γ

DIRECT PHASE ESTIMATION FROM PHASE DIFFERENCES USING FAST ELLIPTIC PDE SOLVERS

Dennis C. Ghiglia, and Louis A. Romero
Sandia National Laboratories
Albuquerque, NM 87185

SAND--89-0157C
DE89 007085

Abstract

Obtaining robust phase estimates from phase differences is a problem common to several areas of importance to the optics and signal processing community. Specific areas of application include speckle imaging and interferometry, adaptive optics, compensated imaging, and coherent imaging such as synthetic-aperture radar. The purpose of this paper is to relate the equations describing the phase estimation problem to the general form of elliptic partial differential equations, and illustrate results of reconstructions on large M by N grids using existing, published, and readily available Fortran subroutines.

An important optical and signal processing problem is that of estimating wavefront (phase) distortions and then compensating for these distortions to obtain near diffraction-limited performance. For example, wavefront distortions occur when imaging through turbulent media[1], or when phase errors exist because of uncompensated platform motion or ionospheric turbulence in the case of synthetic aperture radar (SAR)[2-3].

Since it is generally not possible to measure the phase directly in incoherent imaging systems, operations on the incoming wavefront (e.g., with shearing interferometers etc.) can provide wavefront slopes or phase difference measurements. In coherent imaging systems it is generally only possible to obtain phase estimates modulo 2π. Operations on the complex signal can yield wrapped phases or phase differences from which the overall phase can be estimated.

The problem is then to reconstruct an estimate of the wavefront (phase) to within an arbitrary linear and constant term that is consistent in some sense with the noisy phase difference measurements[4-8]. Much of the previous work has been applied to the solution of the phase estimation problem on relatively small grids (i.e. 64 by 64 or less) using iterative numerical schemes such as Jacobi iterations or successive overrelaxation (SOR). It is to be noted that because of their slow convergence, these popular iteration schemes are only feasible for small problems and that their extension to much larger grids requires a more direct approach. For example, it is well known that for a fixed degree of accuracy, Jacobi iteration requires $O(N^4)$ operations and SOR requires $O(N^3)$ [9].

The purpose of this paper is to relate the equations describing the phase estimation problem to the general form of elliptic partial differential equations (PDE's), which can be solved with only $O(N^2 \ln N)$ operations[10] using fast algorithms. A recent paper[11] develops a direct solution method but does not relate the method to fast PDE solvers. Results of

reconstructions on large N by N ($N = 512$) grids using existing, published, and readily available Fortran subroutines will be shown.

Suppose that we know the phase, ϕ modulo 2π, of a function on a discrete grid of points:

$$\begin{aligned} \psi_{i,j} &= \phi_{i,j} + 2\pi k \\ k &= integer \\ & \quad 0 \le \psi_{i,j} < 2\pi \\ & \quad i = 1...M;\ j = 1...N \end{aligned} \tag{1}$$

Given the values $\psi_{i,j}$ we wish to determine the phase $\phi_{i,j}$ at the same grid locations. To do this we require that the phase differences of the $\phi_{i,j}$ agree with those of the $\psi_{i,j}$ in some "best" sense. In particular, let

$$\begin{aligned} \alpha_{i+1/2,j} &= \psi_{i+1,j} - \psi_{i,j} \\ & \quad i = 1...M-1;\ j = 1...N \end{aligned} \tag{2}$$

$$\begin{aligned} \beta_{i,j+1/2} &= \psi_{i,j+1} - \psi_{i,j} \\ & \quad i = 1...M;\ j = 1...N-1 \end{aligned} \tag{3}$$

Thus we would like to solve

$$\begin{aligned} \phi_{i+1,j} - \phi_{i,j} &= \alpha_{i+1/2,j} \\ & \quad i = 1...M-1;\ j = 1...N \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_{i,j+1} - \phi_{i,j} &= \beta_{i,j+1/2} \\ & \quad i = 1...M;\ j = 1...N-1 \end{aligned} \tag{5}$$

Equations 4 and 5 constitute an overdetermined system and will be solved in a least squares sense. We will find a solution $\phi_{i,j}$ that minimizes

$$\sum_{i=1}^{M-1}\sum_{j=1}^{N}(\phi_{i+1,j} - \phi_{i,j} - \alpha_{i+1/2,j})^2 + \sum_{i=1}^{M}\sum_{j=1}^{N-1}(\phi_{i,j+1} - \phi_{i,j} - \beta_{i,j+1/2})^2.$$

It can be shown that the least squares solution to this problem is identical to the solution of the following linear system of equations:

$$a_i(\phi_{i+1,j} - \phi_{i,j}) - a_{i-1}(\phi_{i,j} - \phi_{i-1,j}) + b_j(\phi_{i,j+1} - \phi_{i,j}) - b_{j-1}(\phi_{i,j} - \phi_{i,j-1}) =$$

$$\begin{aligned} & a_i\alpha_{i+1/2,j} - a_{i-1}\alpha_{i-1/2,j} + b_j\beta_{i,j+1/2} - b_{j-1}\beta_{i,j-1/2}. \\ & i = 1...M;\ j = 1...N \\ & a_i = 1;\ i = 1...M-1 \\ & a_0 = a_M = 0 \\ & b_j = 1;\ j = 1...N-1 \\ & b_0 = b_N = 0 \end{aligned} \tag{6}$$

This system of equations can be considered to be a discretization of Poisson's equation with Neumann boundary conditions[9]. Therefore, fast, direct methods for the solution of specialized elliptic equations on rectangular grids can be applied[12]. We used the subroutine BLKTRI that is available in the SLATEC mathematical subroutine library[13-15]. This routine is capable of solving equations other than Poisson's, so it may be somewhat slower than some fast Poisson solvers, but it is still extremely fast.

The images depicted in Figure 1 represent the results of simulations with noiseless data. Figure 1a is the 512 by 512 image of an arbitrary phase function scaled between black and white for display. Peak-to-peak dynamic range is approximately 250 radians. Figures 1b and 1c represent the phase differences of Figure 1a in the x and y directions respectively. The phase differences were used to construct the driving term (right hand side) of the PDE and BLKTRI was called with appropriate coefficient arrays analagous to those in Eq. 6. Figure 1d is the reconstructed phase superimposed with a few constant phase contours to illustrate the quality of the solution.

Figures 2a through 2d represent the same sequence except that noise was added to the phase differences. The signal-to-noise ratio of each phase difference array was approximately unity. Figure 2d is the solution superimposed with the same constant phase contours to depict the quality of reconstruction (compare with Figure 1d).

We have shown that the basic problem of estimating phase from phase differences is equivalent to solving Poisson's equation on a rectangular grid with Neumann boundary conditions. All the speed and power of fast elliptic PDE solvers can be brought to bear on this problem with the result that robust phase estimates on large M by N grids are obtainable in $O(MN \ln N)$ time. Existing, high quality Fortran subroutine libraries are readily available that contain several fast Poisson solvers. Experiments have shown, however, that significant precision can be lost on large problems even when using state-of-the-art algorithms. One must use caution when solving large problems on computers with relatively short word lengths (i.e. 32 bits), especially when N is a power of two (because of the high degree of cyclic reduction performed).

This research was performed at Sandia National Laboratories and supported by the U. S. Department of Energy under contract DE-AC04-76DP00789.

References

1. J.W. Hardy, J.E. Lefebvre, C.L. Koliopoulos, J. Opt. Soc. Am., Vol. 67, 360-369, (1977).
2. W. D. Brown, D. C. Ghiglia, J. Opt. Soc. Am. A, Vol. 5, p 924, June, 1988.
3. D. C. Ghiglia, W. D. Brown, J. Opt. Soc. Am. A, Vol. 5, p 942, June, 1988.
4. D.L Fried, J. Opt. Soc. Am., Vol. 67, 370-375, (1977).
5. R.H. Hudgin, J. Opt. Soc. Am., Vol. 67, 375-378, (1977).
6. R.J. Noll, J. Opt. Soc. Am., Vol. 68, 139-140, (1978).
7. B.R. Hunt, J. Opt. Soc. Am., Vol. 69, 393-399, (1979).
8. H. Takajo, T. Takahashi, J. Opt. Soc. Am. A, Vol. 5, No. 3, 416-425, (1988).
9. W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, Numerical Recipes: The Art of Scientific Computing, Cambridge, 1986.
10. P. Swartztrauber, R. Sweet, Rep. NCAR-TN/1A-109, National Center for Atmospheric Research, Boulder, Colorado, 1975.
11. H. Takajo, T. Takahashi J. Opt. Soc. Am. A, Vol. 5, No. 11, 1818-1827, (1988).
12. B.L. Buzbee, G.H. Golub, C.W. Nielson, SIAM J. Numer. Anal., Vol. 7, No. 4, 627-655, Dec. 1970.

13. B. Buzbee, "SLATEC Common Mathematical Library," in Sources and Development of Mathematical Software, W. R. Cowell, Editor, Prentice-Hall, 1984.
14. W.H. Vandevender, Sandia National Laboratories, SAND84-0281, Apr. 1984.
15. Software is available through: National Energy Software Center (NESC), Department of Energy, Argonne National Laboratory, 9700 South Cass Ave., Argonne, Illinois 60439

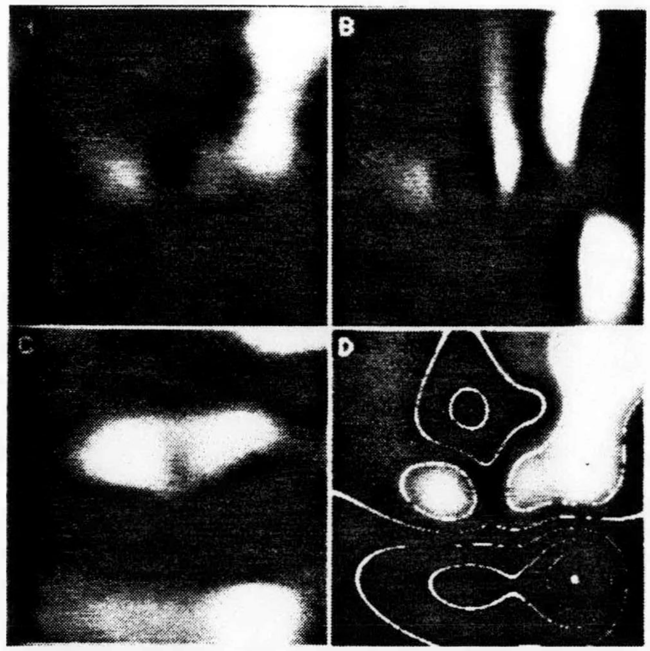

Figure 1. a) Arbitrary phase function scaled for display. b) Phase differences in the x direction. c) Phase differences in the y direction. d) Reconstructed phase from phase differences. A few constant phase contours are superimposed.

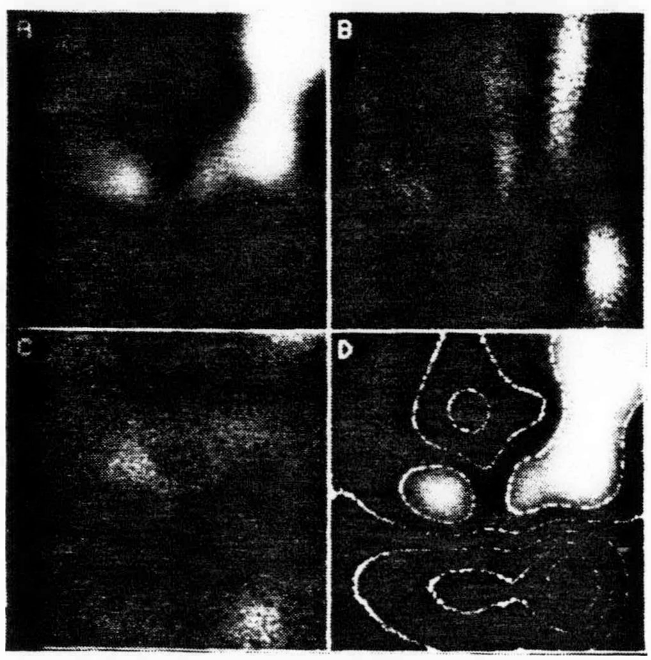

Figure 2. a) Arbitrary phase function scaled for display. b) Phase differences in the x direction with added noise. c) Phase differences in the y direction with added noise. d) Reconstructed phase from noisy phase differences. A few constant phase contours are superimposed.