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An Algebraic Formulation of Duality

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Abstract

Two dimensional lattice spin (chiral) models over (possibly non-abelian) compact groups are formulated in terms of a generalized Pauli algebra. Such models over cyclic groups are written in terms of the generalized Clifford algebra. An automorphism of this algebra is shown to exist and to lead to the duality transformation.

It is known that the two-dimensional Ising model is very closely related to the Clifford algebra [1,2]. In fact the solubility of the Ising model is a consequence of this relation [2]. A similar algebraic formalism is constructed for models defined over more general groups [3]. This includes chiral models over compact groups as well as Potts models [4]. The special case of cyclic groups is interesting because they appear in strong interaction physics [5]. In this case, the model can be formulated in terms of a generalized Clifford algebra that has already been studied in the literature [6-11] in a different context. The 't Hooft algebra [5] is a special case of this Generalized Clifford Algebra (G.C.A.).

My emphasis will be on, as Onsager said [1], "the abstract properties of relatively simple operators rather than their explicit representation in terms of unwieldy matrices."

The most general translation invariant and globally symmetric Hamiltonian for a two-dimensional lattice is

$$H=\sum_{ij}\{f_1(g_{ij}g^{-1}_{i+1,j})+f_2(g_{ij}g^{-1}_{i,j+1})\} \qquad (1)$$

Here, g_{ij} is the "spin" or "field" variable at the lattice point (i,j) and it belongs to a compact group G. f_1 and f_2 are real functions on G satisfying

$$f(hgh^{-1}) = f(g) \qquad (2a)$$

$$f(g^{-1}) = f(g) \qquad (2b)$$

The problem of finding the partition function

$$z_N = \int \prod_{ij} dg_{ij} e^{-H} \qquad (3)$$

can be reduced to that of finding the largest eigenvalue of the matrix $\hat{M}$ defined by

$$<g_1'..g_N'|\hat{M}|g_1..g_N>=\exp-\{\sum_{ij}[f_1(g_j g_j'^{-1})+f_2(g_j g_{j+1})]\} \qquad (4)$$

The factor $\frac{1}{KT}=\beta$ has been absorbed into the definition of f_1 and f_2. $\hat{M}$ is an operator in the space of square integrable functions on the group

$G^N = GxG...G.$

In order to express the transfer matrix $\hat{M}$ in a representation invariant way, it is helpful to define following two sets of operators

$$\hat{L}_K(h)|g_1..g_k..g_N> = |g_1..hg_k..g_N> \qquad (5a)$$

$$\hat{g}_{k,\alpha\beta}(h)|g_1..g_k..g_N> = D_{\alpha\beta}(g_k)|g_1..g_k..g_N> \qquad (5b)$$

Thus, $\hat{L}_k(h)$ is the left-translation operator. Here $D_{\alpha\beta}(g_k)$ is the matrix representing $g_k \varepsilon G$ in some faithful irreducible representation of G. Define also the class function $\tilde{f}_1$ by the condition

$$e^{-f_1(gg'^{-1})} = <g'|\exp\{-\int dh\ \hat{L}(h)\tilde{f}_1(h)\}|g> \qquad (6)$$

This function can be explicitly computed by first expanding the exponential in (6) to get

$$e^{-f_1(gg'^{-1})}=\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\tilde{f}_1*..\tilde{f}_1(gg'^{-1})$$

where * denotes convolution [12]. If we again expand f_1 and $\tilde{f}_1$ in terms of the character functions $\chi_r(g)$ we can get

$$\tilde{f}_1(g) = -\sum_{r\varepsilon\hat{G}}\ln I_r\cdot\chi_r(g) \qquad (7)$$

Here,

$$I_r=\int dg\ e^{-f_1(g)}\chi_r(g) \qquad (8)$$

and $\hat{G}$ denotes the set of all irreducible representations of G.

Using (5) and (6) we get

$$\hat{M}=e^{-\sum_k \int dh \tilde{f}_1(h)\hat{L}_k(h)}\, e^{-\sum_k f_2(\hat{g}_k\hat{g}_{k+1}^{-1})} \quad (9)$$

The operators $\hat{L}_k(h)$ and $\hat{g}_k$ may be thought of as being defined by the following algebra:

$$\hat{L}_k(h)\hat{g}_k = D(h^{-1})\hat{g}_k\hat{L}_k \quad (10a)$$

$$\hat{L}_k(h_1)\hat{L}_k(h_2) = \hat{L}_k(h_1h_2) \quad (10b)$$

$$\hat{L}_k(h)\hat{g}_{k'} = \hat{g}_{k'}L_k;\ k\neq k' \quad (10c)$$

D(h) is the matrix representing $h\varepsilon G$ in some faithful irreducible representation of G. This algebra reduces in the case $G=Z_2$ to the Pauli algebra, so we call it the generalized Pauli algebra (G.P.A.). The one-dimensional quantum Hamiltonian of a lattice model on G is, then [3]

$$H = \sum_k \int dh\ \tilde{f}_1(h)\hat{L}_k(h)+\sum_k f_2(\hat{g}_k\hat{g}_{k+1}^{-1}) \quad (12)$$

In the special case $G=Z_2$ we recover the Ising model. If $G=Z_d$, the additive group of integers modulo d, we get the Potts model. If $0<r<d$ is coprime to d, there is a one-dimensional faithful representation of G,

$$D^{(r)}(h)=\exp \frac{2\pi irh}{d}\quad ,\ 0\leqslant h\leqslant d$$

If we call

$$P_k = \hat{L}_k(1) \quad (13a)$$

$$Q_k = \hat{g}_k \quad (13b)$$

we recover the algebra that has already been introduced for a similar model in Ref.[13]. The transfer matrix becomes

$$M = e^{-\sum_k f_1'(P_k)}\, e^{-\sum f_2(Q_k Q_{k+1}^{-1})} \qquad (14)$$

where f_1' is the "dual" for f_1, defined by

$$f_1'(s) = -\ln \sum_r e^{-f_1(r)}\, e^{2\pi \frac{irs}{d}} \qquad (15)$$

Clearly $(f_1')' = f_1$.

If we proceed analogously to the Ising model, we are led to the Generalized Clifford Algebra. Define

$$\Gamma_{2m-1} = P_1 P_2 \dots P_{m-1} Q_m \qquad (16a)$$

$$\Gamma_{2m} = P_1 \dots P_{m-1} P_m Q_m \qquad (16b)$$

These satisfy the G.P.A. C^d_{2N} [10]

$$\Gamma_\mu \Gamma_\nu = e^{\frac{2\pi i r}{d}}\, \Gamma_\nu \Gamma_\mu ; \quad \mu > \nu \qquad (17a)$$

$$\Gamma_\mu^d = 1; \quad \mu,\nu = 1 \dots 2N \qquad (17b)$$

This algebra was introduced by Yamazaki [6] and studied by A. O. Morris [7] and A. Ramakrishnan et.al. [8-11]. For d=2 this reduces to the Clifford algebra with 2N generators. This is the famous relation of the Ising model to the Clifford algebra. Note that

$$P_k = \Gamma_{2k} \Gamma_{2k-1}^{-1} \qquad (18a)$$

$$Q_k Q_{k+1}^{-1} = \Gamma_{2k} \Gamma_{2k+1}^{-1} \qquad (18b)$$

So that

$$M = e^{-\sum f_1'(\Gamma_{2k}\Gamma_{2k-1}^{-1})}\, e^{-\sum f_2(\Gamma_{2k}\Gamma_{2k+1}^{-1})} \qquad (19)$$

In the case of the Ising model, the exponents in (19) are bilinear in the Clifford elements, so that M can be represented by an element of the group O(2N). It is not known what the analogous

Lie group is, for the general case.

It is shown in [10] that the G.C.A. $C_{2N}^{d,r}$ has only one irreducible representation, the one given in (16). Matrices representing P_k and Q_k are given in [8] and [10].

In the limit of large N, the transformation

$$\Gamma_\mu \to \Gamma_{\mu+1} \tag{20}$$

is an automorphism of (23). Under this

$$\sum_k f(\Gamma_{2k}\Gamma_{2k-1}^{-1}) \leftrightarrow \sum_k f(\Gamma_{2k}\Gamma_{2k+1}^{-1}) \tag{21}$$

Applied to the transfer matrix (19) we find that the free energy of the system satisfies

$$F(f_1,f_2) = F(f_2',f_1') \tag{22}$$

so the map

$(f_1,f_2) \mapsto (f_2',f_1')$ must map critical points to critical points. This is the famous duality transformation for the model. The self-dual case is of interest because it may be soluble exactly [13].

It is not possible to define a self-dual model for a general group G. This is because the dual of the function $f_1: G\to R$ is a function on $\hat{G}$. For a cyclic group of finite order, $\hat{G}\approx G$ so the problem does not arise. In this case duality is an automorphism of the algebra of P_k and Q_k, which can be implemented by an equivalence transformation

$$DP_kD^{-1}=Q_kQ_{k+1}^{-1};\ DQ_kD^{-1}=\prod_{\ell=0}^{k} P_\ell^{-1} \tag{23}$$

Note that D^2 is the translation operator on the lattice.

Some other special cases may be of interest. If $G=U(1)$ and

$f_1(\theta)=f_2(\theta)=J\cos\theta$

we recover the planar model, for which

$$\tilde{f}(\theta) = -\sum_{\ell=-\infty}^{+\infty} I_\ell(J)e^{i\ell\theta} \tag{24}$$

$I_\ell(J)$ being the modified Bessel function of the ℓ^{th} order. The G.P.A. (10) reduces to the exponentiated version of the canonical commutation relations (Weyl algebra). Thus the 'ℓ' in (24) may be interpreted as the "momentum" of a particle moving in a box at site 'k'. Pushing this interpretation further, we can write

$$\hat{M} = e^{-\sum_k \varepsilon(P_k)} \; e^{-\sum_k f_2(\theta_k - \theta_{k+1})} \tag{25}$$

where $\varepsilon(\ell) = -\ell n \; I_\ell$ is a function defined on the set of integers. It is the "kinetic energy" of a particle of "momentum" ℓ. This is the dual of the function f_1.

For G=U(N) and

$$f_1(U) = f_2(U) = J\{\mathrm{Tr}\; U + \mathrm{Tr}\; U^{-1}\} \tag{26}$$

$U \varepsilon U(N)$, we recover the lattice chiral model with Wilson action. The function f_1 can be calculated and is a combination of modified Bessel functions.

Most of the general framework can be readily generalized to the case where the "spin" belongs to a symmetric space like $S^{(n-1)}$ or CP^{n-1}. This will give the O(n) σ-model and the CP^{n-1} model respectively. It is clear that the statistical mechanical problem considered here is very closely related to the theory of random walks and quantum mechanics of a particle moving in the space G. The G.P.A. is really the analogue of canonical commutation relations for a particle moving in a space curvature The duality transformation would have to be a canonical transformation, but such a canonical transformation does not exist since the co-tangent space of a curved manifold is not of the same structure as the manifold itself.

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