TITLE: **ON THE MINIMUM WEIGHT STEINER TRIANGULAR TILING PROBLEM**

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On the Minimum Weight Steiner Triangular Tiling problem

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Abstract

In this paper, we introduce the *Minimum Weight Steiner Triangular Tiling* problem, which is a generalization of the *Minimum Weight Steiner Triangulation*. Contrary to the conjecture of Eppstein that the Minimum Weight Steiner Triangulation of a convex polygon has the property that the Steiner points all lie on the boundary of the polygon [Epp94], we show that the Steiner points of a Minimum Weight Steiner Triangular Tiling could lie in the interior of a convex polygon.

1   Introduction

Triangulating a set of planar points has many applications in practice, for instance, spatial data processing, mesh generation and network design. There are many optimization criteria in obtaining "good" triangulations. Among them, the *Minimum Weight Triangulation (MWT)*, which minimizes the sum of edge lengths, is the most longstanding open problem in computational geometry. It is neither known to be NP-hard nor polynomially solvable, and moreover, no approximation algorithm is known to have approximation ratio better than $O(\log n)$. In [Epp94], the *Minimum Weight Steiner Triangulation (MWST)*, which allows Steiner points, is studied. Although the complexity of MWST is also not known (in fact not known even for convex polygons), Eppstein proposes an algorithm which approximates MWST with a constant factor.

In this paper, we introduce the *Minimum Weight Steiner Triangular Tiling (MWSTT)* problem, which is a generalization of the MWST. Given a set of points S, the *Minimum Weight Steiner Triangular Tiling* is defined as the decomposition of the interior of the convex hull of S, with the help of some Steiner points, into non-overlap triangles such that the sum of the edge lengths is minimized. The major difference between these two problems is the way collinear points are treated. Clearly, for MWSTT, we allow Steiner points to be on the boundary of a triangle. We feel that this problem is also interesting

since tiling also has applications in practice [GS87].

In [Epp94], Eppstein conjectures that the MWST of a convex polygon has the property that the Steiner points all lie on the boundary of the polygon. It is still not known whether his conjecture is true or not. We show below that for MWSTT, this conjecture is not true. An example and its proof is shown in the following section.

2 The example

Before we get into the detail, let us make some clarifications. A Steiner point which is not a vertex of any triangle in the tiling is called a *dull* Steiner point. We do not allow dull Steiner points when dealing with MWSTT since a dull Steiner point has no effect on the weight of a tiling. With this rule, clearly, the MWSTT of a triangle is itself. We use $(a_1, ..., a_n)$ to denote a convex polygon with vertices $a_1, ..., a_n$ in counterclockwise order and we use $|a_1 a_2|$ to denote the Euclidean distance between a_1, a_2. Now, we show following lemma on the MWSTT of a convex quadrilateral.

Lemma 1. The Steiner points in the MWSTT of a convex quadrilateral must lie on the boundary of the convex quadrilateral.

Proof. Suppose in the MWSTT of a convex quadrilateral there are some Steiner points within the interior of it. Take the shortest diagonal (x, y) of the convex quadrilateral, the shortest path from x to y and visiting some Steiner points is at least as long as that of (x, y). Therefore the weight of such a MWSTT is not the minimum, which is a contradiction. $\square$

Second, we show that the MWSTT of the convex quadrilateral of Fig. 1 has the following property.

Lemma 2. The only Steiner point of the MWSTT of the convex quadrilateral (a, b, c, d) in Fig. 1 (I) is f, which is the midpoint of the edge (a, d).

Proof. Following Lemma 1, we only need to consider the Steiner points which are on the boundary of the convex quadrilateral (a, b, c, d). Let the sum of the edge lengths of the convex quadrilateral be $C = |ab| + |bc| + |cd| + |da|$. The weight of this so claimed MWSTT is $C + |bf| + |cf| = C + 4\delta$. δ can be set very small and throughout this paper we set $\delta = 0.1$. We show that as long as there are some Steiner points on $(a, b), (b, c)$ or (c, d) the weight of a Steiner tiling is larger than $C + 4\delta$. We have several cases and for simplicity we ignore symmetric cases. (Since most cases are similar, we only show one typical case in the figure. The readers are encouraged to draw the corresponding figures and follow the proof.)
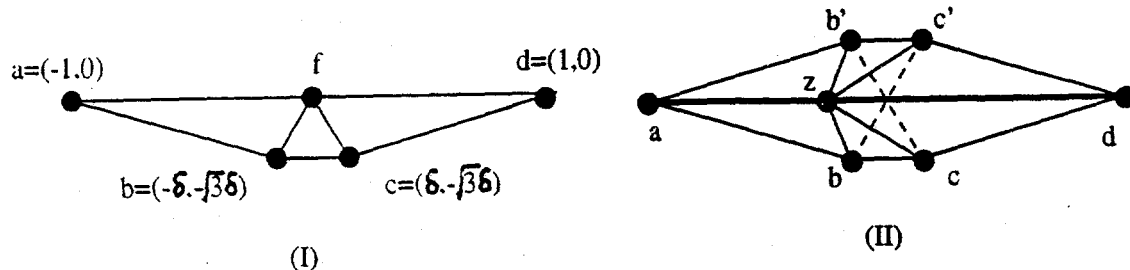
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Figure 1: The MWSTT of a convex quadrilateral.

(1) a is connected with some Steiner points on (b,c) or (c,d). This Steiner tiling has weight at least $C + |ab| = \sqrt{(1-\delta)^2 + 3\delta^2}$. It is clear that this is greater than $C + 4\delta$.

(2) x is the highest Steiner point on (a,b) and y is the highest Steiner point on (c,d) and (x,y) is an edge in the tiling. There must be some $z \in (a,d)$ such that $\triangle xyz$ is a triangle in the Steiner tiling. The weight of this Steiner tiling is at least $C + |xy| + |yz| + |xz|$. By triangular inequality, $C + |xy| + |yz| + |xz| > C + 2|xy| > C + 2|bc| = C + 4\delta$.

(3) There is no Steiner point on (a,b), but there is some Steiner points z on (b,c). Clearly there are some edges with one endpoint on (a,d) connecting b, c and z. The weight of this Steiner tiling is at least $C + |gf| + |gf| + |gf|$, which is equal to $C + 3\sqrt{3}\delta$ and is greater than $C + 4\delta$ (g is the midpoint of (b,c) and $|gf|$ is the vertical distance between $(a,d), (b,c)$).

(4) There is no Steiner point on (a,b) and (b,c). Clearly there must be a Steiner point on (a,d) such that $\triangle bcz$ is a triangle in the Steiner tiling. The weight of this Steiner tiling is at least $C + |bz| + |cz|$. We reflect the convex quadrilateral with line (a,d) being a mirror (Fig. 1 (II)). By triangular inequality, $|b'z| + |cz| > |b'c|$, $|c'z| + |bz| > |bc'|$. This implies that $|bz| + |cz| > 2|bf| = 4\delta$ and consequently, $C + |bz| + |cz| > C + 4\delta$.

(5) There is no Steiner point on $(a,b), (b,c), (c,d)$ and there are some Steiner points on (a,d). Clearly in this case any Steiner point w on (a,d) which is not connected to both b and c can be eliminated to decrease the weight of the tiling. If z is the only Steiner point on (a,d) then we can show, by mimicking Step (4), that the weight of the tiling is minimized when $z = f$. $\square$

Now we construct a pentagon (a,b,c,d,e) based on the convex quadrilateral (a,b,c,d). Note that $\angle bad$ is equal to $\angle def$ (Fig. 2 (I)). We have the following result regarding the

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MWSTT of this pentagon.

Theorem 3. The only Steiner point of the MWSTT of the pentagon (a, b, c, d, e) in Fig. 2 (I) is f, which is the midpoint of the diagonal (a, d). Moreover, the internal edges of the MWSTT are $(a, f), (b, f), (c, f)$ and (d, f).

Proof. Let the sum of the edge lengths of the pentagon be $C' = |ab| + |bc| + |cd| + |de| + |ae|$. The weight of the so claimed MWSTT is $C' + |af| + bf| + |cf| + |df| = C' + 2 + 4\delta$. First we prove that the Steiner points can not lie in the interior of $\triangle abe$ and (a, b, c, d). Suppose there are some Steiner points in $\triangle ade$ and in the convex quadrilateral (a, b, c, d). We claim that the weight of the tiling is greater than $C' + 2 + 4\delta$. This is due to the fact that if (a, d) is an edge of the tiling then following Lemma 2 the MWSTT of (a, b, c, d) is obtained by introducing f; otherwise, if (a, d) is not an edge of the tiling then consider the shortest path between b and e and visiting some Steiner points, the weight of the tiling is at least $C' + |be| > C' + |ef|$ which is greater than $C' + 2 + 4\delta$.

Let the vertical distance from a to (d, e) be $|aa^*|$ and the vertical distance from b to (d, e) be $|bb^*|$, we have $|aa^*| = \frac{2}{\sqrt{1 + 3\delta^2/(1-\delta)^2}} > 1.9$ and $|bb^*| = (\frac{1+\delta}{1-\delta})\sqrt{(1-\delta)^2 + 3\delta^2} > 1$. Now we prove that if the Steiner points all lie on the boundary of the pentagon (a, b, c, d, e) then the resulting tiling has a weight greater than $C' + 2 + 4\delta$. Clearly if (a, d) is an edge of the tiling then we are done following Lemma 2. Similarly, if e connects to either b, c or some Steiner points not on $(a, e), (d, e)$ then the weight of the resulting tiling is greater than $C' + 2 + 4\delta$. Therefore, we will not consider these cases anymore. We again go through the case analysis and ignore some symmetric cases.

(1) If there are two Steiner points $x \in (a, b), y \in (d, e)$ such that (x, y) is an edge in the tiling and there are two Steiner points w on (a, e) and z on either (b, c) or (c, d) such that $\triangle xyw, \triangle xyz$ are triangles in the tiling (Fig. 2 (II)). Clearly the weight of this tiling is at least $C' + |xy| + |xw| + |yw| + |yz| + |xz| > C' + 3|xy|$, which is greater than $C' + 3|bb^*|$. Moreover $C' + 3|bb^*| > C' + 2 + 4\delta$.

(2) If there are two Steiner points $x \in (a, b), y \in (d, e)$ such that (x, y) is an edge in the tiling and there are two Steiner points w on (a, x) and z on (x, b) such that $\triangle xyw, \triangle xyz$ are triangles in the tiling. Similar to the argument in Case (1), the weight of this tiling is at least $C' + |xy| + |yw| + |yz| > C' + 3|bb^*|$, which is greater than $C' + 2 + 4\delta$.

With respect to a combination of Case (1) and (2), the case when w is on (a, x) and z is on either (b, c) or (c, d) and the case when w is on (a, e) and z is on (x, b), can be all dealt with similarly.

(3) If there are two Steiner points $x \in (b, c), y \in (d, e)$ such that (x, y) is an edge in the tiling and there are two Steiner points w on (a, e) or (a, b) and z on (c, d) such

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$e=(0,(1-\delta)/\sqrt{3\delta})$

$a=(-1.0)$ f $d=(1,0)$

$b=(-\delta,-\sqrt{3\delta})$ $c=(\delta,-\sqrt{3\delta})$

(I)

e

w y

a b^*

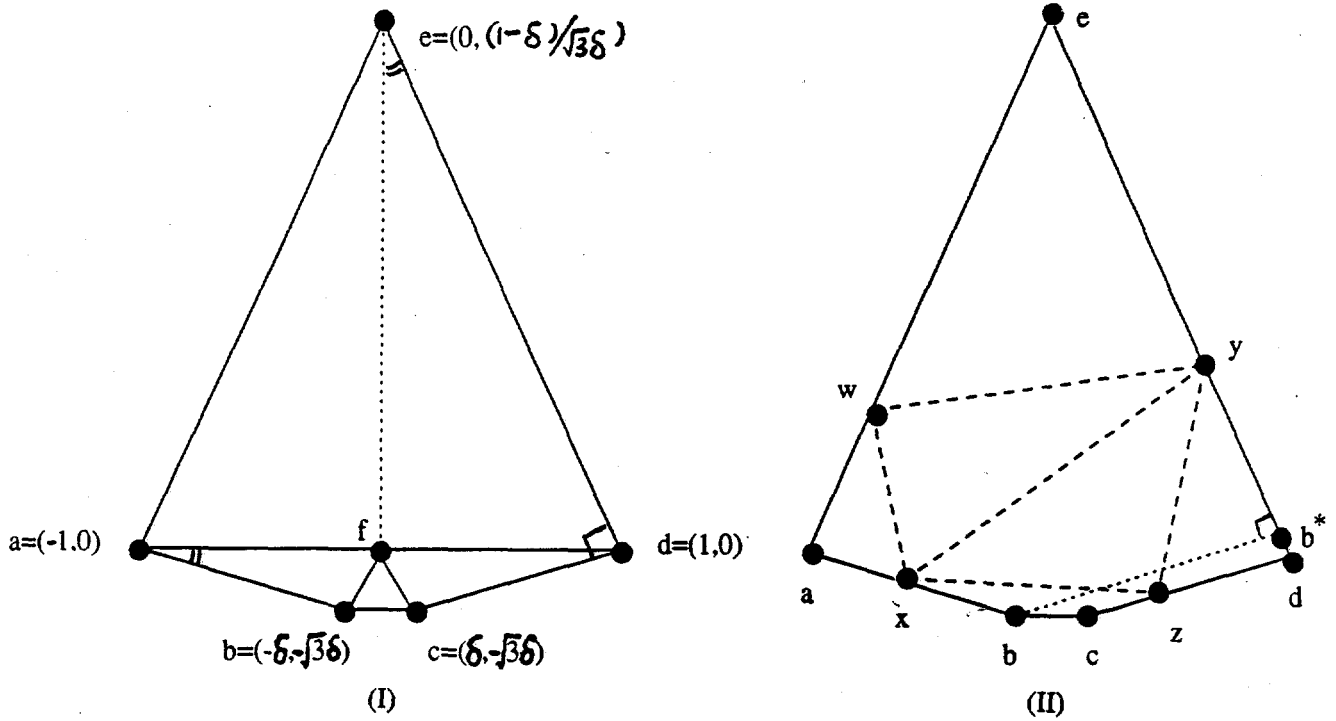x d

b c z

(II)

Figure 2: The MWSTT of a pentagon.

that $\triangle xyw, \triangle xyz$ are triangles in the tiling. The weight of this tiling is at least $C' + |xy| + |xw| + |yw| + |yz| + |xz| > C' + 3|xy|$, which is greater than $C' + 3|cd|$. Moreover $C' + 3|cd| > C' + 2 + 4\delta$ since $|cd| > 0.9$.

(4) If there are two Steiner points $x \in (b,c), y \in (d,e)$ such that (x,y) is an edge in the tiling and there are two Steiner points w on (b,x) and z on (x,c) such that $\triangle xyw, \triangle xyz$ are triangles in the tiling. The weight of this tiling is at least $C' + |xy| + |yw| + |yz| > C' + 3|cd|$, which is greater than $C' + 2 + 4\delta$ since $|cd| > 0.9$.

Again, with respect to a combination of Case (3) and (4), the case when w is on (b,x) and z is on (c,d) and the case when w is on (a,e) or (a,b) and z is on (x,c), can be dealt with similarly.

(5) For the above four cases, if there are some Steiner points on (a,d) as well, we can show similarly that the weight of the resulting Steiner tiling is greater than $C' + 2 + 4\delta$.

(6) Finally, if there is no Steiner point on (a,b), (b,c) and (a,d) then there must be some Steiner points either on $(a,e), (d,e)$ or (c,d) which connect a, b, c. The weight of such a tiling, is clearly greater than either $C' + |aa^*| + |bb^*| + |bc|$ or $C' + |bb^*| + 2|cd|$, which are both greater than $C' + 2 + 4\delta$.

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From the above discussions, we see that the Steiner points can only lie on (a, d). Following Lemma 2, the MWSTT of (a, b, c, d) is obtained by introducing only one Steiner point f, which is the midpoint of (a, d). Together with $(a, e), (d, e)$, the MWSTT of (a, b, c, d) forms the MWSTT of the pentagon (a, b, c, d, e). $\square$

Now we can claim that with respect to the pentagon in Fig. 2, the only Steiner point of the MWSTT lies in the interior of the pentagon.

3   Closing Remark

It seems necessary to prove some property of the MWSTT of a convex polygon before having an algorithm. Parallel to Eppstein's conjecture on the MWST of a convex polygon, we feel that the Steiner points of the MWSTT of a convex polygon lie either on the boundary or diagonals of the polygon. We make this as a conjecture to conclude this paper.

The Steiner points of the MWSTT of a convex polygon lie either on the boundary or on the diagonals of the polygon.

References

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Date: Mon, 27 Mar 1995 10:00:10 -0700
From: "Srinivas R. Doddi" <srinu@gardener.lanl.gov>
To: archyy@lanl.gov
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The conference is "Seventh Canadian Conference on Computational Geometry".
It will be held in Quebec City, Quebec, Canada from August 10 to August
14, 1995.

regards,
Srini