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TRANSIENT ELASTODYNAMIC RESPONSE OF A CIRCULAR CRACK IN A THICK PLATE UNDER TORSION

BY

E. P. Chen*
Systems Safety Technology
Division 4442
Sandia Laboratories
Albuquerque, New Mexico 87185

ABSTRACT

The elastodynamic response of a thick plate under torsion is considered in this study. A penny-shaped crack is assumed to exist in the center of the plate such that the problem is axisymmetric in nature. The crack is pressurized suddenly along its surfaces resulting in transient conditions. This problem is also equivalent to that of sudden appearance of a crack in the loaded plate. Hankel and Laplace transforms are used to reduce the problem to the solution of a pair of dual integral equations. A numerical Laplace inversion routine is used to recover the time-dependence of the solution. The dynamic stress intensity factor is determined and its dependence on time and geometry is discussed.

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*Formerly of Institute of Fracture and Solid Mechanics, Lehigh University, Bethlehem, Pennsylvania 18015

INTRODUCTION

One of the important problems in design concerns with the effect of rapid loading on a structural member. The action is transmitted through the member in the form of stress waves. If the structure contains flaws, this action could lead to the propagation of such flaws and eventually to the failure of the structural component at a load level well below that under static considerations.

The effect of transient loads on structures containing flaws has been treated by many authors, e.g., [1-5]. The axisymmetric torsion problem was considered by Embley and Sih [2] for an elastic medium of infinite extent. In practical problems, however, the boundary of the specimen also interacts with the stress waves. Depending upon the loading conditions on the bounding surfaces, the load transfer to the crack may be lowered or intensified. This is the subject of investigation of the present paper.

Laplace and Hankel transform techniques are used to reduce this elastodynamic problem to the solution of a pair of dual integral equations. The solution is then obtained in terms of a Fredholm integral equation of the second kind. A numerical Laplace inversion scheme is used to recover the time dependence of the solution. Numerical results on the dynamic stress intensity factor are obtained and presented in a graphical form. The dependence of the stress intensity factor on the time and geometry is discussed.

FORMULATION OF THE PROBLEM

Consider the thick plate of height 2h as shown in Figure 1. The plate is assumed to be infinite in extent in the x and y directions. Let a circular crack of radius a be situated at the center of the mid-plane of the plate and a set of cylindrical coordinates (r,θ,z) is attached to the center of the crack. The displacement components in the r, θ and z directions are designated as u_r, u_θ and u_z, respectively. In view of the axisymmetry of the problem, the displacements may be taken as

$$u_r = u_z = 0, \ u_\theta = u_\theta(r,z,t) \tag{1}$$

Through the strain-displacement relationship and the Hooke's Law, the stress field in the plate is found as

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \ \tau_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} \tag{2}$$

while other stress components vanish everywhere. In equations (2), μ denotes the shear modulus of elasticity of the plate material. Substituting equations (2) into the nontrivial equation of motion renders the governing differential equation of the problem:

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u_\theta}{\partial t^2} \tag{3}$$

where $c^2 = \mu/\rho$ is the shear wave speed of the material with ρ being the mass density. It is intended to solve equation (3) under the following two sets of boundary conditions:

(A) Free surface condition

$$\tau_{\theta z}(r,\pm h,t) = 0 \qquad (4)$$

$$\tau_{\theta z}(r,o,t) = -\frac{\tau_o r}{a} H(t) \qquad 0<r<a \qquad (5)$$

$$u_\theta(r,o,t) = 0 \qquad r\geq a \qquad (6)$$

(B) Clamped surface condition

$$u_\theta(r,\pm h,t) = 0 \qquad (7)$$

$$\tau_{\theta z}(r,o,t) = -\mu \frac{wr}{ha} H(t) \qquad 0<r<a \qquad (8)$$

$$u_\theta(r,o,t) = 0 \qquad r\geq a \qquad (9)$$

in which τ_o and w are constants having the dimensions of stress and displacement, respectively, while $H(t)$ represents the Heaviside unit step function in time. The set of boundary conditions (A) refers to either the problem of sudden twisting of the crack while the bounding surfaces are traction free or the sudden appearance of a crack in a plate whose surfaces are twisted by the stress of magnitude $\tau_o r/a$. The problem of sudden twisting of the crack while the bounding surfaces are clamped and that of the sudden appearance of a crack in a plate whose surfaces are twisted by an amount wr/a are covered by the set of boundary conditions (B).

SOLUTION PROCEDURE

Define a Laplace transform pair by

$$f^*(p) = \int_0^\infty f(t)e^{-pt}dt, \quad f(t) = \frac{1}{2\pi i}\int_{Br} f^*(p)e^{pt}dp \tag{10}$$

where Br denotes the Bromwich path of integration. Applying (10) and Hankel transform to equation (3) yields

$$u_\theta^*(r,z,p) = \int_0^\infty [A(s,p)e^{-\gamma z} + B(s,p)e^{\gamma z}]J_1(rs)ds \tag{11}$$

in which $\gamma = (s^2 + \frac{c^2}{p^2})^{1/2}$ and J_1 is the first order Bessel function of the first kind. In equation (11), A(s) and B(s) are the unknown functions to be determined from the boundary conditions. For clarity, the two sets of boundary conditions will be dealt with separately.

(A) *Free surface problem*

The satisfaction of the boundary conditions (4), (5) and (6) leads to a pair of dual integral equations of the following form:

$$\int_0^\infty c(s,p)J_1(rs)ds = 0 \qquad r \geq a \tag{12}$$

$$\int_0^\infty sF(s,p)c(s,p)J_1(rs)ds = \frac{\tau_0 r}{\mu p a} \qquad r < a \tag{13}$$

where

$$F(s,p) = \gamma(1 - e^{-2\gamma h})/[s(1 + e^{-2\gamma h})] \tag{14}$$

and the original unknowns A(s,p) and B(s,p) are related to the new one C(s,p) through

$$A(s,p) = (1 + e^{-2\gamma h})^{-1}\ C(s,p) \tag{15}$$

$$B(s,p) = e^{-2\gamma h}\ A(s,p) \tag{16}$$

Following a procedure developed by Copson [6], a solution to equations (12) and (13) is

$$C(s,p) = \sqrt{\frac{2sa}{\pi}}\ \frac{2\tau_o a^2}{3\mu p} \int_0^1 \sqrt{\xi}\ \Phi(\xi,p) J_{3/2}(sa\xi) d\xi \tag{17}$$

with $\Phi(\xi,p)$ being governed by a Fredholm integral equation of the second kind:

$$\Phi(\xi,p) + \int_0^1 \Phi(\eta,p) M(\xi,\eta,p) d\eta = \xi^2 \tag{18}$$

The symmetric kernel $M(\xi,\eta,p)$ is given by

$$M(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^\infty s[F(\tfrac{s}{a},p) - 1] J_{3/2}(s\xi) J_{3/2}(s\eta) ds \tag{19}$$

(3) Clamped surface problem

Enforcing equations (7), (8) and (9) renders the following pair of dual integral equations:

$$\int_0^\infty D(s,p) J_1(rs) ds = 0 \qquad r \geq a \tag{20}$$

$$\int_0^\infty sG(s,p)D(s,p)J_1(rs)ds = \frac{wr}{hpa} \qquad r<a \tag{21}$$

where

$$G(s,p) = \gamma(1 + e^{-2\gamma h})/[s(1 - e^{-2\gamma h})] \tag{22}$$

$$A(s,p) = (1 - e^{-2\gamma h})^{-1} \; D(s,p) \tag{23}$$

$$B(s,p) = - e^{-2\gamma h} \; A(s,p) \tag{24}$$

Following the same procedure as before, it is found that

$$D(s,p) = \sqrt{\frac{2sa}{\pi}} \frac{2wa^2}{3hp} \int_0^1 \sqrt{\xi} \, \Psi(\xi,p)J_{3/2}(sa\xi)d\xi \tag{25}$$

$$\Psi(\xi,p) + \int_0^1 \Psi(\eta,p)N(\xi,\eta,p)d\eta = \xi^2 \tag{26}$$

$$N(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^\infty s[G(\tfrac{s}{a},p) - 1]J_{3/2}(s\xi)J_{3/2}(s\eta)ds \tag{27}$$

Once Φ and Ψ are determined, the unknowns C(s,p) and D(s,p) are known and through which A(s,p) and B(s,p) can be obtained. Once A(s,p) and B(s,p) are found, the stress and displacement field in the Laplace transform plane are determined. The question now is how to bring these answers back to the physical plane.

CRACK BORDER DYNAMIC STRESS DISTRIBUTION

The unstable crack growth is controlled by the intensity of the stress field near the crack front, measured by the stress intensity factor. Thus, it is not necessary to determine the stress distribution everywhere in the plate. Following [3], the dynamic stress intensity factor may be determined by first obtaining the asymptotic stresses near the crack border in the Laplace transform domain and then performing a Laplace inversion.

By substituting the unknowns $A(s,p)$ and $B(s,p)$ into equation (11) and making use of equation (2) yield the stresses in integral form. Expanding the integrand asymptotically near the crack border and then carrying out the integration renders

$$\tau^{*}_{\theta z}(r_1,\theta_1,p) = \frac{k^{*}_{3}(p)}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} + O(r_1^{0}) \tag{28}$$

$$\tau^{*}_{r\theta}(r_1,\theta_1,p) = \frac{k^{*}_{3}(p)}{(2r_1)^{1/2}} \sin \frac{\theta_1}{2} + O(r_1^{0}) \tag{29}$$

where

$$k^{*}_{3}(p) = \begin{cases} \dfrac{4}{3\pi}\, \tau_0 \sqrt{a}\, \dfrac{\Phi(1,p)}{p} & \text{for case (A)} \\[2em] \dfrac{4}{3\pi}\, \dfrac{\mu w}{h}\, \sqrt{a}\, \dfrac{\Psi(1,p)}{p} & \text{for case (B)} \end{cases} \tag{30}$$

and r_1 and θ_1 are a set of polar coordinates with its origin at the crack border. Note that the familiar inverse square-root singularity is preserved in the dynamic case. Since r_1 and θ_1 are independent of p, only $k_3^*(p)$ will be affected by the Laplace inversion. Applying equations (10) to equations (28) and (29) yields the crack border stress field:

$$\tau_{\theta z}(r_1,\theta_1,t) = \frac{k_3(t)}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} + O(r_1^0) \tag{31}$$

$$\tau_{r\theta}(r_1,\theta_1,t) = \frac{k_3(t)}{(2r_1)^{1/2}} \sin \frac{\theta_1}{2} + O(r_1^0) \tag{32}$$

in which the dynamic stress intensity factor $k_3(t)$ is given by

$$k_3(t) = \frac{1}{2\pi i} \int_{Br} k_3^*(p)e^{pt}dp \tag{33}$$

In order to determine $k_3(t)$, equations (18) and (26) have to be solved first to obtain $\Phi(1,p)$ and $\Psi(1,p)$. Then a numerical scheme in [3] may be used to evaluate the integral in equation (33).

RESULTS AND DISCUSSION

Figure 2 shows a plot of the normalized dynamic stress intensity factor $\bar{k}_3 = 3\pi k_3(t)/(4\tau_0\sqrt{a})$ against the normalized time variable $T = ct/a$ for a/h ratios of 0.5, 1.0 and 2.0. This figure refers to the free surface condition of the plate. It is seen that $\bar{k}_3$ increases with increasing T, reaching a peak and then oscillates about its static value. This is typical of transient problems. The dynamic stress intensity factor increases with increasing a/h ratio. For a/h = 2.0, the peak $\bar{k}_3 \simeq 1.21$ which means a 21% increase from the static solution for an infinite medium. The opposite trend is observed for the clamped boundary case in Figure 3. There the normalized dynamic stress intensity factor $\bar{k}_3 = 3\pi h k_3(t)/(4\mu w\sqrt{a})$ decreases with increasing a/h ratio. The $\bar{k}_3$ versus T behavior is similar to that in the free surface case. Undoubtedly, these phenomena are associated with the diffraction characteristics of the waves.

In summary, the dynamic torsion problem of a thick plate containing a penny-shaped crack has been treated in this study. It is found that the maximum values of the dynamic stress intensity factor are higher than their static counterparts. Also, depending on the loading conditions on the plate surfaces, the stress intensity can either increase or decrease with the crack radius to plate thickness ratio, a/h.

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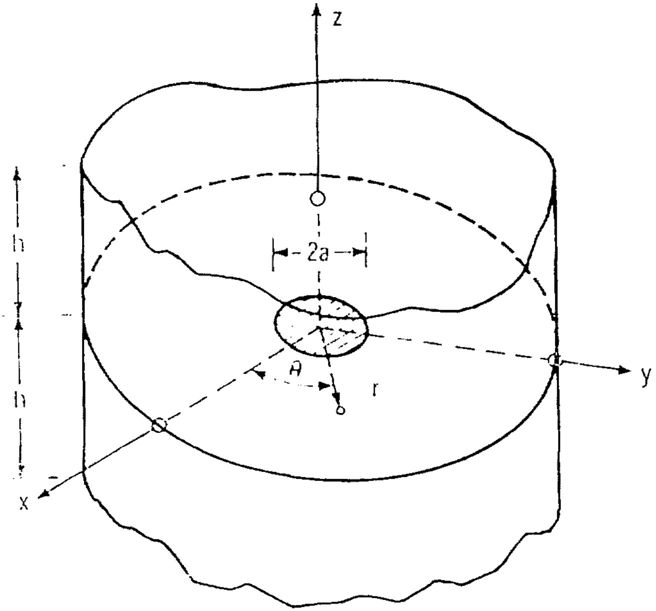

Figure 1. The geometry of the problem.

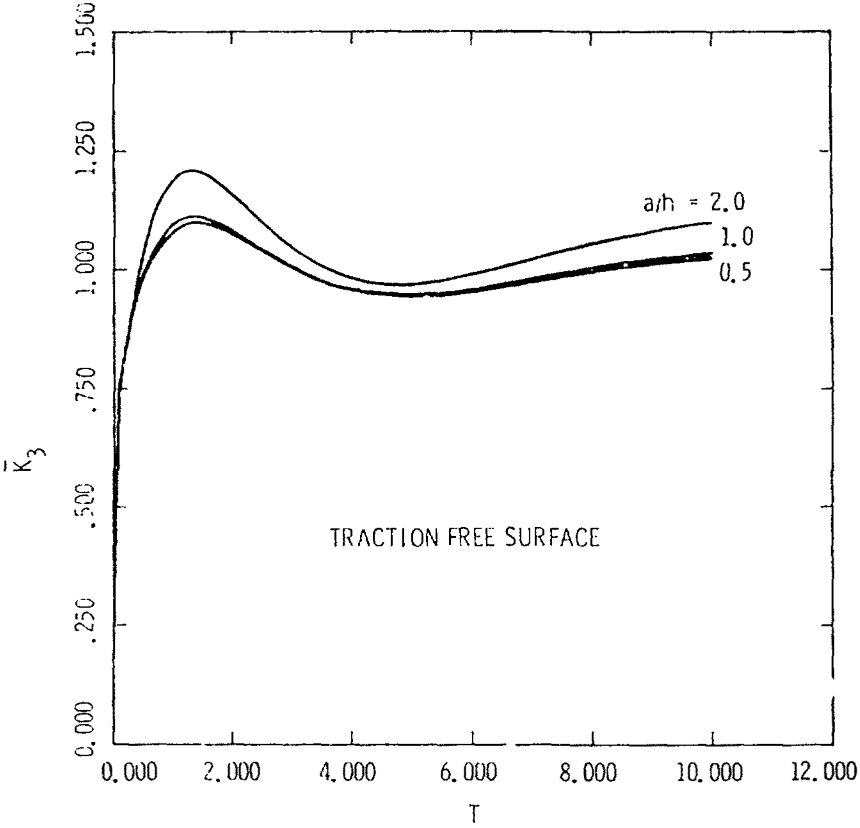

Figure 2. Normalized stress intensity factor as a function of time for traction free surface conditions

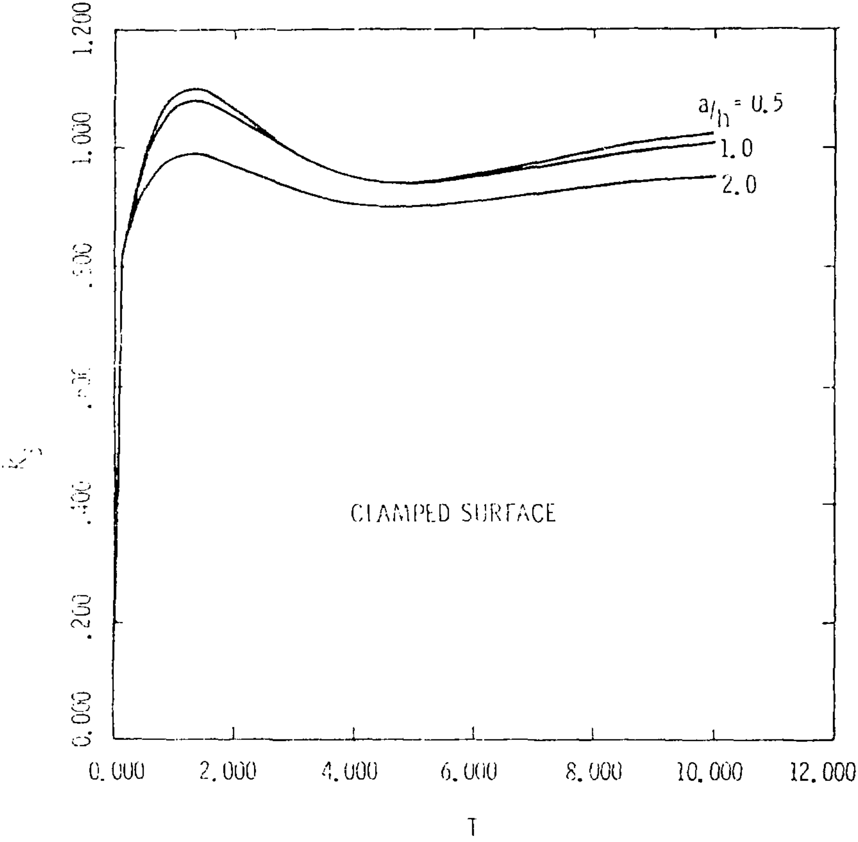

Figure 3. Normalized stress intensity factor as a function of time for clamped surface conditions