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NONLINEAR ERGODIC THEOREMS FOR ABEL MEANS

by

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Applied Mathematics Division

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ABSTRACT

This report is concerned with ergodic theorems for Abel means of nonlinear contraction mappings and nonlinear contraction semigroups in a Hilbert space.

In a recent note [1], Baillon proved an ergodic theorem for Cesàro means of nonlinear contraction mappings in a Hilbert space. A similar theorem was obtained by Baillon and Brézis [2] in the case of nonlinear contractive semigroups. In this note we give the analogous ergodic theorems for Abel means. Our approach is based on the methods developed by Brézis and Browder in [3].

Throughout this note, H denotes a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. C is a closed bounded convex subset of H. The notations $\to$ and $\rightharpoonup$ refer to convergence in the norm topology and weak topology in H, respectively.

1. Let T be a nonlinear contractive mapping of C into itself. For $r \in [0,1)$, $n = 0,1,\ldots, x \in C$, let $\sigma_n(r)x \in C$ be defined by the expression

$$\sigma_n(r)x := (1-r)(1-r^n)^{-1} \sum_{i=0}^{n-1} r^i T^i x \ .$$

The boundedness of C implies that the sequence $\{\sigma_n(r)x\colon n=0,1,\ldots\}$ is Cauchy in the norm topology; let $\sigma(r)x$ denote its limit. As C is closed, $\sigma(r)x \in C$; $\sigma(r)x$ has the following representation,

$$\sigma(r)x = (1-r) \sum_{i=0}^{\infty} r^i T^i x \ .$$

Let F denote the set of fixed points of T in H, which is closed, convex and nonempty, cf. [4], Lemma 2.7. We first prove the following lemma.

<u>Lemma 1</u>. Let $\{r_n : n=0,1,\ldots\}$ be a sequence of real numbers in $[0,1)$ tending to 1. If $\sigma(r_n)x \rightharpoonup \ell$, then $\ell \in F$.

<u>Proof</u>. Take any $u \in H$. Then

$$|\sigma(r)x-u|^2 = (1-r)^2 \sum_{i,j=0}^{\infty} r^{i+j}(T^i x-u, T^j x-u) .$$

Using the law of cosines, $2(a-b,a-c) = |a-b|^2 + |a-c|^2 - |b-c|^2$ for any three vectors $a,b,c \in H$, we find

$$(*) \qquad |\sigma(r)x-u|^2 = (1-r) \sum_{i=0}^{\infty} r^i |T^i x-u|^2 - \frac{1}{2}(1-r)^2 \sum_{i,j=0}^{\infty} r^{i+j} |T^i x-T^j x|^2 .$$

Taking $u = \sigma(r)x$ we obtain the identity

$$\frac{1}{2}(1-r) \sum_{i,j=0}^{\infty} r^{i+j} |T^i x-T^j x|^2 = \sum_{i=0}^{\infty} r^i |T^i x-\sigma(r)x|^2 ,$$

which we use to replace the double sum in (*) by a single sum. Then, taking $u = T\sigma(r)x$ in (*) we find, after a rearrangement of terms,

$$|\sigma(r)x-T\sigma(r)x|^2 = (1-r)\left\{|x-T\sigma(r)x|^2 + \sum_{i=0}^{\infty} r^{i+1} |T^{i+1}x-T\sigma(r)x|^2 - \sum_{i=0}^{\infty} r^i |T^i x-\sigma(r)x|^2\right\} .$$

Since T is a contraction it follows that

$$|\sigma(r)x-T\sigma(r)x|^2 \leq (1-r)\left\{|x-T\sigma(r)x|^2 + \sum_{i=0}^{\infty} (r^{i+1}-r^i) |T^i x-\sigma(r)x|^2\right\}.$$

Each term in the sum is negative. Hence,

$$|\sigma(r)x-T\sigma(r)x|^2 \leq (1-r)|x-T\sigma(r)x|^2 \leq (1-r)(\text{diam } C)^2 .$$

Now, let $\{r_n : n=0,1,\ldots\}$ be a sequence of real numbers in $[0,1)$ tending to 1. Then $\sigma(r_n)x-T\sigma(r_n)x \to 0$ as $n \to \infty$. Since T is a contraction, $I-T$ is demiclosed -- cf. Opial [5], Lemma 2 -- i.e., for any sequence $\{x_n\} \in C$ which converges weakly to x_0 in C, the strong convergence of the sequence $\{x_n - Tx_n\}$ to a y_0 in C implies that $x_0 - Tx_0 = y_0$. Hence, if $\sigma(r_n)x \rightharpoonup \ell$, then $\ell - T\ell = 0$. □

Let Proj_F denote the projection of H on F. We have the following theorem.

<u>Theorem 2</u>. The Abel means $\{\sigma(r)x: r \in [0,1)\}$ of x converge weakly as $r \to 1$ to a fixed point of T; this fixed point is also the strong limit of the sequence $\{\mathrm{Proj}_F\, T^n x: n=0,1,\ldots\}$ as $n \to \infty$.

<u>Proof</u>. Since the set $\{\sigma(r)x: r \in [0,1)\}$ is sequentially weakly compact, there exists at least one weakly convergent subsequence of $\{\sigma(r_n)x: n=0,1,\ldots\}$; we assume that this subsequence coincides with the sequence itself. Let ℓ denote its weak limit; then by Lemma 1, $\ell \in F$. Define $y_n := \mathrm{Proj}_F\, T^n x$ for $n=0,1,\ldots$. The sequence $\{y_n: n=0,1,\ldots\}$ converges strongly to an element $y \in F$, cf. [1], Lemma 4. Consequently, the Abel means $(1-r_n)\sum_{i=0}^{\infty} r_n^i y_i$ converge strongly to y as $n \to \infty$. In order to prove that $\ell = y$ it suffices to show that $(f-y, \ell-y) \leq 0$ for all $f \in F$. The latter inequality holds true if

$$(**) \qquad \lim_{n\to\infty} (1-r_n) \sum_{i=0}^{\infty} r_n^i (f-y, T^i x - y_i) \leq 0 .$$

The function $\phi(t) := |tf + (1-t)y_i - T^i x|^2$, $t \in [0,1]$, attains its minimum value at $t = 0$ by virtue of the definition of y_i. Hence, $\phi'(0) \geq 0$ and therefore $(f-y_i, y_i - T^i x) \geq 0$, so

$$(f-y, T^i x - y_i) < (y_i - y, T^i x - y_i) \leq |y_i - y|\,|T^i x - y_i| .$$

Given any $\varepsilon > 0$, there exists a $i(\varepsilon)$ such that $|y_i - y| < \frac{1}{2}(\text{diam } C)^{-1}\varepsilon$ for all $i > i(\varepsilon)$, and a $n(\varepsilon)$ such that $1 - r_n^{i(\varepsilon)} < \frac{1}{2}(\text{diam } C)^{-2}\varepsilon$ for all $n > n(\varepsilon)$. Then,

$$(1-r_n)\sum_{i=0}^{\infty} r_n^i (f-y, T^i x - y_i)$$

$$\leq (1-r_n)\left(\sum_{i=0}^{i(\varepsilon)-1} + \sum_{i=i(\varepsilon)}^{\infty}\right) r_n^i |y_i - y||T^i x - y_i| < \varepsilon$$

for all $n > n(\varepsilon)$, which proves (**).

Since every subsequence of $\{\sigma(r_k)x\}$ converges weakly to the same limit y, y is the only weak accumulation point, so $\{\sigma(r_k)x\}$ itself converges weakly to y. □

Let $\sigma(1)x$ denote the weak limit of $\sigma(r)x$ as $r \to 1$. Then we have the following corollary.

<u>Corollary 3</u>. The operator $\sigma(1)\colon x \mapsto \sigma(1)x$ is a contractive mapping of C into F, which satisfies $T\sigma(1)x = \sigma(1)Tx = \sigma(1)x$ for all $x \in C$.

<u>Proof</u>. It follows immediately from Theorem 2 that $\sigma(1)$ maps C into F, and that $T\sigma(1)x = \sigma(1)x$. Furthermore, $\sigma(r)Tx - \sigma(r)x = (1-r)r^{-1}(\sigma(r)x - x) \rightharpoonup 0$ as $r \to 1$; hence, $\sigma(1)Tx = \sigma(1)x$. Since the mapping $x \mapsto |x|$ is weakly lower semi-continuous and $\sigma(r)$ is a contraction, $\sigma(1)$ is also a contraction. □

2. Let $\{S(t)\colon t \geq 0\}$ be a continuous semigroup of nonlinear contraction mappings of C into itself. Consider the real interval $[0,R]$ for any $R > 0$. For $\lambda > 0$, $x \in C$, let $\sigma_R(\lambda)x$ be defined by the expression

$$\sigma_R(\lambda)x := \lambda(1-e^{-\lambda R})^{-1}\int_0^R e^{-\lambda t} S(t)x\,dt ,$$

where the intgral is interpreted as the strong limit of Riemann sums. Since C is convex, $\sigma_R(\lambda)x \in C$ for every $R > 0$. The sequence $\{\sigma_{R_n}(\lambda)x: n=0,1,\dots\}$ is Cauchy in the norm topology of H for any increasing sequence $\{R_n: n=0,1,\dots\}$; let $\sigma(\lambda)x$ denote its limit. As C is closed, $\sigma(\lambda)x \in C$; $\sigma(\lambda)$ has the following representation,

$$\sigma(\lambda)x = \lambda \int_0^\infty e^{-\lambda t} S(t)x\,dt .$$

Let F denote the set of fixed points of S in H, which is closed, convex and nonempty, cf. [4], Remark 2.5. By [4], Remark 3.4 and Theorem A2, there exists a unique maximal dissipative set $A \subset H \times H$ such that its minimal section A^0 is the generator of S on the domain D(A) of A, where D(A) is dense in C. We have the following lemma.

<u>Lemma 4</u>. Let $\{\lambda_n: n=0,1,\dots\}$ be a sequence of positive real numbers tending to zero. If $x \in D(A)$ and $\sigma(\lambda_n)x \rightharpoonup \ell$, then $\ell \in F$.

<u>Proof</u>. Take any $[v,w] \in A$, then

$$(w, \sigma(\lambda)x - v) = \lambda \int_0^\infty e^{-\lambda t}(w, S(t)x - v)\,dt .$$

Now, observe that $(w,S(t)x-v) = (A^0 S(t)x, S(t)x-v) + (w - A^0 S(t)x, S(t)x-v)$, where the last inner product is positive, because of the dissipativity of A. Thus,

$$(w, \sigma(\lambda)x - v) \geq \lambda \int_0^\infty e^{-\lambda t}(A^0 S(t)x, S(t)x - v)\,dt .$$

By [4], Corollary 3.1, $A^0 S(t)x = (d/dt)^+ S(t)x$ for $t \geq 0$, where $(d/dt)^+$ denotes the right derivative, so

$$(w, \sigma(\lambda)x - v) \geq \frac{1}{2} \lambda \int_0^\infty e^{-\lambda t}(d/dt)^+ |S(t)x - v|^2\,dt$$

$$= -\frac{1}{2}\lambda|x-v|^2 + \frac{1}{2}\lambda^2 \int_0^\infty e^{-\lambda t}|S(t)x - v|^2\,dt .$$

Now, let $\{\lambda_n: n=0,1,\ldots\}$ be a sequence of positive real numbers tending to zero. C is bounded; hence, $\lambda_n|x-v|^2 \leq \lambda_n(\text{diam } C)^2 \to 0$ and $\lambda_n^2\int_0^\infty e^{-\lambda_n t}|S(t)x-v|^2 dt \leq \lambda_n(\text{diam } C)^2 \to 0$ as $n \to \infty$, so if $\sigma(\lambda_n)x \rightharpoonup \ell$ then $(w,\ell-v) \geq 0$ for all $[v,w] \in A$. Since A is maximal dissipative, $A = \{[\phi,\psi] \in H\times H: (y-\psi,x-\phi) \leq 0 \text{ for all } [x,y] \in A\}$, cf. [4], Lemma 2.2. It follows that $0 \in A\ell$ and, hence, $\ell \in F$. □

Let Proj_F denote the projection of H on F. We have the following theorem.

Theorem 5. The Abel means $\{\sigma(\lambda)x: \lambda > 0\}$ of any $x \in C$ converge weakly as $\lambda \to 0$ to a fixed point of S; this fixed point is also the strong limit of $\{\text{Proj}_F\, S(t)x: t > 0\}$ as $t \to \infty$.

Proof. Suppose $x \in D(A)$. Because the set $\{\sigma(\lambda)x: \lambda \geq 0\}$ is sequentially weakly compact, there exists a weakly convergent sequence $\{\sigma(\lambda_n)x: n=0,1,\ldots\}$ with a weak limit ℓ; by Lemma 3, $\ell \in F$. Now, suppose $x \in C$. Then, since $D(A)$ is dense in C, there exists a sequence $\{x_i: i=1,2,\ldots\}$ with $x_i \in D(A)$, such that $x_i \to x$. Then we know that $\sigma(\lambda)x_i$ is weakly convergent as $\lambda \downarrow 0$ to ℓ_i, say, and $\ell_i \in F$. For any $w \in H$ with $|w| = 1$ we have

$$|(\sigma(\lambda)x-\sigma(\mu)x,w)| \leq |\sigma(\lambda)x-\sigma(\lambda)x_i| + |(\sigma(\lambda)x_i-\sigma(\mu)x_i,w)| + |\sigma(\mu)x-\sigma(\mu)x_i|.$$

Since σ is a contraction, the first and last term of the right member can each be estimated by $|x-x_i|$, so

$$|(\sigma(\lambda)x-\sigma(\mu)x,w)| \leq 2|x-x_i| + |(\sigma(\lambda)x_i-\sigma(\mu)x_i,w)|$$

and, hence,

$$\limsup_{\lambda,\mu \downarrow 0}|(\sigma(\lambda)x-\sigma(\mu)x,w)| \leq 2|x-x_i| .$$

This implies that $\{\sigma(\lambda)x: \lambda \geq 0\}$ is weakly Cauchy and, hence, weakly convergent as $\lambda \downarrow 0$ to ℓ, say. We now show that $\ell \in F$. We have the inequality

$$|(\sigma(\lambda)x-\ell,w)| \leq 2|x-x_i| + |(\sigma(\lambda)x_i-\ell_i,w)| ,$$

whence

$$|(\ell_i-\ell,w)| \leq 2|(\ell_i-\sigma(\lambda)x_i,w)| + |(\sigma(\lambda)x_i-\sigma(\lambda)x,w)| + 2|x-x_i|$$

$$\leq 2|(\ell_i-\sigma(\lambda)x_i,w)| + 3|x-x_i| .$$

Given any $\varepsilon > 0$, there exists a $i(\varepsilon)$ such that $|x-x_i| < \varepsilon/6$ for all $i \geq i(\varepsilon)$. With i thus fixed, we can choose a $\lambda(\varepsilon,i)$ such that $|(\ell_i-\sigma(\lambda)x_i,w)| < \varepsilon/4$ for all $\lambda < \lambda(\varepsilon,i)$. It follows that $\lim_{i\to\infty}(\ell_i-\ell,w) = 0$ for all $w \in H$, so $\ell_i \rightharpoonup \ell$ and, since F is closed, $\ell \in F$.

It remains to be shown that ℓ is the strong limit of $\text{Proj}_F\, S(t)x$ as $t \to \infty$. Define $y(t) := \text{Proj}_F\, S(t)x$ for $t > 0$. Then $y(t)$ converges in norm as $t \to \infty$ to an element $y \in F$, cf. [2], Lemma 3. As in the proof of Theorem 2, it suffices to show that

$$(***) \qquad \lim_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t}(f-y,S(t)x-y(t))dt \leq 0$$

for all $f \in F$. By virtue of the definition of $y(t)$ we have the inequality $(f-y(t),S(t)x-y(t)) \leq 0$ for any $f \in F$, so

$$(f-y,S(t)x-y(t)) \leq (y(t)-y,S(t)x-y(t))$$

$$\leq |y(t)-y||S(t)x-y(t)| .$$

Given any $\varepsilon > 0$, there exists a $t(\varepsilon)$ such that $|y(t)-y| < \frac{1}{2}(\text{diam } C)^{-1}\varepsilon$ for all $t \geq t(\varepsilon)$, and a $\lambda(\varepsilon)$ such that $\lambda \int_0^{t(\varepsilon)} e^{-\lambda t}dt < \frac{1}{2}(\text{diam } C)^{-2}\varepsilon$ for all $\lambda < \lambda(\varepsilon)$. Then

$$\lambda \int_0^\infty e^{-\lambda t}(f-y,S(t)x-y(t))dt < \varepsilon$$

for all $\lambda \leq \lambda(\varepsilon)$, which proves (***).

Since y is the only weak limit, it follows that $\ell = y$. □

Let $\sigma(0)x$ denote the weak limit of $\sigma(\lambda)x$ as $\lambda \to 0$. Then we have the following corollary.

Corollary 6. The operator $\sigma(0): x \mapsto \sigma(0)x$ is a contractive mapping of C into F, which satisfies $S(t)\sigma(0)x = \sigma(0)S(t)x = \sigma(0)x$ for all $x \in C$, $t \geq 0$.

Proof. It follows immediately from Theorem 5 that $\sigma(0)$ maps C into F, and that $S(t)\sigma(0)x = \sigma(0)x$. Furthermore, $\sigma(\lambda)S(t)x = \lambda \int_0^\infty e^{-\lambda\tau}S(\tau)S(t)x d\tau = \lambda \int_0^\infty e^{-\lambda\tau}S(\tau+t)x d\tau = e^{\lambda t}\left\{\sigma(\lambda)x - \lambda \int_0^t e^{-\lambda\tau}S(\tau)x d\tau\right\}$, which converges weakly to $\sigma(0)x$ as $\lambda \to 0$; hence $\sigma(0)Tx = \sigma(0)x$. Since the mapping $x \mapsto |x|$ is weakly lower semi-continuous and each $\sigma(\lambda)$ is a contraction, $\sigma(0)$ is also a contraction. □

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