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M. P. Ekstrom

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DIGITAL, REALIZABLE WIENER FILTERING IN TWO-DIMENSIONS

Michael P. Ekstrom *
Lehrstuhl für Nachrichtentechnik
Universität Erlangen - Nürnberg
Cauerstr. 7
8520 Erlangen, F.R. Germany

Abstract

This paper deals with the extension of Wiener's classical mean-square filtering theory to the estimation of two-dimensional (2-D), discrete random fields. In analogy with the 1-D case, the optimal realizable filter is derived by solution of a 2-D discrete Wiener-Hopf equation using a spectral factorization procedure. Computational algorithms for performing the required calculations are discussed.

1. INTRODUCTION

Despite the rather classical nature of Wiener's minimum mean-square error (MMSE) filtering [1], its extensions to new applications remain topics of active research. The representative problem addressed by Wiener was that of optimally estimating an unobserved time signal, s(t), given a noise corrupted observation, s(t) + w(t), where w(t) is a noise process. Both signal and noise are taken to be wide-sense stationary processes, and the estimator is chosen to be linear, and derived optimal in the sense of achieving the MMSE. Two classes of estimators were described by Wiener: the so-called noncausal (unrealizable, bilateral) filter which uses past, present and future observations in forming the estimate, and the causal (realizable, unilateral) filter which uses only past and present observations.

With appropriate generalization, fundamental estimation problems of the type described above occur in many applications involving two-dimensional (2-D) signals. These include atmospheric physics [2], x-ray astronomy [3], biomedical imaging [4], etc. - that is, they occur in most scientific fields in which 2-D data are measured and signals are to be estimated or inferred from the data. As a consequence of its broad applicability, efforts have been made for sometime to extend Wiener's formalism to 2-D problems. Gabor [5] apparently first developed the Wiener "noncausal" filter for 2-D continuous fields, with the discrete version being developed by Helstrom [6]. Both filters have been extensively and successfully used in optical and digital signal processing applications, respectively.

In contrast, despite the almost forty year lapse since Wiener's original work, the 2-D generalization of his "causal" filter has not been previously presented. The purpose of this paper is to develop this generalization. Because of applications interests, the problem of estimating 2-D discrete

* Presently on leave from the Lawrence Livermore Laboratory, University of California, Livermore, California

random fields is addressed. This leads to so-called digital Wiener filters. Following a discussion of realizability in 2-D, the filtering problem is formulated and the optimal filter is derived by solving a 2-D discrete Wiener-Hopf equation. A recently developed 2-D spectral factorization procedure is used for this purpose [7]. Canonical versions of the filter are described along with the computational algorithms for performing the required calculations.

2. PHYSICAL REALIZABILITY IN TWO-DIMENSIONS

In this section the issues of "causality" and "realizability" are discussed as they apply in the 2-D case. As this seems to be a topic of some confusion in the relevant signal processing literature, the presentation will be informal, stressing motivation and plausibility. A related and more formal characterization is to be found in [8].

2.1 GENERAL CONDITIONS

Typically in 1-D, the causality and realizability descriptors are used synonymously to distinguish systems functions having the property of being physically realizable, that is, they can be realized by stable systems constructed with physically-real components. When dealing with time domain signals/systems, this implies that the system unit sample response h(n) be real, stable, and causal,

$$h(n) = 0 \qquad n < 0 \qquad (1)$$

where n is the time index (independent variable). Systems (or filters) satisfying (1) are said to be causal, realizable, and/or unilateral, because of their one-sided response.

Now, the underlying physical situation in 2-D is quite different from the 1-D case, the most important difference being that usually neither of the two independent variables involved are time. In most imaging applications, for example, the 2-D signal is a light-intensity field defined over a spatial region. Hence, the independent variables are space-space. When processing such signals, it is well-known that systems having bilateral or double-sided response functions are, in fact, physically realizable by using optical components [9]. Consequently, one must excercise care in establishing the physical realizability of an arbitrary system simply from the support of its system response.

Some interesting and useful conditions can be developed, however, when considering systems having only digital filtering components *. In this case the 2-D signal is taken to be the finite-extent bi-sequence $f(m,n)\ \forall\ 0 \le m < M,\ 0 \le n < N$. Its processing is characterized by the discrete convolution

$$g(m,n) = \sum_{k,\ell=-\infty}^{+\infty}\sum f(m-k,n-\ell)h(k,\ell) \qquad (2)$$
$$= f(m,n) * h(m,n)$$

where g(m,n) is the output and h(m,n) is the processor unit sample response. All the bi-sequences in (2) are assumed absolutely summable, i.e. $\in \ell_1$. (Regarding the continuous imaging problem, these bi-sequences can be obtained by appropriate discretization of that model.) Although (2) is defined completely in the spatial domain, there is a time sequency associated with its implementation. When actually performing the processing, the 2-D signal is first mapped to a 1-D sequence by the mapping $T: I^2 \to I$, the system output is computed, and the result then inverse mapped to 2-D via T^{-1}. It is the form of the mapping T which imposes a physical realizability condition on the spatial support of the filter response.
This is easily demonstrated by considering a specific mapping, one commonly used in image processing practice and some recent 2-D system analysis (e.g. see [7] and [10]).

Definition: The 2-D bi-sequence f(m,n) is said to be row-sequentially mapped by the transformation

$$T:\ i = m+nM \quad \forall\ 0 \le m < M,\ 0 \le n < N,$$

resulting in the 1-D sequence,

$$f_1(i) = f_1(m+nM) = f(m,n)\ . \qquad (3)$$

* This discussion can be appropriately generalized to include any physical components having their input/output characteristics described in the time-domain.

This mapping involves the simple concatenation of the rows of f(m,n) (see Fig. 1). It is sometimes referred to as a line-by-line or lexicographic ordering.

Now, this row-sequential mapping totally orders the bi-sequence, with i indexing the time sequency of the processing. According to (1), any physically realizable processing will use only past and present values of the input sequence $f_1(i)$. These values are shown in Fig.2 for the 1-D and 2-D (under T^{-1}) ordering. Thus, in analogy with the 1-D case, the following theorem can be stated.

Theorem 1. Under row-sequential mapping, the system in (2) is physically realizable iff h(m,n) is real, absolutely summable and its support is contained in the region $\mathcal{R}_{\oplus +}$, where

$$\mathcal{R}_{\oplus +} = \{(m,n): 0 \le m \le \infty, 0 \le n \le \infty\} \cup \{(m,n): -\infty \le m < 0, 0 < n \le \infty\} \quad . \tag{4a}$$

The "$\oplus +$" notation is chosen to be consistent with that in [7]. Here, the $\mathcal{R}_{\oplus +}$ region is called the upper half-plane and is illustrated in Fig.2. The lower half-plane $\mathcal{R}_{\ominus -}$ is correspondingly defined:

$$\mathcal{R}_{\ominus -} = \{(m,n): -\infty \le m \le 0, -\infty \le n \le 0\} \cup \{(m,n): 0 < m \le \infty, -\infty \le n < 0\} \ . \tag{4b}$$

2.2 TRANSFORM CONDITIONS

Recall that 1-D physical realizability as described in (1) also invokes certain requirements on the region of analyticity for the system transfer function, H(z) (e.g. see [11]). As might be expected, this also generalizes to 2-D, the conditions of interest being those for the half-plane bi-sequences defined above.

The 2-D Z transform of h(m,n) is written as

$$H(z_1,z_2) = \sum_{m,n=-\infty}^{+\infty}\sum h(m,n) z_1^{-m} z_2^{-n} \ , \tag{5}$$

and is defined over a region of convergence in C^2 such that

$$\sum_{m,n}\sum |h(m,n) z_1^{-m} z_2^{-n}| < \infty \quad .$$

A transform pair is indicated by the relation

$$h(m,n) \longleftrightarrow H(z_1,z_2) \quad .$$

First, consider the case of $\mathcal{R}_{\oplus +}$ bi-sequences. It has been previously shown that the transform of an ℓ_1 bi-sequence taking support on the full, upper half-plane, i.e. for all $m,n \in \{(m,n): -\infty \le m \le \infty, 0 \le n \le +\infty\}$, must have an open region of holomorphy which includes $\{(z_1,z_2): |z_1|=1, 1 \le |z_2|\}$ [7].
This is, in essence, a "one-sideness" condition associated only with the variable n. An additional condition is also involved, however, because h(m,o) is a unilateral sequence. This requires that $H(z_1,\infty)$ be analytic on a region including $\{z_1: 1 \le |z_1|\}$. Corresponding conditions can be stated for the lower half-plane ($\mathcal{R}_{\ominus -}$) bisequences by noting the simple transformation of variables between the upper and lower half planes.

These relations can be summarized in the following theorem:

Theorem 2. An absolutely summable bi-sequence, h(m,n), taking support on $\mathcal{R}_{\oplus +}$($\mathcal{R}_{\ominus -}$) has a Z transform satisfying the set of holomorphic conditions $\mathcal{S}_{\oplus +}$($\mathcal{S}_{\ominus -}$), where

$\mathcal{S}_{\oplus +}$: Condition 1. $H(z_1,z_2)$ is holomorphic on a region including $\{(z_1,z_2): |z_1|=1, 1 \le |z_2|\}$
Condition 2. $H(z_1,\infty)$ is analytic on a region including $\{z_1: 1 \le |z_1|\}$.

Likewise,

$\mathcal{S}_{\ominus -}$: Condition 1. $H(z_1,z_2)$ is holomorphic on a region including $\{(z_1,z_2): |z_1|=1, |z_2| \le 1\}$
Condition 2. $H(z_1,\infty)$ is analytic on a region including $\{z_1: |z_1| \le 1\}$.

The results of Theorems 1 and 2 will assume a fundamental role in developing the 2-D MMSE filtering formalism. The first theorem formally extends the concepts of causality/physical realizability to 2-D in a manner consistent with Wiener's approach, and in so doing, establishes the relevant canonical filter form. On the other hand, the conditions of the second theorem serve as a basis for both deriving the optimal filter and formulating the operations required in its calculation. Because of the general irreducibility of 2-D polynomials, this last step must by necessity involve procedures which do not explicity depend on the root-finding techniques employeed by Wiener.

With these results in hand, the filtering problem can now be addressed.

3. OPTIMAL WIENER FILTERING

3.1 PROBLEM FORMULATION

The 2-D formulation of the discrete MMSE estimation problem is as follows. An arbitrary bi-sequence $a(m,n)$ is assumed to have been observed or measured, and it is desired to infer or estimate a signal, $s(m,n)$, from this observation. Random field models are adopted. Both $a(m,n)$ and $s(m,n)$ are taken to be sample bi-sequences from homogeneous random fields with 2-D autocovariances $R_a(m,n)$ and $R_s(m,n)$, respectively. Their crosscovariance is $C_{sa}(m,n)$.

The estimator is chosen to be discrete, linear, and shift-invariant (convolutional), hence, each point estimate is of the form

$$\hat{s}(m,n) = \sum_{k,\ell}\sum a(m-k,n-\ell)h(k,\ell) \ , \qquad (7)$$

where the weights $h(m,n)$ constitute the unit sample response of the estimator or filter. For implementation purposes, it is also required that the filter be physically realizable, in the sense of Theorem 1. Hence, (6) can be written as

$$\hat{s}(m,n) = \sum_{k,\ell \in \mathcal{R}_{\oplus+}}\sum a(m-k,n-\ell)h(k,\ell) \ . \qquad (8)$$

Commensurate with the statistical models adopted, the classical mean-square error criteria is used for design.

This error is

$$\xi = E\{[s(m,n) - \hat{s}(m,n)]^2\} \ \forall\, m,n \ , \qquad (9)$$

where $E\{\cdot\}$ is an expectation operator taken over the ensemble of possible measurement and signal bi-sequences. The optimal filter is designed to minimize this error, subject to the constraint that $h(m,n)$ be physically realizable.

The most concise method of performing this minimization involves use of the so-called orthogonal projection theorem [12].

Theorem 3. With $s(m,n)$ as given in (7), the optimal filter $h_o(m,n)$ which minimizes ξ in (9) is such that

$$E\{[s(m,n)-\hat{s}(m,n)]\, a(m-k,n-\ell)\} = 0 \qquad (10)$$
$$\forall\, k,\ell \in \mathcal{R}_{\oplus+}$$

This states that the filter weights are optimal when the estimator error is orthogonal to the observations used in forming the estimate - a quite well-known result.

As a first step in deriving $h_o(m,n)$, $s(m,n)$ in (8) is substituted into (10) yielding

$$E\{s(m,n)\, a(m-k,n-\ell)\}$$
$$= E\{\sum_{\tau,\lambda \in \mathcal{R}_{\oplus+}}\sum a(m-\lambda,n-\lambda)h(\tau,\lambda)a(m-k,n-\ell)\} \qquad (11)$$
$$\forall\, k,\ell \in \mathcal{R}_{\oplus+}$$

Interchanging the order of expectation and summation and using the covariance relations

$$C_{sa}(k,\ell) = E\{s(m,n)a(m-k,n-\ell)\} \qquad (12a)$$

$$R_a(k-\tau,\ell-\lambda) = E\{a(m-\tau,n-\lambda)a(m-k,n-\ell)\}, \qquad (12b)$$

(11) can be rewritten as

$$C_{sa}(k,\ell) = \sum_{\tau,\lambda \in \mathcal{R}_{\oplus+}}\sum R_a(k-\tau,\ell-\lambda)h_o(\tau,\lambda) \qquad (13)$$
$$\forall\, k,\ell \in \mathcal{R}_{\oplus+} \ .$$

In accordance with traditional nomenclature, this is called a 2-D discrete Wiener-Hopf equation of the first kind. It must, of course, be solved to obtain $h_o(m,n)$. Because the equality holds only over a projection of the plane, its solution is nontrivial, although straightforward (given the 1-D precedent).

3.2 SOLUTION FOR THE OPTIMAL FILTER

The procedure adopted here for solving (13) makes use of a recently developed spectral factorization [7]. This factorization decomposes a 2-D power spectral density into the form

$$\Phi(z_1,z_2) = \overset{\oplus+}{\Phi}(z_1,z_2)\overset{\ominus-}{\Phi}(z_1,z_2) \qquad (14)$$

where $\Phi^{\oplus+}(z_1,z_2)$ and $1/\Phi^{\oplus+}(z_1,z_2)$ satisfy the holomorphic conditions $\mathcal{D}_{\oplus+}$, and $\Phi^{\ominus-}(z_1,z_2)$ and $1/\Phi^{\ominus-}(z_1,z_2)$ satisfy the holomorphic conditions $\mathcal{D}_{\ominus-}$. Thus, for example, $\Phi^{\oplus+}(z_1,z_2)$ has no zeros or poles (nonessential singularities of the first kind) in a region including $\{(z_1,z_2):|z_1|=1,\ 1 \le |z_2|\}$, and $\Phi^{\oplus+}(z_1,\infty)$ has no zeros or poles in a region including $\{z_1: 1 \le |z_1|\}$. Consistent with the even symmetry of $\Phi(z_1,z_2)$, the constraint

$$\Phi^{\ominus-}(z_1,z_2) = \Phi^{\oplus+}(z_1^{-1},z_2^{-1}) \tag{15}$$

is chosen to ensure the uniqueness of (14). Now, this decomposition is similar in spirit to Wiener's classical spectral factorization in that it equivalently factors the corresponding covariance into the convolutional product of two terms, one of which is "causal" or physically realizable and the other which is "anticausal" (i.e. in the sense of Theorem 1). The holomorphic conditions of the individual terms in (14) guarantee this behaviour.

As a first step in solving for the optimal filter, (13) is rewritten

$$e(m,n) = -C_{sa}(m,n) + \sum_{\tau,\lambda}\sum R_a(m-\tau,n-\lambda)h_o(\tau,\lambda) \tag{16}$$

$$= 0 \quad \Big| \quad \forall\, m,n \in \mathcal{R}_{\oplus+}$$

Thus, the bi-sequence $e(m,n)$ takes support on $\mathcal{R}_{\ominus-}-(o,o)$. Using the Z transform pairs

$$e(m,n) \longleftrightarrow E(z_1,z_2)$$
$$R_s(m,n) \longleftrightarrow \Phi_s(z_1,z_2)$$
$$R_a(m,n) \longleftrightarrow \Phi_a(z_1,z_2)$$
$$C_{sa}(m,n) \longleftrightarrow \Phi_{sa}(z_1,z_2)$$
$$h_o(m,n) \longleftrightarrow H_o(z_1,z_2)$$

and the convolution property of Z transforms, it follows that the transform

$$E(z_1,z_2) = -\Phi_{sa}(z_1,z_2) + \Phi_a(z_1,z_2)H_o(z_1,z_2) \tag{17}$$

satisfies the holomorphic conditions $\mathcal{D}_{\ominus-}$.

Spectrally factoring $\Phi_a(z_1,z_2)$ as in (14), $E(z_1,z_2)$ may be expanded in the form

$$E(z_1,z_2) = \Phi_a^{\ominus-}(z_1,z_2)\ \Big\{-\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_2)} + \Phi_a^{\oplus+}(z_1,z_2)H_o(z_1,z_2)\Big\} \tag{18}$$

$$= \Phi_a^{\ominus-}(z_1,z_2)B(z_1,z_2) \tag{19}$$

where

$$B(z_1,z_2) = -\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_2)} + \Phi_a^{\oplus+}(z_1,z_2)H_o(z_1,z_2) \ . \tag{20}$$

As $\Phi_a^{\ominus-}(z_1,z_2)$ satisfies the holomorphic conditions $\mathcal{D}_{\ominus-}$, it follows from (18) that $E(z_1,z_2)$ must likewise satisfy the conditions $\mathcal{D}_{\ominus-}$. Inverse transforming (19), leads to

$$e(m,n) = \phi^{\ominus-}(m,n) * b(m,n) \tag{21}$$

where

$$\phi^{\ominus-}(m,n) \longleftrightarrow \Phi_a^{\ominus-}(z_1,z_2)$$
$$b(m,n) \longleftrightarrow B(z_1,z_2) \ .$$

Because of the construction of $\Phi_a^{\ominus-}(z_1,z_2)$, $\phi^{\ominus-}(m,n)$ takes support on $\mathcal{R}_{\ominus-}$. Hence, for $e(m,n)$ to vanish on $\mathcal{R}_{\oplus+}$, $b(m,n)$ must also vanish on $\mathcal{R}_{\oplus+}$.

One further expansion leads to the desired result. Let the first term of $B(z_1,z_2)$ in (20) be expanded as

$$\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_2)} = \left[\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_2)}\right]_{\oplus+} + \left[\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_2)}\right]_{\sigma-} \tag{22}$$

where the notation $[\cdot]_{\oplus+}$ indicates the component of $\Phi_{sa}(z_1,z_2)/\Phi_a^{\ominus-}(z_1,z_2)$ which contributes to $b(m,n)$ in the region $\mathcal{R}_{\oplus+}$. Notice that the second term contributes to $b(m,n)$ in the region $\mathcal{R}_{\ominus-}-(o,o)$, hence the $\sigma-$ notation. Thus,

$$B(z_1,z_2) = -\left[\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_1)}\right]_{\oplus+} - \left[\frac{\Phi_{sa}(z_1,z_2)}{\Phi_a^{\ominus-}(z_1,z_2)}\right]_{\sigma-} + \Phi_a^{\oplus+}(z_1,z_2)H_o(z_1,z_2) \tag{23}$$

Clearly, only the first and last terms in $B(z_1,z_2)$ contribute to $b(m,n)$ on $\mathcal{R}_{\oplus+}$. In fact, because of the construction of $\Phi^{\oplus+}(z_1,z_2)$ and the physical realizability constraint on $H_o(z_1,z_2)$, the inverse transform of their product (the last term) only takes values on $\mathcal{R}_{\oplus +}$. Consequently, for $b(m,n)$ (hence, $\varepsilon(m,n)$) to vanish on $\mathcal{R}_{\oplus +}$, these two contributions must identically cancel:

$$-\left[\frac{\Phi_{sa}(z_1,z_2)}{\overset{\ominus -}{\Phi_a}(z_1,z_2)}\right]_{\oplus +} + \overset{\oplus +}{\Phi_a}(z_1,z_2)H_0(z_1,z_2) = 0 \quad (24)$$

Solving (24), the transfer function of the 2-D optimal, physically realizable Wiener filter is given by

$$H_0(z_1,z_2) = \frac{1}{\overset{\oplus +}{\Phi}(z_1,z_2)} \left[\frac{\Phi_{sa}(z_1,z_2)}{\overset{\ominus -}{\Phi_a}(z_1,z_2)}\right]_{\oplus +} . \quad (25)$$

This is the main result of the paper. The operations required in deriving $H_0(z_1,z_2)$ are i) a spectral factorization of the measurement field spectrum, and ii) a subsequent partitioning to obtain the physically realizable (half-plane) part of the signal and measurement cross-spectrum divided by the "co-analytic" term of the factorization. With this optimal filter, expressions for the MMSE can be derived and compared with the unrealizable error, although this will not be pursued here.

4. REMARKS

Remark 1. The model, criteria, and general viewpoint adopted here for the 2-D linear filtering problem are essentially identical in context to those employed by Wiener in his classical work. With a careful generalization of physical realizability to 2-D, both the canonical, realizable filter form and subsequent 2-D spectral factorization follow as natural extensions of their 1-D antecedents. Thus, it is not surprising that the form of the optimal, 2-D realizable filter in (25) appears functionally identical to the 1-D result [11].

The traditional motivations for using the MMSE formalism (compatible statistical characterization of many estimation problems, availability of the required covariance information, global optimality for the gaussian case, ease of derivation and implementation for the linear filter, etc.) substantially extend to the 2-D case. The possible exception, observed by some researchers, seems to be the potentially limited utility of the mean-square error criteria in 2-D image processing applications which involve the human visual system [13]. However, most 2-D signal processing problems either to not have as their objective improvement of the "visual" quality of measured data, or simply do not involve image data bases. Thus, this does not appear to be a serious limitation, although it does point out the need to determine general applicability of the MMSE criteria as a first step to applying the formalism.

Remark 2. A remarkable difference exists between the 1-D and 2-D cases, however, which was mentioned above. This difference arises from the absence of an equivalent Fundamental Theorem of Algebra for 2-D polynomials. As a consequence of this irreducibility of 2-D polynomials, the spectral factorization and realizability projection operations called for in (25) generally do not result in rational functions, although the underlying spectral densities involved may themselves be rational.

Despite this complication, it is possible to perform the two individual operations of (25) in approximation, thereby obtaining suitable rational approximates. It turns out that both operations can be implemented with algorithms based on sectioning cepstral bi-sequences.

5. ACKNOWLEDGEMENT

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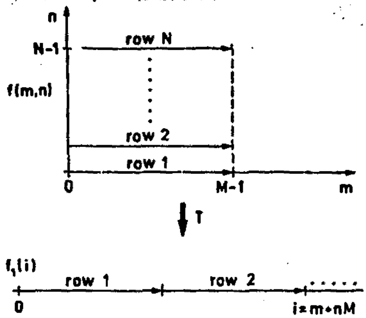

Figure 1. Row-sequential mapping of 2-D bi-sequence $f(m,n)$ to 1-D sequence $f_1(i)$.

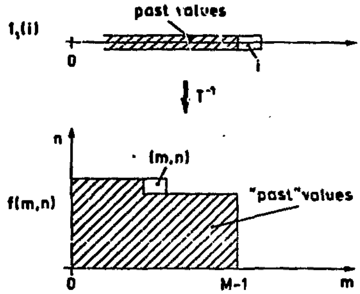

Figure 2. Inverse row-sequential mapping of past values of $f_1(i)$ to "past" values of $f(m,n)$.

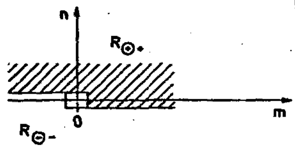

Figure 3. Region of support for $h(m,n)$ of a physically realizable system (using the row-sequential map).