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IDENTIFICATION OF DISTRIBUTED FORCES ON A STRUCTURE*

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Abstract

This paper presents a combined analytical and experimental method for establishing a set of equations to evaluate the equivalent forces acting on a structure. The method requires that a finite element model of the structure be established. It further requires that the acceleration responses to the external forces be measured at a number of points on the structure. The equivalent forces established in the analysis are a representation of the actual forces. The equivalent forces concentrate the effects of the external forces at the degrees of freedom where the acceleration responses are measured.

Introduction

In the analysis of structural response it is essential that the forces applied to the structure be known well in order to insure accurate results. For this reason, direct measurements are made to determine the forces applied to structures, whenever possible, and studies that use fundamental principles of mechanics to determine the theoretical forces applied to structures are used when direct measurements cannot be made. A better alternative to the latter approach, when direct force measurements cannot be made, is to use measurements of structural response to infer the external forces applied to a structure. Several studies have taken this approach.

Some examples of the traditional approach to the identification of forces applied to structures are those presented in References 1 and 2. The approach described in those papers establishes the frequency response functions between external forces applied at specific points on a structure and the responses excited at specific internal points. Then the system is subjected to field environments, and the structural responses are measured. The internal measurements are multiplied times the inverse of the frequency response functions to estimate the applied forces. While this approach can yield accurate results under certain circumstances, it is inherently ill-conditioned. It cannot be very accurate at frequencies where substantial response is not excited by the external forces. Further, if external forces act at locations other than where the frequency response functions are measured, the responses due to these forces create errors in the force estimation.

Another approach that has shown great promise in the identification of external forces applied to a structure is the "sum of weighted accelerations" method summarized in References 3 and 4. This technique makes use of the fact that internal forces in a structure that is free in space completely cancel one another when the structure is excited by external forces. The measured accelerations in such a system can be used to establish the external force. A description of the technique using the finite element method in Reference 5 shows that this result is to be expected, and it can be shown that the technique can be also explained in terms of the orthogonality of modes in a linear structure. Though the technique appears quite robust, a limitation is that it only provides the resultant force acting on a system (and the resultant moment, according to Reference 4). That is, the spatial distribution of the external forces is not established. A substantial advantage of the method is that it can be applied using only experimental data.

A technique is proposed in this paper to establish a set of equivalent external forces acting on a structure at a collection of points where response accelerations are measured. The technique is a combined experimental and analytical method, requiring that a finite element model be established for the structure with degrees of freedom at the points where the response accelerations are measured. The finite element model must be used to compute as many modal frequencies and mode shapes of the system as there are measurement points. In addition, as

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many modal frequencies, modal damping factors and mode shapes as there are measurement points must be estimated experimentally.

Development of Fundamental Equations

The objective of the development in this section is to establish a set of equations to approximately characterize the forces that act on a linear structure. The development uses several assumptions, and these are summarized first.

In practical situations, the forces acting on any structure are spatially distributed. When the forces acting on a structure are distributed over limited areas they are sometimes approximated as point loads, for the purposes of analysis. When the forces on a structure are broadly distributed, then simplifying assumptions regarding the spatial distribution of the loads are often made for use in analysis. The reason for the simplifying assumptions in both these cases is that it is impossible to exactly define the true distribution of loads on a structure using a finite collection of transducers. Further, it is usually felt by structural analysts that the load approximations used in a structural analysis can lead to reliable results when applied with care. The force identification technique developed in this paper leads to a characterization of the dynamic forces applied to a structure that is consistent with the approximations used in current practice.

It is assumed that the structure under consideration is linear and can be described by the governing differential equation

$$[m]\{\ddot{z}\} + [c]\{\dot{z}\} + [k]\{z\} = \{q\} \qquad (1)$$

where $[m]$, $[c]$, and $[k]$ are $n \times n$ matrices representing mass, damping and stiffness, respectively. $\{q\}$ is the $n \times 1$ vector of forces applied to the structure. $\{z\}$ is the $n \times 1$ vector of absolute structural displacements (and rotations), and dots refer to differentiation with respect to time. With this description, structural motion is assessed at a collection of node points, and forces on the structure are applied at these points.

Because the structure is linear, it possesses classical modes, and its modal frequencies can be denoted ω_j, $j=1,\ldots,n$. The mode shapes associated with the individual modes are contained in the $n \times n$ modal matrix $[u]$. Each column in $[u]$ represents an individual mode shape. It is assumed here that the modes are orthonormal with respect to $[m]$, therefore, the modal matrix, $[u]$, diagonalizes the mass and stiffness in the following way.

$$[u]^T [m] [u] = [I] \qquad (2a)$$

$$[u]^T [k] [u] = [\omega^2] \qquad (2b)$$

where $[I]$ is the $n \times n$ identity matrix, and $[\omega^2]$ is the $n \times n$ diagonal matrix of the squares of the modal frequencies of the structure. It is further assumed in this analysis that the damping matrix of the structure is diagonalized by the modal matrix. Therefore, we write

$$[u]^T [c] [u] = [2\zeta\omega] \qquad (2c)$$

where $[2\zeta\omega]$ is the $n \times n$ diagonal matrix whose elements, $2\zeta_j\omega_j$, $j=1,\ldots,N$, involve the modal damping factors.

Transformations that reduce the number of degrees of freedom in the present problem are possible, and an example of such a transformation is a modal transformation based on the modal information described in the above paragraph. Another transformation that permits us to consider the present problem at a subset of node points (or, in the present case, at a number of points where measurements will be made in the field) is the transformation

$$[T]\{z_p\} = \{z\} \qquad (3)$$

where $\{z_p\}$ consists of a portion of the elements in $\{z\}$. It is the $N \times 1$ vector (where $N<n$) that contains a collection of elements that is a subset of the elements in $\{z\}$. $[T]$ is an $N \times n$ transformation matrix. The first element in $\{z_p\}$ equals the element indexed j_1 in $\{z\}$. The second element in $\{z_p\}$ equals the element indexed j_2 in $\{z\}$, etc. And finally, the Nth element in $\{z_p\}$ equals the element indexed j_N in $\{z\}$.

To establish a method for uniquely defining the transformation $[T]$, we use the following procedure. Let $[u_p]$ be a partial modal matrix with dimension $N \times N$. The first element in each column of $[u_p]$ is the j_1th element in the corresponding column of $[u]$. The second element in each column of $[u_p]$ is the j_2th element in the corresponding column of $[u]$, etc. Finally, the Nth element in each column of $[u_p]$ is the j_Nth element in the corresponding column of $[u]$. We multiply the transformation matrix $[T]$ times the partial modal matrix $[u_p]$, then equate the result to the first N columns of the modal matrix $[u]$. That is,

$$[T]\ [u_p] = [u_N] \qquad (4)$$

where the subscript N in $[u_N]$ has been included to denote the fact that this matrix contains only the first N columns from $[u]$. Based on this, $[T]$ can be defined as

$$[T] = [u_N]\ [u_p]^{-1} \qquad (5)$$

where it is assumed that $[u_p]$ possesses an inverse. (The degrees of freedom where the elements of $\{z_p\}$ are chosen must guarantee that $[u_p]$ can be inverted.) Based on the definition of $[u_p]$ in terms of $[u_N]$, it is guaranteed that the first column element in row j_1 of $[T]$ equals one, and all the other elements in that row are zero. The second column element in row j_2 of $[T]$ equals one, and all the other elements equal zero, etc. Finally, the Nth column element in row j_N of $[T]$

equals one, and all the other elements in the row equal zero. This guarantees the accuracy of Equation 3 at the N selected degrees of freedom, and establishes an approximation that interpolates deformations at the other degrees of freedom using the N mode shape functions.

The next step is to use the transformation defined in Equation 3 in Equation 1. Premultiply all terms in the resulting equation by the transpose of [T]. This yields

$$[m_p]\{\ddot{z}_p\} + [c_p]\{\dot{z}_p\} + [k_p]\{z_p\} = \{q_p\} \qquad (6)$$

where the following notation has been used.

$$[m_p] = [T]^T [m] [T] \qquad (6a)$$

$$[c_p] = [T]^T [c] [T] \qquad (6b)$$

$$[k_p] = [T]^T [k] [T] \qquad (6c)$$

$$\{q_p\} = [T]^T \{q\} \qquad (6d)$$

In these expressions $[m_p]$, $[c_p]$ and $[k_p]$ might be called partial mass, partial damping and partial stiffness matrices, respectively. $\{q_p\}$ might be called the partial force, and it is this vector function that we will identify.

Because the columns in $[u_N]$ are orthonormal with respect to [m], [c], and [k], and $[u_p]$ is defined as in Equation 4, the columns of $[u_p]$ are orthonormal with respect to $[m_p]$, $[c_p]$, and $[k_p]$. This means that we can write

$$[u_p]^T [m_p] [u_p] = [I_p] \qquad (7a)$$

$$[u_p]^T [c_p] [u_p] = [2\zeta\omega_p] \qquad (7b)$$

$$[u_p]^T [k_p] [u_p] = [\omega_p^2] \qquad (7c)$$

where $[I_p]$ is simply the N×N identity matrix, $[\omega_p^2]$ is the N×N diagonal matrix whose elements are the squares of the first N modal frequencies, and $[2\zeta\omega_p]$ is the N×N diagonal matrix whose elements involve the damping factors of the first N modes.

Because the orthogonality in Equations 7 exists, it is possible to uncouple the equations represented by the matrix Equation 6. We define the modal coordinates $\{\xi_p(t)\}$ as follows

$$[u_p]\{\xi_p(t)\} = \{z_p(t)\} \qquad (8)$$

Use of Equation 8 in Equation 6 and premultiplication of the result by $[u_p]^T$ yields the set of uncoupled equations

$$\{\ddot{\xi}_p\} + [2\zeta\omega_p]\{\dot{\xi}_p\} + [\omega_p^2]\{\xi_p\} = [u_p]^T\{q_p\} \qquad (9)$$

Now that the equations have been uncoupled, we note from Equation 8 that the vector of modal coordinates can be replaced by the inverse of $[u_p]$ multiplied times $\{z_p\}$. We do this in Equation 9 to obtain

$$[u_p]^{-1}\{\ddot{z}_p\} + [2\zeta\omega_p] [u_p]^{-1}\{\dot{z}_p\} + [\omega_p^2] [u_p]^{-1}\{z_p\} = [u_p]^T\{q_p\} \qquad (10)$$

This equation relates the accelerations, velocities and displacements at a collection of N degrees of freedom on a structure, and the mode shapes of the structure at the corresponding points, and the first N modal frequencies and damping factors to forces applied to the structure at the N degrees of freedom. (We could also replace $[u_p]^{-1}$ on the left side in Equation 10 with $[u_p]^T[m_p]$. The resulting expression would yield a clear interpretation of the computations performed in the "sum of weighted accelerations" method of Referecnes 3 and 4.)

It can be shown that if a structure is linear with known modal frequencies and orthonormal mode shapes, with a damping matrix that can be diagonalized and known modal damping factors, and if the structure has external forces applied only at the measurement points, then the external forces are identically characterized by Equations 6 and 10.

If we could establish all the terms on the left hand side of Equation 10 through experiment or analysis, or through a combination of the two, then we could determine the forces applied to a structure at the degrees of freedom $\{z_p\}$.

Practical Application of the Equations

In fact, it is generally only convenient to measure the absolute accelerations that a structure executes in response to the application of external forces. The accelerations can be integrated to obtain velocities, and then integrated again to establish displacements, but these integrations generally yield results with substantial inaccuracies.

To circumvent this problem, the following approach is taken. First, Equation 6 is used in the identification of forces. The matrices $[m_p]$, $[c_p]$ and $[k_p]$ are established by inverting Equations 7a, 7b and 7c. This yields

$$[m_p] = [u_p]^{-T}[I_p] [u_p]^{-1} \qquad (11a)$$

$$[c_p] = [u_p]^{-T}[2\zeta\omega_p] [u_p]^{-1} \qquad (11b)$$

$$[k_p] = [u_p]^{-T}[\omega_p^2] [u_p]^{-1} \qquad (11c)$$

The quantities on the right side in Equation 11 can be established through analysis or a combination of analysis and experiment. The latter approach can be implemented in the following way. (1) Form a finite element model of the structure. (2) Evaluate its modal frequencies and mode shapes. (3) Experimentally estimate the modal frequencies, modal damping factors, and mode shapes. (4) Adjust the finite element model until its modal characteristics accurately match the experimentally obtained modal characteristics. (5) Orthonormalize the mode shapes using the finite element model. (6) Evaluate the right hand sides of Equations 11 using the orthonormal mode shapes, the experimental modal frequencies, and the experimental modal damping factors. This approach is probably better than a purely analytical approach because it incorporates the actual behavior of the system, and this may be especially important in establishing the modal damping factors.

The Fourier transform of Equation 6 can be taken to obtain

$$[m_p]\{Z_p(f)\} + (i2\pi f)^{-1}[c_p]\{Z_p(f)\} + (i2\pi f)^{-2}[k_p]\{Z_p(f)\} = \{Q_p(f)\} \qquad (12)$$

where $\{Q_p(f)\}$ is the Fourier transform of $\{q_p(t)\}$, $\{Z_p(f)\}$ is the Fourier transform of $\{z_p(t)\}$, and we have used the facts that the Fourier transform of velocity is $(i2\pi f)^{-1}$ times the Fourier transform of acceleration, and the Fourier transform of displacement is $(i2\pi f)^{-2}$ times the Fourier transform of acceleration.

It must be recognized that in an actual analysis using field measured data, Equation 12 cannot be evaluated at $f=0$. This is precisely the region where problems occur in the numerical integration of measured acceleration data. In going to the frequency domain, the problems of numerical integration are avoided, but the trade-off is that the quasi-static elements in the force are not evaluated. In a numerical analysis, the continuous Fourier transforms shown in Equation 12 are replaced by discrete Fourier transforms (DFT).

Equation 12 provides a formula for the Fourier transform of the equivalent force applied to the structure at the degrees of freedom where response acceleration measurements are made. Once the response accelerations are measured, they can be Fourier transformed, and the left side of Equation 12 can be completely evaluated.

The relation between the external forces and the structural responses involve modal information up to the Nth mode. In view of this, it appears that the relation can only be accurate up to the frequency range between the Nth and the (N+1)th mode. For this reason, the acceleration response measurement used on the left side in Equation 12 should be filtered between the Nth and (N+1)th modal frequencies before the equivalent forces are computed.

Once it is evaluated, $\{Q_p(f)\}$ can be used in a number of ways. First, it can simply be inverse Fourier transformed to obtain the vector force time history, $\{q_p(t)\}$. Second, when the applied force is a stationary random process, the vector $\{Q_p(f)\}$ can be used to estimate the spectral density matrix of the applied equivalent forces. Third, when the applied force is governed by a parametric model, then $\{Q_p(f)\}$ or $\{q_p(t)\}$ can be used in a least squares or a maximum likelihood framework to estimate the parameters of the model.

Example 1

The first example is simply an analytical demonstration aimed at showing what equivalent forces are estimated for a structure when the character of the actual external force is known. Consider Equation 6d. If the actual external force distribution is known, then it is related to the equivalent forces as shown. In actual applications $\{q(t)\}$ will not be known, but it is interesting to see how closely $\{q_p(t)\}$ and $\{q(t)\}$ resemble one another in a controlled situation.

Therefore, in this example we consider the uniform rod shown in Figure 1, excited by distributed loads in the axial direction. The loads are described later. The orthonormal mode shapes of the system are known, and are given by

$$u_0(x) = (mL)^{-1/2}, \quad 0 \le x \le L \qquad (13a)$$

$$u_j(x) = (2/mL)^{1/2}\cos(j\pi x/L), \quad 0 \le x \le L \qquad (13b)$$

where m is the mass per unit length of the rod, and L is the length of the rod. These expressions can be used to populate the matrices $[u_N]$ and $[u_p]$, and once these are known, the transformation matrix [T] can be established. This was done using 100 rows and 10 columns in the $[u_N]$ matrix, and 10 rows and 10 columns in the $[u_p]$ matrix. The discretization of the modal vectors was uniform, and the measurement locations were placed uniformly along the rod. The constant m was taken as 1.0, and the rod length was set at 1.0.

A sequence of load distributions is considered in this example. The first load is constant, as a function of x. The other loads, five in number, vary sinusoidally, as a function of x. That is

$$q_1(x) = 1, \quad 0 \le x \le L \qquad (14a)$$

$$q_r(x) = \sin(s\pi x/L), \quad 0 \le x \le L, \quad r=2,3,4,5,6 \qquad (14b)$$

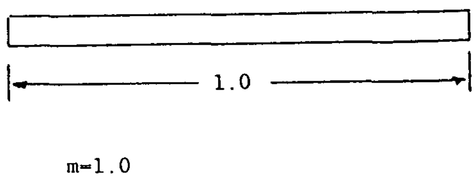

Figure 1. Uniform rod used in Example 1.

As noted above, the length is 1.0, and in the five examples where the load varies, the constant s is chosen as 1, 2, 5, 10, and 15.

The results are shown in Figures 2a through 2f. Both the actual load and the identified load are shown in each plot. Because the identified equivalent loads at 10 points represent the actual loading at 100 discrete points (that is, each modal vector is generated using 100 points), the identified loads are 10 times as great as the idealized loads. To compare the loads, the identified loads were divided by 10, and straight lines were drawn between the load values in each set.

It is clear that when the actual load is smooth and has little variation, then the identified load approximates it very well. However, as the actual load varies more rapidly, the identified load yields a poorer approximation. This seems to be a sort of spatial Nyquist sampling effect.

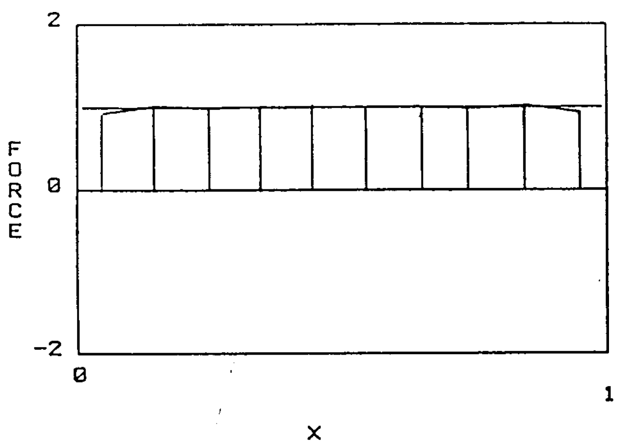

Figure 2a. Actual and estimated force distributions on uniform rod of Example 1.

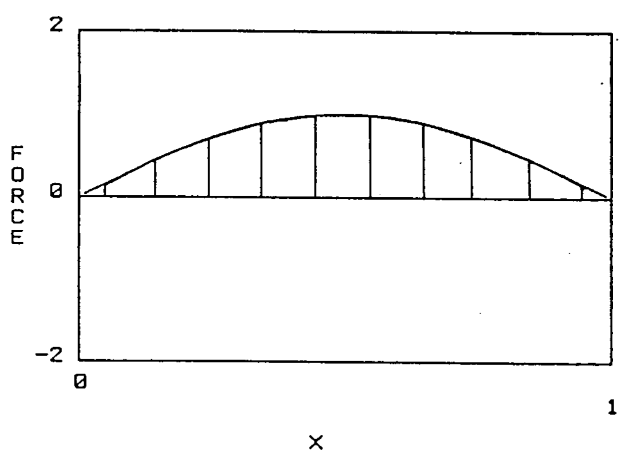

Figure 2b. Actual and estimated force distributions on uniform rod of Example 1.

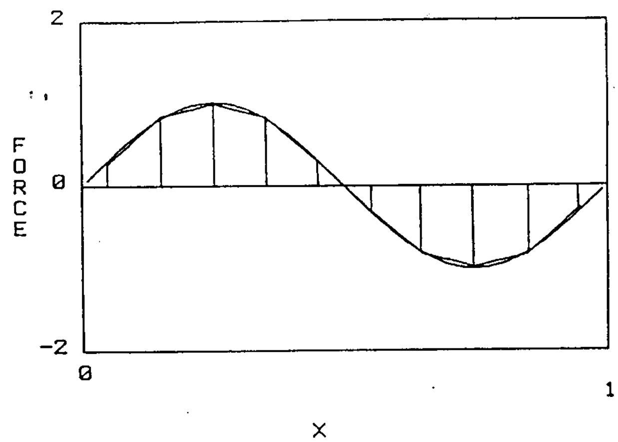

Figure 2c. Actual and estimated force distributions on uniform rod of Example 1.

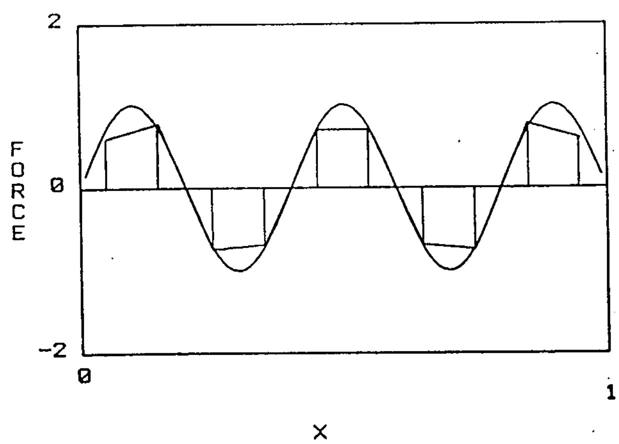

Figure 2d. Actual and estimated force distributions on uniform rod of Example 1.

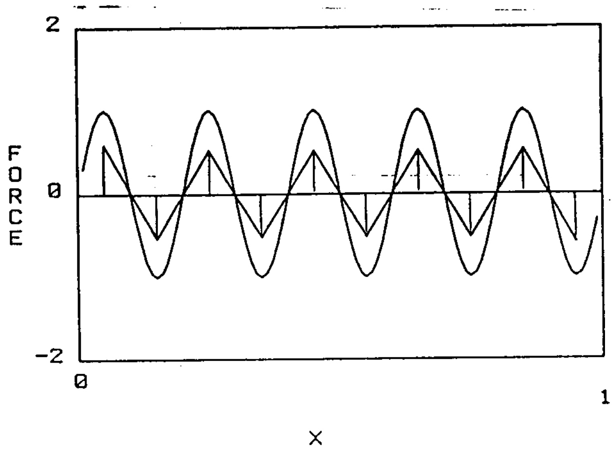

Figure 2e. Actual and estimated force distributions on uniform rod of Example 1.

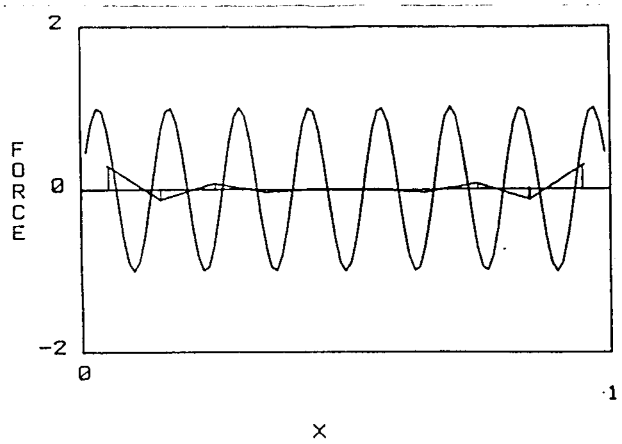

Figure 2f. Actual and estimated force distributions on uniform rod of Example 1.

Example 2

The second example is a combined analytical and experimental example. In this example, the lucite beam shown schematically in Figure 3 was suspended on elastic tubing and excited at the two locations shown in the figure. The excitations were band limited white noise force random processes. The acceleration responses were measured at the three locations shown in the figure. The measured acceleration responses were used with the method developed in this paper to estimate the applied excitation forces.

To obtain the input force estimates, the modal damping factors, modal frequencies, and orthonormal mode shapes were required. The first was obtained from excitation and response spectral densities estimated using test data. The second and third were obtained using analytical forms from Reference 6. (These are not precisely correct for the experimental system, but were used as approximations.) The analytical mode shapes are given by

$$u_1(x) = (mL)^{-1/2}, \quad 0 \le x \le L \tag{15a}$$

$$u_2(x) = (12/mL^3)^{1/2} (x-L/2), \quad 0 \le x \le L \tag{15b}$$

$$u_j(x) = (mL)^{-1/2} (\cos(\beta_j x) - \sin(\beta_j x)), \quad 0 \le x \le L, \quad j=3,4,\ldots \tag{15c}$$

$$\text{where } \beta_j^4 = \omega_j^2 m/(EI), \quad j=1,2,3,\ldots \tag{15d}$$

The modal frequencies and modal damping factors are listed in the following table.

Table 1. Modal frequencies and modal damping factors for example two.

Mode Number	Modal Frequency (rad/sec)	Modal Damping Factor
1	0	0
2	0	0
3	510	0.05

The results are shown in Figures 4a and 4b. The measured excitation forces and response accelerations were filtered at 940 rad/sec (150 Hz). The plots compare the estimated excitation forces to the measured excitation forces. Reasonable agreement between the estimated and measured excitation forces appears to exist. Differences in the measured and estimated forces arise from measurement noise, system nonlinearity, and the differences between the model and the actual system modal frequencies, modal damping factors, and mode shapes.

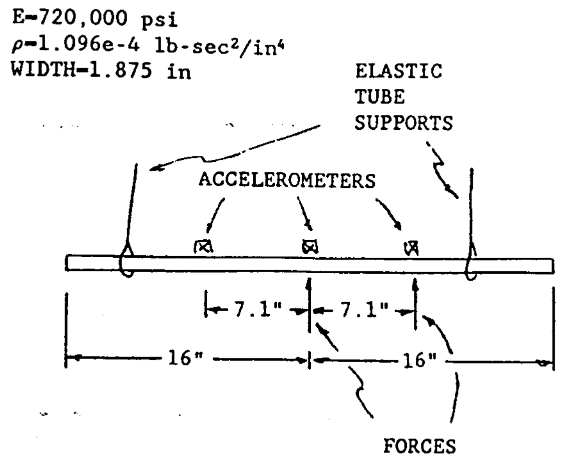

Figure 3. Schematic diagram of the lucite beam used in the experimental Example 2.

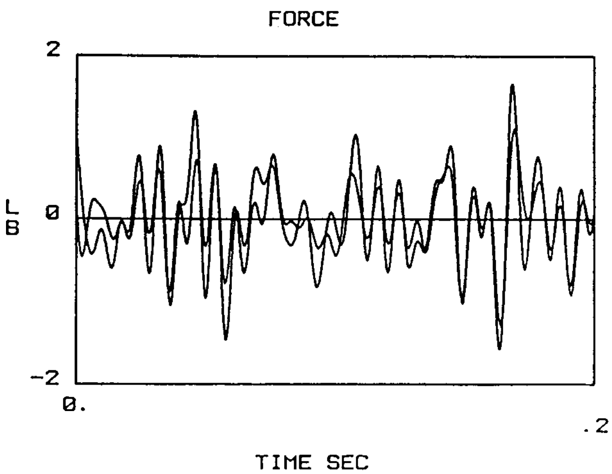

Figure 4a. Actual and estimated forces at the center of the beam in Example 2.

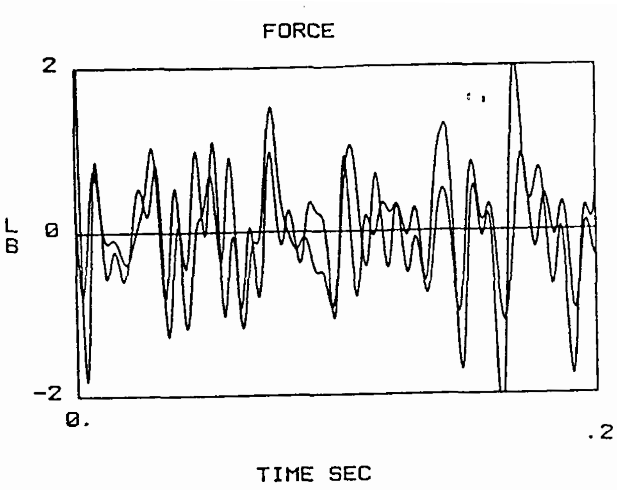

Figure 4b. Actual and estimated forces at a point 7.1 inches to the right of the center of the beam in Example 2.

Conclusion

A technique for the estimation of the external forces applied to a linear structure has been presented in this paper. The technique requires the establishment of a finite element model for the structure under consideration and permits the use of both analytical and experimental data in the force estimation.

The results indicate that reasonable estimates of the applied forces can be established in a combined analytical and experimantal framework. Therefore, the technique has the potential for being useful in the estimation of loads applied to structures in the field.

The analysis presented in this paper permits more than the estimation of distributed forces on a structure. It also permits the interpretation of the computations done in the sum of weighted accelerations method for force identification. A separate paper could be devoted to this subject. Further, the present technique can be applied in random vibration force distribution analyses and parametric force excitation studies. These subjects are left for later investigations.

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