approximation by convex intervals exists. This definition of volume, although superseded by the definition of measure (Lebesque) for purely mathematical reasons in the modern theory of integration, is entirely adequate for all practical applications. For the latter, any region occurring in a real problem will have the same volume measured by the two definitions, content and measure. It is true that unbounded regions can arise where the above result does not hold, but usually in practice unbounded regions either can be approximated by a finite set of convex subregions or else the unbounded region can be replaced, without any appreciable loss of accuracy, by a bounded region.

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