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OHM'S LAW FOR MEAN MAGNETIC FIELDS

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OHM'S LAW FOR MEAN MAGNETIC FIELDS

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ABSTRACT

Spatially complicated magnetic fields are frequently treated as the sum of a large, slowly varying, mean field and a small, rapidly varying, field. The primary effect of the small field is to modify the Ohm's law of the mean field. A set of plausible assumptions leads to a form of the mean field Ohm's law which is fundamentally different from the conventional alpha effect of dynamo theory.

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I. INTRODUCTION

The description of complex magnetic fields by mean field methods has become common in both astrophysical and laboratory plasma physics literature. The usual assumption is that the exact magnetic field is the sum of a slowly varying mean field $\vec{B}_m$ and a small field $\vec{b}$, which has rapid spatial variation. The small, rapidly varying field $\vec{b}$ is also called the turbulent field. The primary effect of the turbulent field is the modification of the mean field Ohm's law from the standard form

$$\vec{E}_m + \frac{\vec{v}}{c} \times \vec{B}_m = \eta \vec{j}_m \quad . \tag{1}$$

This modification is conventionally assumed to be of the form of an additive term $\alpha \vec{B}_m$, the so-called alpha effect.[1,2]

In this paper, certain assumptions will be made about the turbulent field which lead to a different and unique form for the modification to the mean field Ohm's law. The basic assumptions are:

1. The magnetic field energy and helicity are accurately given by the energy and the helicity of the mean field.

2. The turbulent field can lead to differential transport of both field energy and helicity.

3. The dissipation associated with the turbulent field can lead to enhanced dissipation of field energy but not of field helicity.

These assumptions will be explained and to a certain extent justified in Sec. III. Although the form of the modification to Ohm's law, which is dictated by these assumptions, is qualitatively different from the alpha effect, the form has been given for specific models by Schmidt and Yoshikawa,[3] and by Jacobson and Moses.[4]

II. MEAN FIELD EQUATIONS

This section considers the equations which follow from the definition *that a mean magnetic field has zero divergence*

$$\vec{\nabla} \cdot \vec{B}_m = 0 \quad , \tag{2}$$

and the assumption that the exact field energy is accurately approximated by the mean field energy. These equations are Faraday's law, a portion of Ohm's law, Amphere's law, and force balance.

The form of Faraday's law follows from the zero divergence condition on $\vec{B}_m$. That is, $\partial \vec{B}_m / \partial t$ is divergence-free so that it must be the curl of some vector

$$\frac{\partial \vec{B}_m}{\partial t} = -c \vec{\nabla} \times \vec{E}_m \quad . \tag{3}$$

This definition of the mean electric field, $\vec{E}_m$, automatically gives part of Ohm's law. The mean electric field $\vec{R}$ in a frame of reference moving with velocity $\vec{v}(x,t)$ is

$$\vec{E}_m + \frac{\vec{v}}{c} \times \vec{B}_m = \vec{R} \quad . \tag{4}$$

In the usual plasma equations, $\vec{v}$ is identified with the plasma velocity and $\vec{R}$ = $\eta\vec{j}$ with η the resistivity of the plasma. Using mean magnetic fields, $\vec{v}$ may not be the plasma velocity nor is $\vec{R}$ so simply related to the resistivity. However, the form of Eq. (4) implies that only $\vec{R}_{\parallel}$, the component $\vec{R}$ along the mean field $\vec{B}_m$, is important to the structure of the mean magnetic field since the other components can be eliminated by a proper choice of reference frame.

By assumption, the mean field energy W accurately approximates the exact magnetic field energy with

$$W = \int \frac{B_m^2}{8\pi} d^3x \quad . \tag{5}$$

Differentiating W with respect to time,

$$\frac{\partial W}{\partial t} = -\int[\vec{j}_m \cdot \vec{E}_m + \vec{v} \cdot (\frac{c}{4\pi} \vec{E}_m \times \vec{B}_m)]d^3x \quad . \tag{6}$$

The mean current is defined by

$$\vec{v} \times \vec{B}_m = \frac{4\pi}{c} \vec{j}_m \quad , \tag{7}$$

which is Amphere's law. In the cases of interest $\vec{R}_{\parallel}$ will be very small compared to characteristic values of $|\vec{v} \times \vec{B}|/c$ (for example, with $\vec{v}$ the sound speed). Using $E_m = \vec{v} \times \vec{B}_m/c$, one can show that $\vec{j}_m \times \vec{B}_m/c$ is the force on a current carrying medium which is moving with velocity $\vec{v}$.

III. JUSTIFICATION OF ASSUMPTIONS

There are three assumptions which require discussion. First, the turbulent magnetic field will be assumed to dissipate field energy

significantly but to dissipate helicity negligibly. Second, enhanced field energy dissipation is assumed to occur even when the turbulent field makes a negligible contribution to the field energy. Third, the transport of helicity and energy is assumed to be differential. That is, local changes in the helicity or energy are given by the divergence of a locally defined flux. While discussing these assumptions we will use the exact fields $\vec{E}$ and $\vec{B}$ not just the mean fields $\vec{E}_m$ and $\vec{B}_m$.

The two most important characteristics of a magnetic field are its helicity K and its energy W. These are defined in any region of space, bounded by magnetic surfaces, by

$$K = \int \vec{A} \cdot \vec{B}\, d^3x \qquad \text{and} \qquad W = \frac{1}{8\pi} \int B^2 d^3x \tag{8}$$

with $\vec{B}$ the exact magnetic field and $\vec{A}$ its vector potential. Faraday's law implies that the rate of change of K and W in a region of space, which contains a consistent toroidal magnetic flux $2\pi\psi$, is

$$\frac{dK}{dt} = (2\pi)^2 \psi \frac{d\chi}{dt} - 2c \int \vec{E} \cdot \vec{B}\, d^3x \tag{9}$$

$$\frac{dW}{dt} = -\frac{c}{4\pi} \int \vec{E} \times \vec{B} \cdot d\vec{a} - \int \vec{j} \cdot \vec{E}\, d^3x \quad . \tag{10}$$

The notation $2\pi d\chi/dt$ is the rate of change of the poloidal flux outside the bounding magnetic surface, $\vec{j}$ is the exact current, $\vec{E}$ is the exact electric field, c is the speed of light, and the area integral is over the bounding surfaces. The destruction of helicity per unit volume is determined by $2c\vec{E} \cdot \vec{B}$ and the destruction of field energy by $\vec{j} \cdot \vec{E}$. The other terms in dK/dt and dW/dt represent the flux of helicity and energy across the boundaries of the

region.

To evaluate the rate at which K and W are destroyed, a relation between $\vec{j}$ and $\vec{E}$ is required. We will use the standard model relation, Ohm's law,

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{j} \tag{11}$$

with $\vec{v}$ the velocity of the medium carrying the current and η the resistivity. The rate of helicity destruction per unit volume is

$$2c \vec{E} \cdot \vec{B} = 2c \eta \vec{j} \cdot \vec{B} \quad . \tag{12}$$

The total rate of destruction of helicity K in the region is

$$\dot{K} \equiv -2c \int \eta \vec{j} \cdot \vec{B} \, d^3x \quad . \tag{13}$$

The expression for $\vec{j} \cdot \vec{E}$ contains two terms

$$\vec{j} \cdot \vec{E} = \eta j^2 + \vec{v} \cdot \left(\frac{\vec{j}}{c} \times B\right) \quad . \tag{14}$$

The term $\vec{v} \cdot (\vec{j} \times \vec{B})/c$ is the mechanical work done by the field on the current carrying medium. Only the term ηj^2 represents dissipation of field energy. The rate of field energy destruction W in the region is then

$$\dot{W} \equiv -\int \eta j^2 d^3x \quad . \tag{15}$$

Using the Schwartz inequality one can justify the first assumption by showing that if either K or W is dissipated faster than that of a

characteristic rate, then the enhancement of helicity dissipation is always less than that of energy dissipation. The characteristic dissipation rates $\dot{K}_c$ and $\dot{W}_c$ are defined so they would be the exact destruction rates if the current in the region were $\vec{j} = \mu \vec{B}$ with μ a constant. They are

$$\dot{W}_c \equiv -\frac{W^2}{K^2}\left(\frac{c}{4\pi}\right)^2 \int \eta\, B^2 d^3x \tag{16}$$

and

$$\dot{K}_c \equiv 8\pi \frac{K}{W}\dot{W}_c \quad . \tag{17}$$

The Schwarz inequality implies

$$\left(\int \eta\, j^2 d^3x\right)\left(\int \eta\, B^2 d^3x\right) \geq \left(\int \eta\, \vec{j}\cdot\vec{B} d^3x\right)^2 \tag{18}$$

which is equivalent to the desired inequality

$$\frac{\dot{W}}{\dot{W}_c} \geq \left(\frac{\dot{K}}{\dot{K}_c}\right)^2 \quad . \tag{19}$$

A heuristic argument makes plausible the second assumption that a turbulent field $\vec{b}$ with negligible energy content can give significant energy dissipation. Let $1/k$ be the spatial scale of the turbulent field, so the current associated with it is

$$\tilde{j} = \frac{c}{4\pi} k\, b \quad . \tag{20}$$

The dissipation due to $\tilde{j}$ is

$$\eta \tilde{j}^2 \approx \frac{\eta c^2}{4\pi} k^2 b^2 \quad . \tag{21}$$

The mean field and the mean field current j_m have a similar relationship, but with the macroscopic spatial scale a. Therefore

$$\frac{\eta \tilde{j}^2}{\eta j_m^2} \approx k^2 a^2 \frac{b^2}{B_m^2} \quad . \tag{22}$$

If $(ka)^2$ is large, then the assumption is valid.

Finally, consider the assumption that the helicity and the energy transport are differential. There are two reasons for making this assumption. First, if ka, the ratio of the macroscopic to the turbulent spatial scale, is large, then it is plausible that the turbulent field is coherent only in small regions. This implies only neighboring macroscopic regions can influence each other, which is equivalent to saying the transport is differential. The second reason is one of convenience. To avoid unphysical results, conservation laws are required for helicity and energy. The simplest nontrivial form for these laws is differential.

VI. MODIFICATION TO OHM'S LAW

In this section only the mean fields will be considered. Therefore, the subscript m will be dropped to simplify the notation. The equation, which is required, is the mean field Ohm's law. From Sec. II, we know the Ohm's law is of the form

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \vec{R} \tag{23}$$

with only $R_\parallel$ being of interest.

Consider first $\vec{R}_t$, the contribution of the turbulence to $\vec{R}_\parallel$. By assumption the turbulence should not destroy helicity only transport it. Therefore, $\vec{R}_t$ must be of form such that $\vec{R}_t \cdot \vec{B}$ integrated over any volume becomes a surface integral. This means, $\vec{R}_t$ must have the form

$$\vec{R}_t = \frac{\vec{B}}{B^2} \vec{\nabla} \cdot \vec{h} \quad . \tag{24}$$

The second assumption is that the turbulence can enhance the field energy dissipation. In other words, it must always dissipate field energy and never create it. This means that the volume integral of $\vec{j} \cdot \vec{R}_t$ must be the sum of a positive definitive volume integral plus a surface term. Using Eq. (24)

$$\int \vec{j} \cdot \vec{R}_t \, d^3x = -\int \vec{h} \cdot \nabla\left(\frac{j_\parallel}{B}\right) d^3x + \int \frac{j_\parallel}{B} \vec{h} \cdot d\vec{a} \quad . \tag{25}$$

Therefore $\vec{h}$, the helicity flux, must have the form

$$\vec{h} = -\lambda \vec{\nabla} \frac{j_\parallel}{B} \tag{26}$$

with $\lambda \geq 0$. In summary, the turbulent contribution to Ohm's law has the form

$$\vec{R}_t = -\frac{\vec{B}}{B^2} \vec{\nabla} \cdot \left(\lambda \vec{\nabla} \frac{j_\parallel}{B}\right) \quad . \tag{27}$$

The additional field energy dissipation due to the turbulence is $\lambda(\vec{\nabla} j_\parallel/B)^2$ and the field energy flux due to the turbulence is $(j_\parallel/B)\lambda \vec{\nabla}(j_\parallel/B)$.

In addition to the turbulence contribution to $\vec{B}$, the plasma resistivity also contributes. That is, we expect

$$\vec{R}_{\parallel} = \vec{R}_t + \eta_{\parallel}\vec{j}_{\parallel} \quad . \tag{28}$$

Since the turbulence is assumed not to enhance the helicity dissipation, $\eta_{\parallel}$ should be the classical, Spitzer value.

V. DISCUSSION

The most important difference between the treatment of this paper and other treatments of the mean field Ohm's law[1,2] is the inclusion of conservation laws for field energy and helicity. In the conventional alpha effect, the turbulent medium can either create or dissipate mean field energy and helicity. The fundamental conservation laws only arise through the details of the evaluation of alpha.

The application of the proposed form for Ohm's law to the dynamo action of the reversed field pinch plasma confinement device is obvious using Taylor's theory[5] of the device. If $\lambda >> \eta B^2 a^2$ with a the plasma radius and B a typical value of the field strength, then the proposed Ohm's law reduces the field energy while conserving the helicity in a region containing fixed toroidal flux. This is the energy minimization principle used by Taylor.

The application of the proposed Ohm's law to the Earth's dynamo is less obvious. However, Cowling's proof of the absence of an axisymmetric dynamo does not apply with the proposed mean field Ohm's law.

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