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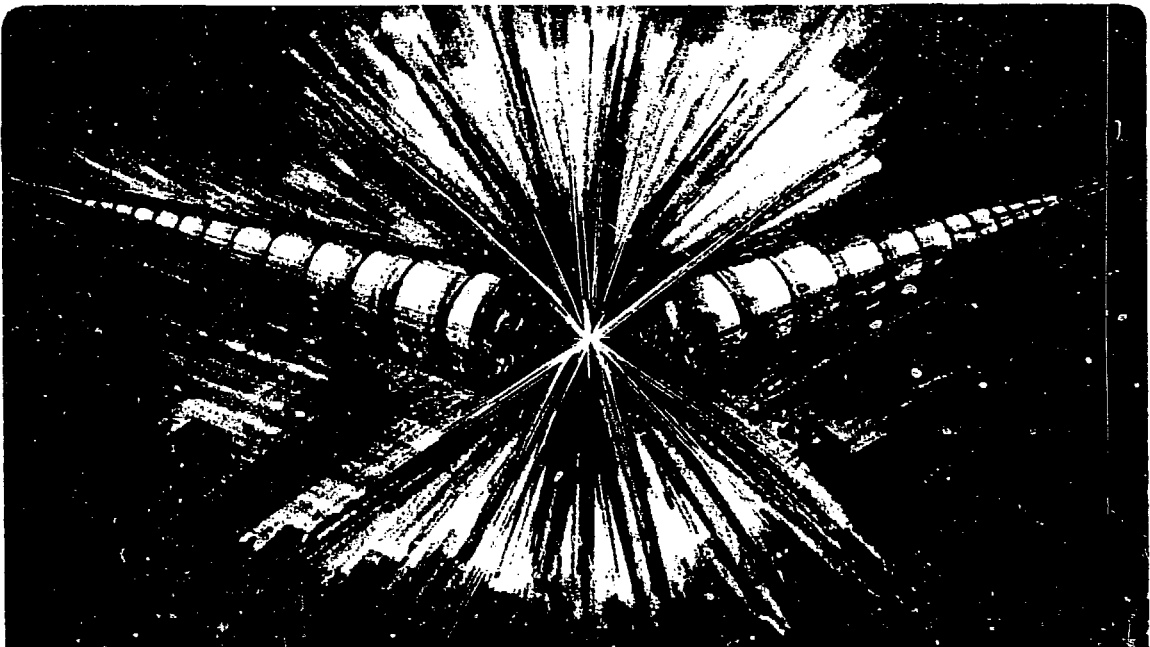
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A NEW FORMULATION OF SYNCHROTRON RADIATION
OPTICS USING THE WIGNER DISTRIBUTION

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MASTER

A new formulation of synchrotron radiation optics using the Wigner distribution

Kwang-Je Kim

Abstract

It is shown that the Wigner distribution provides a proper framework in which the propagation and imaging characteristics of synchrotron radiation can be analyzed, taking fully into account the effect of diffraction and electron beam emittance. The Wigner distribution can be interpreted as a phase-space density of photon flux, i.e., the brightness, and transforms through a general optical medium in the same way as in the case of collection of geometric rays. The brightness due to a collection of electrons can be calculated by a simple convolution of the brightness of a single electron with the electron phase space distribution function. Expressions for the brightness of bending magnets, wigglers and undulators are given.

1. Introduction

In calculating the properties of synchrotron radiation, one usually assumes that the distance from the source is far greater than the source dimensions. The radiation pattern is then entirely specified by the angular distribution, which can be calculated from well-known textbook formulas.¹ However, the finite distance assumption is not valid for many uses of practical importance, such as the intensity distribution at, for example, 10 m away from a 3 m long undulator. The situation is even more extreme when the radiation is focused and one wishes to compute the intensity distribution at the image plane. In these cases, the description of the synchrotron radiation in terms of the angular distribution is clearly inadequate.

A brute force solution of the above problem is to evaluate the radiation formula without the infinite distance assumption. While rigorous, this method is rather cumbersome. The symmetry property of the source, for example the periodicity of the electron motion in the undulator, cannot in general be exploited to simplify the expression. The amplitudes due to different electrons have to be first computed by separate numerical calculations and then added to obtain the total amplitude at a given distance. The whole procedure needs to be repeated for amplitudes at different distances.

A simpler approach which also give more insight into the problem will be explained in this paper,² which is based on the Wigner distribution $\mathcal{B}(x, \theta)$. \mathcal{B} can be interpreted as a flux density in the (x, θ) phase space and therefore will be called as the brightness throughout this paper. In this approach, the characterization of the radiation source is logically separated from the optical properties of the medium through which the radiation propagates. Also, the effects due to electron beam emittance are taken into account in a intuitively appealing way. The method involves the following steps:

- (i) One derives an expression for the source brightness \mathcal{B}_s^0 due to a reference electron.
- (ii) The source brightness \mathcal{B}_s due to a collection of electrons randomly distributed in their phase space is calculated by a convolution of \mathcal{B}_s^0 with the electron distribution function (the brightness convolution theorem).
- (iii) The brightness \mathcal{B}_f at the image plane is obtained by a transformation of the source brightness. The transformation is linear when the angular divergence of the radiation can be considered small (the paraxial approximation), and reduce to the well-known results of Gaussian optics.
- (iv) The flux distribution at the image plane is obtained by integrating \mathcal{B}_f with respect to x or θ .

These steps are described in further details in the remainder of this paper. In Section II, the brightness in a general case is defined in terms of the electric fields, and its transformation properties through optical media are described. Section III establishes the brightness convolution theorem, which relates the brightness due to a single electron to that due to a random collection of electrons. Section IV contains a discussion of the source brightness of synchrotron radiation-bending magnets, and wigglers and undulators. Section V gives an approximate but useful formula for the undulator brightness taking into account the beam emittance effect. Finally, Section VI contains concluding remarks.

II. Definition of the brightness and its transformation properties

The general discussion of brightness starts with the electric field E . Throughout this paper, we shall always consider a narrow bandwidth $\Delta\omega$ about a given frequency ω . (In this sense, the brightness we discuss here is in fact the "spectral" brightness). Also, we shall limit our discussion to a single polarization component because, in synchrotron radiation, usually only the horizontal component is important. Effectively, therefore, the electric field is considered to be a scalar quantity.

Electric fields and brightness are always referred to a plane transverse to the optical axis which is labelled by the z-coordinate. For notational simplicity, the z dependence will be suppressed whenever possible. Thus, the electric field will be represented by either $E(x)$ or $E(x; z)$ where x represents the transverse coordinate. It is convenient to introduce the following Fourier transform pair:

$$E(x) = \int \mathcal{E}(\phi) e^{ik\phi \cdot x} d^2\phi \quad , \quad (1)$$

$$\mathcal{E}(\phi) = \frac{1}{\lambda^2} \int E(x) e^{-ik\phi \cdot x} d^2x \quad .$$

Although x and ϕ are two-dimensional vectors, the vector notation will be suppressed whenever possible. In Eq. (1) $k = \omega/c = 2\pi/\lambda$, λ being the radiation wavelength. Throughout this paper we use the paraxial approximation $\phi^2 \ll 1$, so that

$$\sqrt{1 - \phi^2} \approx 1 - \phi^2/2 \quad . \quad (2)$$

The wave propagation in free space is then described by the Fresnel diffraction formula³

$$\mathcal{E}(\phi; z+l) = \mathcal{E}(\phi; z) e^{ik \cdot l \left(1 - \frac{\phi^2}{2}\right)} \quad , \quad (3)$$

$$E(x; z+l) = \frac{-i}{\lambda l} \int d^2x' E(x'; z) e^{ik \left(\frac{(x-x')^2}{2l} + l\right)} \quad .$$

The brightness is defined as a bilinear function of the electric field as follows:

$$\mathcal{B}(x, \phi) = \text{const} \int d^2\xi \langle \mathcal{E}^*(\phi + \frac{\xi}{2}) \mathcal{E}(\phi - \frac{\xi}{2}) \rangle e^{-ikx \cdot \xi} \quad , \quad (4)$$

$$= \frac{\text{const}}{\lambda^2} \int d^2u \langle E^*(x + \frac{u}{2}) E(x - \frac{u}{2}) \rangle e^{ik\phi \cdot u} \quad .$$

The angular brackets here indicate the ensemble average; they are necessary when the fields fluctuate randomly. The constant in Eq. (4) depends on whether the brightness is defined in terms of power or photon numbers, and need not be specified here.

The brightness defined above is real but not positive definite. This is fundamentally due to the wave nature of the radiation, which precludes a simultaneous determination of both position and angle, much as in quantum mechanics. In fact, a phase-space distribution which closely resembles Eq. (4) was first introduced by Wigner⁴ some 50 years ago in connection with statistical quantum mechanics. Since then, the representation has been rediscovered and studied by several authors⁵ in the context of optics.

We shall now establish some useful properties of the brightness which render its interpretation as the phase space density of the flux plausible. First, one obtains by integration that

$$\int \mathcal{B}(x, \phi) d^2\phi = \frac{d^2\mathcal{F}}{dx^2} = \text{const} \langle |E(x)|^2 \rangle \quad , \quad (5)$$

$$\int \mathcal{B}(x, \phi) d^2x = \frac{d^2\mathcal{F}}{d\phi^2} = \lambda^2 \text{const} \langle |\mathcal{E}(\phi)|^2 \rangle \quad . \quad (6)$$

The right-hand sides of Eq. (5) and (6), which are positive definite, can clearly be identified as the flux density $d^2\mathcal{F}/dx^2$ and the angular distribution $d^2\mathcal{F}/d\phi^2$, respectively.

How the brightness transforms under free-space propagation is determined by the Fresnel formula, Eq. (3). Inserting this into Eq. (4), one finds

$$\mathcal{B}(x, \phi; z+l) = \mathcal{B}(x-l\phi, \phi; z) \quad . \quad (7)$$

For a lens of focal length f , we note that the electric field transforms according to

$$E_{\text{after}}(x) = E_{\text{before}}(x) e^{-ikx^2/2f} \quad (8)$$

From Eq. (8) and Eq. (4), one obtains

$$\mathcal{B}_{\text{after}}(x, \theta) = \mathcal{B}_{\text{before}}\left(x, \theta + \frac{x}{f}\right) \quad (9)$$

Equations (7) and (8) can be cast in the following form

$$\mathcal{B}(x, \theta; z_2) = \mathcal{B}(x', \theta'; z_1) \quad (10a)$$

$$\begin{pmatrix} x' \\ \theta' \end{pmatrix} = M^{-1} \begin{pmatrix} x \\ \theta \end{pmatrix} \quad (10b)$$

where M is a 2×2 matrix which characterizes the transformation property of the optical medium given by

$$M = M_l = \begin{pmatrix} 1 & l \\ 0 & 0 \end{pmatrix} \quad \text{for free space of length } l \quad (11)$$

$$M = M_f = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \quad \text{for a lens of focal length } f \quad (12)$$

For a general optical medium consisting of a sequence of lenses and free spaces, M is given by a product of the above matrices. Equation (10b) is identical in structure to the trajectory equation in Gaussian optics (geometrical optics with the paraxial approximation), Eq. (10a) is analogous to the invariance of the ray density in phase space formulation of Gaussian optics.

When slits are present, the diffraction effect become manifest and the analogy to Gaussian optics fails. To see this let $E_i(x)$ and $E_f(x)$ be the electric fields before and after the slit, respectively. One may assume that

$$E_f(x) = E_i(x) S(x) \quad (13)$$

where $S(x)$ is a appropriate function that describes the blocking of the radiation. An example would be

$$S(x) = \theta\left(\frac{\Delta}{2} - |x|\right) \quad (14)$$

where Δ is the width of the slit and $\theta(y)$ is the step function which is unity for $y \geq 0$ and vanishes for $y < 0$. From the definition Eq. (4), it can be shown that the brightness transforms as

$$\mathcal{B}_f(x, \theta) = \int G(x, \theta - \theta') \mathcal{B}_i(x, \theta') d^2\theta' \quad (15)$$

where

$$G(x, \theta) = \frac{1}{\lambda} \int d^2u S^*(x + \frac{u}{2}) S(x - \frac{u}{2}) e^{ik\theta u} \quad (16)$$

For $S(x)$ given by Eq. (14) this becomes

$$G(x, \theta) = \frac{1}{\lambda} \theta(\Delta - |x|) \sin(2k\theta(\Delta - |x|/2)) \quad (17)$$

For small Δ ($k\Delta \ll 1$), the phase space area over which the function $G(x, \theta)$ does not vanish is of the order λ . This indicates that the minimum phase space area that can be measured is of the order λ , a well-known result. At the other extreme, as $k\Delta \rightarrow \infty$, one obtains

$$G(x, \theta) \xrightarrow[k\Delta \rightarrow \infty]{} S(x) \delta(\theta) \quad (18)$$

reproducing the result of geometrical optics.

III. Synchrotron radiation due to a random collection of electrons and the brightness convolution theorem

The discussion in the previous section is of general validity. For synchrotron radiation, one starts from the following expression¹ for the electric field (MKS units) at $z = 0$:

$$\mathcal{E}(\theta; 0) = \frac{e}{4\pi\epsilon_0 c} \frac{2\pi}{\lambda} \int \frac{cdt}{\lambda} \underline{n} \times (\underline{n} \times \underline{\beta}(t)) e^{i\omega(t - \underline{n} \cdot \underline{r}/c)} \quad (19)$$

Here ϵ_0 is the vacuum dielectric constant, \underline{n} is the direction vector whose transverse component is θ and whose axial component is

$$n_z = 1 - \theta^2/2 \quad ,$$

and \underline{r} and $c\beta$ are, respectively, the position and the velocity of the electron trajectory. Inserting Eq. (19) into Eq. (4), we have now an explicit expression for the source brightness due to a single electron. This will be discussed further in later sections, but the purpose here is to discuss an important theorem for a random collection of electrons, as found in an electron storage ring.

The motion of electrons in storage rings is also analogous to the propagation of rays in Gaussian optics. Corresponding to the brightness, we introduce the phase-space distribution function f of electrons, which, at symmetric points around the ring, is of the form

$$f(x_e, \theta_e) = \frac{1}{2\pi\sigma_x\sigma_\theta} e^{-\frac{1}{2} \left(\frac{x_e^2}{\sigma_x^2} + \frac{\theta_e^2}{\sigma_\theta^2} \right)} \quad (20)$$

The product $\sigma_x\sigma_\theta$ is known as the emittance. Equation (20) gives the probability distribution of electrons in phase space (x_e, θ_e) .

The theorem, which will be called the brightness convolution theorem, is as follows; let \mathcal{B}^0 be the source brightness of the reference electron. The source brightness due to all electrons is then given by

$$\mathcal{B}(x, \theta) = N_e \int \mathcal{B}^0(x - x_e, \theta - \theta_e) f(x_e, \theta_e) dx_e d\theta_e \quad (21)$$

where N_e is the total number of electrons. The conditions necessary for Eq. (21) are that different electrons be statistically independent and that the variation of the magnetic guide field across the electron beam dimensions be negligible. Both of these are well-satisfied by the usual sources of synchrotron radiation.

To sketch the proof of the theorem, let the coordinate system be such that the reference electron passes through the plane $z = 0$ at $x = 0$, parallel to the z -direction at time $t = 0$. Assume the i -th electron passes through the plane $z = 0$ at time $t = t^i$. The transverse position and the angle with respect to the z -direction as it passes the plane $z = 0$ will be denoted by x_e^i and θ_e^i . It can be shown from Eq. (19) that the electric field \mathcal{E}^i due to the i -th electron and the electric field \mathcal{E}^0 due to the reference electron, for which $x_e = \theta_e = 0$ at $z = 0$, are related by

$$\mathcal{E}^i(\theta) = \mathcal{E}^0(\theta - \theta_e^i) e^{ik(ct^i - \theta \cdot x_e^i)} \quad (22)$$

Noting that the total electric field is given by the sum

$$\mathcal{E}(\theta) = \sum_i \mathcal{E}^i(\theta) \quad (23)$$

One obtains

$$\langle \mathcal{E}^*(\theta_1) \mathcal{E}(\theta_2) \rangle = \sum_i e^{ikx_e^i \cdot (\theta_1 - \theta_2)} \mathcal{E}^{0*}(\theta_1 - \theta_e^i) \mathcal{E}(\theta_2 - \theta_e^i) \quad (24)$$

Since the electron distribution is assumed to be random, terms containing the cross products $\mathcal{E}^{i*} \mathcal{E}^j$, $i \neq j$, vanish upon ensemble average and do not appear in the right-hand side of Eq. (24). In terms of the electron distribution function f , Eq. (24) becomes

$$\langle \mathcal{G}^*(\theta_1) \mathcal{G}(\theta_2) \rangle = N_e \int dx_e^2 d^2 \theta_e e^{ikx_e(\theta_1 - \theta_2)} \mathcal{G}^{0*}(\theta_1 - \theta_e) \mathcal{G}^0(\theta_2 - \theta_e) f(x_e, \theta_e) \quad (25)$$

Equation (21) is then easily obtained from Eq. (25) and Eq. (4). This completes the proof.

According to Eq. (21), the phase space distribution of radiation defined in terms of the Wigner distribution can be convolved with the electron distribution as if it were a genuine probability distribution.

IV. Source brightness of synchrotron radiation

Let us now turn to a more rigorous derivation of the source brightness of synchrotron radiation due to the reference electron based on Eq. (19) and Eq. (4). The electron trajectory is assumed to lie in the horizontal plane, and the coordinate system is explained in Fig. (1).

Bending magnets and wigglers

For bending magnets and wigglers, one can expand the electron trajectory appearing in the phase factors around the point tangent to the direction of the observation, similar to the approximation Schwinger used in his original derivation of the radiation spectrum from bending magnets.⁷ It is then possible to show that

$$\mathcal{B}(\underline{x}, \theta) = \frac{d^2 \mathcal{F}}{d\theta d\psi} \sum \delta(x - \bar{x}(\theta) + \theta \bar{z}(\theta)) \delta(y + \psi \bar{z}(\theta)) \quad (26)$$

Here, $d^2 \mathcal{F}/d\theta d\psi$ is the well-known angular distribution of the flux, and $\bar{x}(\theta)$ and $\bar{z}(\theta)$ are the coordinates of the point on the trajectory where its slopes coincides with θ , as shown in Fig. (2). For a given θ , there are in general a multiplicity of tangent points, and their contributions must be added in computing the brightness, as indicated by the summation symbol in Eq. (26). According to Eq. (26), photons are emitted incoherently in the tangential direction at each point of the trajectory. However, the formula needs modification when diffraction effects become important, for

$$|y + \psi \bar{z}(\theta)| \text{ and } |x - \bar{x}(\theta) + \theta \bar{z}(\theta)| < \rho (\frac{\lambda}{\rho})^{2/3} \quad (27)$$

where ρ is the instantaneous radius of the curvature. Equation (26) is a reformulation of Green's original analysis⁸, of synchrotron radiation optics.

Undulators

Although the general expression for undulator brightness is rather complicated, one obtains the following approximate expression as $N \rightarrow \infty$:

$$\mathcal{B}(\underline{x}, \theta) = \mathcal{B}(0,0) \frac{1}{\pi} \int_0^1 dz \int_{-(1-z)}^{(1-z)} d \frac{\sin((\theta'z + \underline{x}')^2 / \lambda - \lambda \theta'^2)}{\lambda} \quad (28)$$

where

$$\theta' = \theta / \sqrt{\frac{\lambda}{L\pi}} \quad , \quad \underline{x}' = \underline{x} / \sqrt{\lambda L / 4\pi} \quad (29)$$

The integral in Eq. (28) can be evaluated numerically,⁹ and the result is shown in Fig. (3) for a particular case where θ and \underline{x} are parallel. One can see that a Gaussian approximation to the brightness distribution will be poor. The implication of the non-Gaussian shape of the brightness distribution is currently under investigation.

V. Gaussian approximation of undulator radiation

The expression for the source brightness of the undulator brightness is somewhat involved, and it is useful to have an approximate formula. For this purpose, the brightness for the reference electron will be represented by the following form:

$$\mathcal{B}(x, \phi; 0) = \frac{\mathcal{F}}{(2\pi\sigma_r\sigma_{r'})^2} e^{-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma_r^2} + \frac{\phi^2+\psi^2}{\sigma_{r'}^2}\right)} \quad (30)$$

where \mathcal{F} is the total flux. The source brightness of the fundamental mode of a optical resonator has precisely this form,² with

$$\sigma_r\sigma_{r'} = \frac{\lambda}{4\pi} \quad (31)$$

We will require that the undulator brightness also satisfy Eq. (31). To express σ_r and $\sigma_{r'}$ in terms of the undulator parameters, we first determine $\sigma_{r'}$ from the undulator angular distribution and write¹⁰

$$\sigma_{r'} = \sqrt{\frac{\lambda}{L}} = \sqrt{\frac{1 + K^2/2}{2\gamma^2 Nn}} \quad (32)$$

where L is the length of the undulator, K is the deflection parameter, γ is the electron energy/rest energy, n is the harmonic number, and N is the number of the undulator periods. From Eq. (31), one obtains

$$\sigma_r = \frac{1}{4\pi} \sqrt{\lambda L} \quad (33)$$

which is the diffraction-limited source size. The undulator brightness of a single electron is then approximated by Eq. (30), with σ 's given by Eq. (32) and Eq. (33).

We now use the brightness convolution theorem to obtain the brightness corresponding to a beam of electrons. The integral (21) in this case is a convolution of two Gaussian functions, and one obtains

$$\mathcal{B}(x, \phi) = \mathcal{F} \frac{e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_{Tx}^2} + \frac{y^2}{\sigma_{Ty}^2} + \frac{\phi^2}{\sigma_{T\phi}^2} + \frac{\psi^2}{\sigma_{T\psi}^2}\right)}}{(2\pi)^2 \sigma_{Tx} \sigma_{Ty} \sigma_{T\phi} \sigma_{T\psi}} \quad (34)$$

Here we are using the two-dimensional notation; $x \rightarrow \underline{x} = (x, y)$ and $\phi \rightarrow \underline{\phi} = (\phi, \psi)$, and

$$\begin{aligned} \sigma_{Tx} &= \sqrt{\sigma_r^2 + \sigma_x^2} \quad , \quad \sigma_{Ty} = \sqrt{\sigma_r^2 + \sigma_y^2} \\ \sigma_{T\phi} &= \sqrt{\sigma_{r'}^2 + \sigma_\phi^2} \quad , \quad \sigma_{T\psi} = \sqrt{\sigma_{r'}^2 + \sigma_\psi^2} \end{aligned} \quad (35)$$

An expression for the undulator brightness was first derived by Krinsky¹⁰ using an intuitive approach. His equation is similar to Eq. (34), except that his σ_r is larger by a factor 2π , and his expression includes an additional term in σ_{Tx} and σ_{Ty} which represents the depth of field effects due to electron beam angular divergence. The latter term is implicit in the transformation property of the electron beam phase space, and thus should not appear in Eq. (35).¹¹

VI. Conclusions

In this paper, we have presented a formalism which provides a rigorous theoretical basis for calculating optical properties of synchrotron radiation. The brightness defined here has the same transformation properties as in Gaussian optics. Furthermore, the brightness convolution theorem shows how the distributions in the incoherent phase space of electrons and the coherent phase space of radiation be convolved as if they were both the probability distributions. The formalism also provides a convenient description of the coherence characteristics, as is shown in a previous publication.²

Acknowledgments

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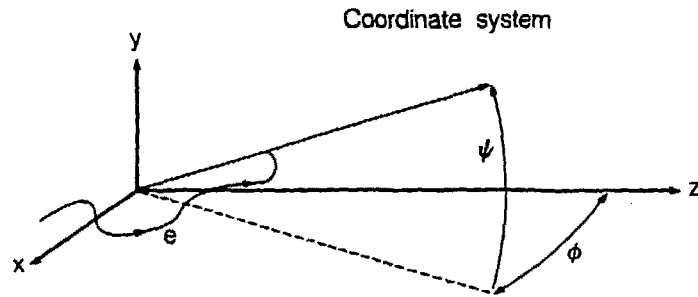


Fig. 1. The coordinate system.

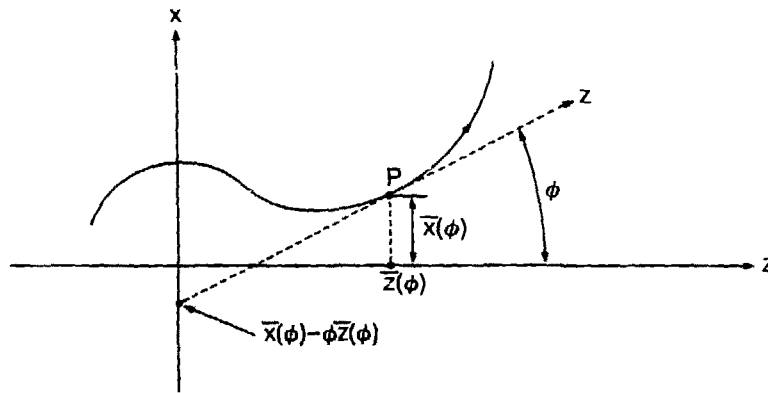


Fig. 2. An illustration of Eq. (26). The radiation emitted at P is projected to the reference plane at $z = 0$ to find the source coordinate.

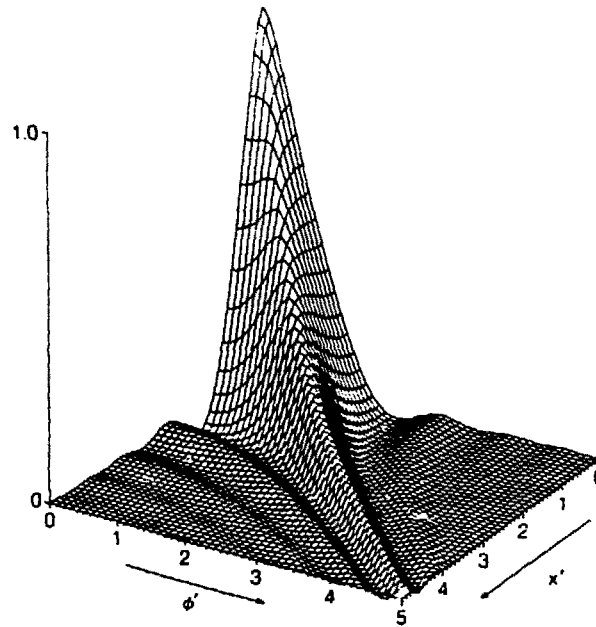


Fig. 3. The function $B(x, \phi) / B(0, 0)$ when \underline{x} and $\underline{\phi}$ are parallel.

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