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RADIATION REACTION IN NONRELATIVISTIC QUANTUM THEORY*

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ABSTRACT

In this article we review some recent work on the quantum theory of radiation reaction. The starting point of this work is the Heisenberg operator equation of motion for a nonrelativistic point electron coupled to the quantized electromagnetic field. It is shown that this equation, in contrast to its classical counterpart, leads to a finite value for the electrostatic self-energy of a point electron and, for values of the fine structure constant $\alpha \lesssim 1$, admits neither runaway behavior nor noncausal motion. Furthermore, the correspondence limit of the solution to the quantum mechanical equation of motion agrees with that of the Lorentz-Dirac theory in the classical regime, but without the imposition of additional conditions and with no possibility of observable noncausality. Thus, a consistent picture of a classical point electron emerges in the correspondence limit of the quantum mechanical theory.

I. INTRODUCTION

I would like to describe some calculations[1,2] which E. J. Moniz and I carried out for the purpose of understanding more clearly the relationship between classical and quantum electrodynamics, particularly in regard to their treatment of radiation reaction. Let us begin by reviewing some of the questions of interest here.

According to the classical theory of radiation reaction due to Abraham, Lorentz and Dirac,[3-5] a nonrelativistic point electron, interacting with its self-field and subject to an external force $\vec{F}(t)$, obeys the equation of motion

$$m_0\ddot{\vec{R}} = \vec{F}(t) - \delta m\ddot{\vec{R}}(t) + (2e^2/3c^3)\dddot{\vec{R}}(t) \quad , \tag{1.1}$$

where δm is the electron's electrostatic self-energy.

This theory of radiation reaction suffers from a number of defects besides the fact the $\delta m = \infty$ for a point electron, a fact which can after all be swept under the rug by working with the experimental mass, according to the philosophy of renormalization.

The first defect is that Eq. (1.1) admits runaway solutions, i.e. solutions for which the acceleration of a particle increases exponentially, even in the absence of external forces.

The second defect is that the solutions violate causality. This comes about when runaways are eliminated from the theory by imposition of a suitable asymptotic condition. To see this, notice that the general solution to Eq. (1.1) is

$$\ddot{\vec{R}}(t) = e^{t/\tau}\left[\ddot{\vec{R}}(0) - (1/\tau m)\int_0^t dt' e^{-t'/\tau}\vec{F}(t')\right] \quad , \tag{1.2}$$

with $\tau = 2e^2/3mc^3$. If $\ddot{\vec{R}}(0)$ is chosen arbitrarily, Eq. (1.2) gives an acceleration which grows asymptotically like $e^{t/\tau}$ even if the force acts only for a finite period of time. This behavior can be avoided if you impose the condition

$$\ddot{\vec{R}}(0) = (1/\tau m)\lim_{t\to\infty} \int_0^t dt' e^{-t'/\tau}\vec{F}(t') \quad , \quad (\text{i.e. } \lim_{t\to\infty} \ddot{\vec{R}}(t) \to 0).$$

But then one can write the solution (1.2) in the form

$$\ddot{\vec{R}}(t) = (1/\tau m) \int_t^\infty dt' e^{-(t'-t)/\tau}\vec{F}(t') \quad ,$$

or, introducing $s = (t'-t)/\tau$, as

$$\ddot{\vec{R}}(t) = (1/m)\int_0^\infty ds\ e^{-s}\vec{F}(t+\tau s) \quad . \tag{1.3}$$

This form of the solution displays clearly the acausal behavior known as preacceleration: the electron accelerates before the force acts.

While these defects mar the internal consistency of classical electrodynamics, the point can be made that Eq. (1.3) does in fact correctly describe classical radiation damping, in so far as it has been tested, and the view is often adopted that, since preacceleration occurs on such a short time scale ($\sim 10^{-23}$ seconds for an electron), the acausal effects would occur in the quantum domain, which is where one has to look for a resolution of the problem.

It is a very reasonable proposition that runaway solutions should not occur in quantum theory. One would not expect a Heisenberg-picture operator to display an exponentially growing dependence on time, since its time

development is given by

$$O(t) = e^{iHt}O(0)e^{-iHt} \quad ,$$

with e^{iHt} unitary. Nevertheless, to date no rigorous proof of the absence of runaways in quantum electrodynamics has been given.

Unfortunately, such a rigorous proof is not the subject of the present paper either. Instead, I will describe some rather straightforward calculations which appear to shed some light on the following questions:

(i) What is the mechanism by which runaway solutions are eliminated in quantum mechanics?

(ii) How does quantum theory manage to suppress the runaways and at the same time give Eq. (1.1) in the correspondence limit?

(iii) What sort of formula do you get for the electrostatic self-energy in quantum theory?

(iv) What about preacceleration?

In other words, we will discuss whether quantum theory resolves any of the problems of consistency which appear to be present already in classical electrodynamics. We will not be addressing the more fundamental questions of the possible finiteness, and overall consistency, of quantum electrodynamics.

II. CLASSICAL ELECTRODYNAMICS OF EXTENDED CHARGES

It will turn out that some aspects of our results on the quantum theory of radiation reaction can be best understood by comparing them to the classical results for the motion of an extended charge. For a spherically symmetric static charge distribution $\rho(\vec{x},t) = \rho(\vec{x} - \vec{R}(t))$, where $\vec{R}(t)$ is the coordinate of the mean position of the charge, Eq.(1.1) is replaced by[6]

$$m_0\ddot{\vec{R}}(t) = \vec{F}(t) - (2e^2/3c^2)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\gamma_n\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}} + (\text{non-linear terms}) \tag{2.1}$$

where

$$\gamma_n = \int\int d\vec{x}d\vec{x}'\rho(\vec{x},t)|\vec{x} - \vec{x}'|^{n-1}\rho(\vec{x}',t) \propto L^{n-1} \quad ,$$

and L is the effective charge radius.

We have shown explicitly in Eq.(2.1) only terms which are linear in the particle's velocity or its time derivatives. These terms all arise from the electric self-field. The non-linear terms, which arise both from the electric and magnetic self-fields, are all of order $|\dot{\vec{R}}/c|^2$ times the linear terms. These are neglected in this discussion, since we are considering the motion of a nonrelativistic electron.

For simple charge distributions, the coefficients γ_n can be explicitly evaluated and the series summed. Thus, for a spherical shell, one obtains

$$\gamma_n = (2e^2)(2L)^{n-1}/(n+1) \quad , \tag{2.2}$$

and the equation of motion can be written in the form[2,7,8]

$$\ddot{\vec{R}}(t) = \vec{F}(t)/m(1 - c\tau/L) + \xi[\dot{\vec{R}}(t - 2L/c) - \dot{\vec{R}}(t)] \quad , \tag{2.3}$$

neglecting non-linear terms, where

$$\xi = (c/2L)(c\tau/L)/(1 - c\tau/L) \quad , \quad \tau = 2e^2/3mc^3 \quad , \quad m = m_0 + 2e^2/3Lc^2 \quad .$$

The solutions to Eq. (2.3) have been analyzed fully.[2,8] One finds that if $L > c\tau (\xi > 0)$ Eq. (2.3) has no runaway nor preaccelerating solutions, while if $L \ll c\tau$ Eq. (2.3) reduces to Eq. (1.1) with $m_0 + \delta m = m$. Runaway and acausal solutions of Eq. (2.3) occur if $L < c\tau$.

The pertinence of these results to the quantum mechanical case is the following. We will find that the structure of the radiation reaction problem for a quantum mechanical point electron, i.e. an electron with zero charge radius, is similar to that of a classical extended charge. Specifically, the quantum mechanical equation of motion for a point charge has the general form of Eq. (2.1), with the electron Compton wavelength λ formally playing the role of the charge radius L. It is the fact that there is a new length scale in the quantum theory which allows this to happen.

III. QUANTUM THEORY OF RADIATION REACTION

Our plan is to first derive the Heisenberg-picture operator equation of motion for a nonrelativistic electron, including the self-force terms, and then to analyze some properties of its solutions. In other words, we want to study the quantum counterparts of Eq. (1.1) or Eq. (2.1).

A. Equation of Motion

To derive the equation of motion, we follow the Abraham-Lorentz procedure for deriving the self-force on an electron, except that we must remember to take proper account of the fact that we are working with operators. For the purposes of the present discussion, this just means that we pay attention to the order of non-commuting quantities.

Our starting point for this calculation is the Hamiltonian[9]

$$H = \frac{1}{2m_0}\left[\vec{P} - \frac{e}{c}\vec{A}(\vec{R})\right]^2 + \frac{1}{8\pi}\int d\vec{r}\{\vec{E}^2(\vec{r},t) + [\vec{\nabla}\times\vec{A}(\vec{r},t)]^2\} \quad , \qquad (3.1)$$

where

$$\vec{A}(\vec{R}) = \int d\vec{r}\rho(\vec{r} - \vec{R}(t))\vec{A}(\vec{r},t)$$

and

$$\vec{E} = \vec{E}_{long} + \vec{E}_{trans} \quad .$$

This describes a nonrelativistic charged particle of mechanical mass m_0 and (spherically symmetric) charge distribution[10] [defined so that $\int d\vec{r}\rho(\vec{r} - \vec{R}) = 1$] interacting with an electromagnetic field, computed in Coulomb gauge. $\vec{P}(t)$ and $\vec{R}(t)$ are, respectively, the Heisenberg-picture momentum and position operators of the particle.

Proceeding in standard fashion, we use Eq. (3.1) to derive the Heisenberg equations of motion, and arrive at the operator form of the Lorentz force equation

$$\frac{d}{dt}\,(m_0\dot{\vec{R}}) = e\vec{E}(\vec{R}) + (e/2c)[\dot{\vec{R}} \times \vec{B}(\vec{R}) - \vec{B}(\vec{R}) \times \dot{\vec{R}}] \tag{3.2}$$

and the usual operator field equations for the electromagnetic potentials $\vec{A}$ and ϕ (here written in Lorentz gauge[11])

$$-\Box\vec{A}(\vec{r},t) = 4\pi e\vec{j}(\vec{r},t) \quad , \tag{3.3a}$$

$$-\Box\phi(\vec{r},t) = 4\pi e\rho(\vec{r},t) \quad . \tag{3.3b}$$

In writing these equations, we have used the following notation:

$$\vec{B}(\vec{R}) = \vec{\nabla} \times \vec{A}(\vec{R}) \quad , \qquad \vec{E}(\vec{R}) = -\vec{\nabla}\phi(\vec{R}) - (1/c)\,\frac{\partial\vec{A}(\vec{R})}{\partial t}$$

and

$\vec{j}(\vec{r},t) = \frac{1}{2}\,[\rho(\vec{r} - \vec{R}(t)),\dot{\vec{R}}(t)]_+$ = the single particle current density operator.

As in the classical case, our goal now is to use the field equations (3.3) to eliminate the self-fields from the Lorentz force equation. To do this, we first observe that the exact solution to Eq. (3.3), satisfying retarded boundary conditions, may be written

$$\vec{A}(\vec{r},t) = \vec{A}_{in}(\vec{r},t) + (e/c)\int d\vec{r}'\,\frac{\vec{j}(\vec{r}',t'_{ret})}{|\vec{r}-\vec{r}'|} \quad , \tag{3.4a}$$

$$\equiv \vec{A}_{in} + \vec{A}_{self}$$

and

$$\phi(\vec{r},t) = \phi_{in}(\vec{r},t) + e\int d\vec{r}' \frac{\rho(\vec{r}',t'_{ret})}{|\vec{r}-\vec{r}'|} \quad , \tag{3.4b}$$

$$\equiv \phi_{in} + \phi_{self}$$

where the retarded time $t'_{ret} = t - (|\vec{r}-\vec{r}'|/c)$. Note that $\lim_{t\to-\infty} \vec{A} \to \vec{A}_{in}$, which is a free field.

We can relate an operator evaluated at the retarded time t'_{ret} to its value at time t by the formula

$$\begin{aligned} \mathcal{O}(t'_{ret}) &= e^{+iH(t'_{ret}-t)}\,\mathcal{O}(t)\,e^{-iH(t'_{ret}-t)} \\ &= \sum_{n=0} \frac{(-i)^n}{n!} \frac{|\vec{r}-\vec{r}'|^n}{c^n} (ad^n H)\mathcal{O}(t) \quad , \end{aligned} \tag{3.5}$$

where

$$(ad\ H)\mathcal{O} = [H,\mathcal{O}]_- \quad , \quad (ad^2 H)\mathcal{O} = [H,[H,\mathcal{O}]_-]_- \quad .$$

Next, we use the expansion (3.5) and Eq. (3.4) to evaluate the electric and magnetic self-fields occurring in the Lorentz force equation. The resulting form of the quantum mechanical equation of motion is

$$\begin{aligned} m_0 \ddot{\vec{R}}(t) &= e\vec{E}_{in} + (e/2c)[\dot{\vec{R}} \times \vec{B}_{in} - \vec{B}_{in} \times \dot{\vec{R}}] \\ &+ (2e^2/3c^2) \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!c^n} \int\int d\vec{r}d\vec{r}' \frac{1}{2}[\rho(\vec{r}-\vec{R}(t))|\vec{r}-\vec{r}'|^{n-1}, (ad^{n+1}H)\vec{j}(\vec{r}',t)]_+ \quad . \end{aligned} \tag{3.6}$$

In writing Eq. (3.6), we have dropped the contributions from the magnetic self-field. These have been evaluated and found to be formally of order

$\dot{\vec{R}}^2/c^2$ times the leading contributions from the electric self-field, as one would expect. Such contributions should therefore be negligible for a slowly moving electron. The same should be true of the nonlinear terms associated with the electric self-field. The statement that these terms are negligible means, in the quantum mechanical context, that there is a subset of states in Hilbert space for which the matrix elements $\langle m|\dot{\vec{R}}^2/c^2|n\rangle$ are "small" and that one can work consistently to a given level of accuracy within this set of states. It is a crucial assumption of this calculation that such a set of states exists.

Equation (3.6) is the starting point for our study of radiation reaction in quantum mechanics. The main labor in this calculation consists in evaluating the nested commutators in Eq. (3.6), so that it can be put in a useful form. The details can be found in Ref. 2, and I won't reproduce them here. However, it might be instructive to see what the first few terms in the series (3.6) look like.

After evaluation of $[H,\vec{J}(\vec{r},t)]$, the n=0 term in Eq. (3.6) gives[12]

$$(m_0\ddot{\vec{R}})_{n=0} = -(2e^2/3c^2)\langle|\vec{r}-\vec{r}'|^{-1}\rangle\ddot{\vec{R}} \quad ,$$

with the notation $\langle|r-r'|\rangle = \int\int d\vec{r}d\vec{r}'\,\rho(\vec{r})\,|\vec{r}-\vec{r}'|\,\rho(\vec{r}')$. This is the same as the classical result. Similarly, the n=1 term gives

$$(m_0\ddot{\vec{R}})_{n=1} = (2e^2/3c^3)\langle 1\rangle\dddot{\vec{R}} \quad ,$$

which is again the same as in the classical case. The first new quantum terms come in when n=2, where you find

$$(m_0\ddot{\vec{R}})_{n=2}$$

$$= -\underbrace{(1/3c^4)\langle|\vec{r}-\vec{r}'|\rangle\dddot{\vec{R}}}_{\text{classical result}} - \underbrace{(1/9c^4)\langle|\vec{r}-\vec{r}'|^{-1}\rangle[3\{\dot{\vec{R}}^2,\ddot{\vec{R}}\} + \{\ddot{\vec{R}}\cdot\vec{R},\dot{\vec{R}}\} + \{\dot{\vec{R}}\cdot\vec{R},\ddot{\vec{R}}\} + \dot{\vec{R}}\cdot\ddot{\vec{R}}\,\dot{\vec{R}} + \dot{\vec{R}}\,\ddot{\vec{R}}\cdot\dot{\vec{R}}]}$$

neglecting order of operators, exactly the $\dot{\vec{R}}^2/c^2$ correction arising from $\vec{E}_{self}$ to the classical result for $\ddot{\vec{R}}$

$$- \underbrace{(8\pi/3c^2)(\hbar/m_0c)^2[\int d\rho^2(\vec{r})]\ddot{\vec{R}}}_{\text{new quantum term}} .$$

Let L denote the particle's charge radius and let $\lambda = (\hbar/m_0c)$ be the electron's Compton wavelength. We see that the new term diverges as $(1/L^3)$ as $L \to 0$, and is $\sim (\lambda/L)^2$ times the classical self-energy term. This suggests a small quantum mechanical correction to the self-energy if $\lambda << L$, but if $\lambda >> L$ (point charge limit), one will have to sum the series.

This is how one proceeds, carrying out the term by term evaluation of Eq. (3.6), until one has inferred the combinatorics governing the general term. The result is[1,2] (dropping all terms of order $\dot{\vec{R}}^2/c^2$ or smaller compared to the leading terms)

$$m_0\ddot{\vec{R}}(t) = e\vec{E}_{in} + (e/2c)[\dot{\vec{R}} \times \vec{B}_{in} - \vec{B}_{in} \times \dot{\vec{R}}]$$

$$- (2e^2/3c^2) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^n} A_n \overset{(n+2)}{\vec{R}}(t) \qquad , \tag{3.7}$$

with

$$A_n = \left[1 + \frac{\lambda}{3(n+2)} \frac{\partial}{\partial\lambda}\right] B_n \quad ,$$

$$B_n = \sum_{k=0}^{\infty} \frac{n!}{(n+2k)!} \binom{n+1+2k}{2k} \left(\frac{-\lambda^2}{4}\right)^k \int\int d\vec{r}\, d\vec{r}\,' \rho(r) |\vec{r}-\vec{r}\,'|^{n-1+2k} (\nabla^2_{\vec{r}\,'})^{2k} \rho(r') \quad .$$

We have used the notation that $\vec{R}^{(m)}(t) = d^m\vec{R}(t)/dt$ and, again, that $\lambda = \hbar/m_0c$. Note that each structure coefficient is a power series in (λ^2/L^2) and that if we retain only the $k = 0$ term in the series for B_n, which corresponds to taking the $\hbar \to 0$ limit of the expression, we recover the equation of motion for a classical extended charge.

B. Evaluation of the Structure Coefficients

a) Electrostatic Self-Energy

The electrostatic self-energy, defined as the coefficient of th acceleration arising from the self-force, is given by the n=0 term in the series in Eq.(3.7). Specifically, one obtains

$$\delta m = (2e^2/3c^2)A_0 = (2e^2/3c^2)(1 + \frac{\lambda}{6}\frac{\partial}{\partial\lambda})(1 + \lambda\frac{\partial}{\partial\lambda})\Omega_0 \tag{3.8}$$

where

$$\Omega_0 = \int \frac{d\vec{k}}{(2\pi)^3} \tilde{\rho}(k)^2 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} \left(\frac{\lambda k^2}{2}\right)^{2\ell} \int \frac{d\vec{r}}{r} e^{i\vec{k}\cdot\vec{r}}\, r^{2\ell} \quad ,$$

and with $\tilde{\rho}(k)$ the Fourier transform of $\rho(r)$. Doing the integral over r and summing the series leads to

$$\Omega_0 = (2/\pi)P \int_0^{\infty} dk \frac{\tilde{\rho}(k)^2}{1 - \lambda^2k^2/4} \quad , \tag{3.9}$$

where the improper k-integral has been regularized by taking the Cauchy principal value.

This formula for the electrostatic self-energy has a number of remarkable features:

i) If one lets $\lambda \to 0$ in Eqs.(3.8)-(3.9), and <u>then</u> goes to the point charge limit, one obtains the classic divergent expression for δm. However, if one first takes the point charge limit $(\tilde{\rho}(k) = 1)$ in Eq.(3.9), <u>keeping λ fixed</u>, one finds $\delta m = 0$. Thus, according to this calculation, one finds that the electrostatic self-energy of a point charge is zero in nonrelativistic quantum electrodynamics. This is a surprising result. To keep things in perspective we emphasize that it is <u>not</u> claimed that this calculation shows that the electron's self-mass is finite. There are contributions to this self-mass other than the one treated here, and these other contributions may well be infinite. Nervertheless, it would be most interesting to see what an essentially non-perturbative calculation, as this is, would tell us about these contributions.

ii) It is also interesting to study the self-energy in the case when the particle has a convergent form factor, such as the Yukawa form factor $\tilde{\rho}(k) = (1 + k^2L^2)^{-1}$. (L is the effective charge radius.) Using Eqs. (3.8)-(3.9), one finds[2] that the maximum value for δm occurs when $L \sim \lambda$, and that then

$$(\delta m)_{MAX} \sim \alpha m_0 \quad , \tag{3.10}$$

where α is the fine structure constant. This seems to be a physically reasonable result. For example, it would lead to hadronic electromagnetic mass shifts on the order of a few MEV if m_0 were chosen to be a few

hundred MEV. Furthermore, Eq.(3.10) excludes the possibility of a purely electromagnetic origin for the electron's mass within the framework of nonrelativistic quantum electrodynamics.

iii) For $0 < L << \lambda$, the electrostatic self-energy can actually become negative.[2]

iv) Additional insight into these results can be obtained by transforming Ω_0, Eq.(3.9), back into coordinate space. One finds that

$$\Omega_0 = \int\int d\vec{r}d\vec{r}' \rho(r) |\vec{r}-\vec{r}'|^{-1} \left[1 + \frac{\lambda^2}{4} \vec{\nabla}^2_{\vec{r}'}\right]^{-1} \rho(r') \quad . \tag{3.11}$$

The integral operator in this expression is defined by

$$[1 + \lambda^2\vec{\nabla}^2/4]^{-1}\rho(x) = \int dy S_\lambda(x-y)\rho(y) \equiv \rho_{eff}(x) \quad ,$$

with

$$\begin{aligned} S_\lambda(r) &= P\int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{1 - \lambda^2k^2/4} \\ &= - \cos(2r/\lambda)/(\pi\lambda^2 r) \quad . \end{aligned} \tag{3.12}$$

Thus we see that in this calculation all of the physics involved in the interaction of the charged particle with its quantized self-field is summarized in the "spreading function" $S_\lambda(r)$ which generates an effective charge distribution $\rho_{eff}(r)$ which is smeared out over a Compton wavelength. In this respect, our results are quite similar to those obtained many years ago by Weiskopf.[13] In his work, however, the spreading out of the charge distribution was caused by virtual electron-positron pairs, whereas in our strictly nonrelativistic treatment there are, of course, no positrons.

Finally, we note that the charge distribution ρ_{eff} generates an effective scalar potential

$$\phi_{eff}(\vec{r}) = \int d\vec{r}' \, |\vec{r}-\vec{r}'| \rho_{eff}(\vec{r}') \quad ,$$

from which the electrostatic self-energy can be calculated as

$$\Omega_0 = \int d\vec{r} \rho(r) \phi_{eff}(r) \quad .$$

b. The Remaining Coefficients

I will simply assert in passing that the other structure coefficients in Eq. (3.7) have been evaluated,[1,2] and that in the point charge limit one finds

$$A_n = \begin{cases} (-1)^{(n-1)/2} \dfrac{2n(4n+5)}{3(n+1)(n+2)} (2n-1)!! \lambda^{n-1} & , \quad n \text{ odd} \\ 0 & , \quad n \text{ even} \end{cases} \quad . \tag{3.13}$$

Thus the equation of motion (3.7) is indeed similar in structure to that of a classical extended charged particle, Eq.(2.1), with the Compton wavelength λ playing the role of a size parameter.

C. Solutions of the Equation of Motion

a) Motion in the Absence of External Forces

Now I want to discusss some properties of the solutions to Eq.(3.7). First I will discuss the motion of a "free" electron, i.e. one that experiences self-interactions, but is not acted upon by any external force. This is the situation in which one encounters runaway solutions classically, and we want to see what happens in the quantum mechanical case.

To investigate this question, we take matrix elements of the equation of motion between the exact stationary states of the Hamiltonian (3.1). We assume that among these states there are ones for which the matrix elements of the in-fields are negligible, and we confine our attention to these states. This is possible owing to the linearity of Eq. (3.7).

While we do not know how to construct the exact stationary states of the Hamiltonian (3.1), we do know that for such states one can write

$$\langle m|\dot{\vec{R}}(t)|n\rangle \quad e^{iE_{mn}t/\hbar} \langle m|\dot{\vec{R}}(0)|n\rangle \quad ,$$

with $E_{mn} = E_m - E_n$. We see that <u>if</u> there were runaway solutions to Eq.(3.7), there would have to be states such that $\langle m|\dot{\vec{R}}(0)|n\rangle \neq 0$ and for which $\beta \equiv iE_{mn}/\hbar$ has a positive real part.

Supposing that $\langle m|\dot{\vec{R}}(0)|n\rangle \neq 0$, and taking the indicated matrix elements of Eq.(3.7) results in a power series in the variable $\eta = \beta\lambda/c$:

$$1 = \frac{2}{3}\alpha\eta \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} (-1)^{(n-1)/2} \frac{(2n-1)!!}{n!} \left[\frac{1}{3}\left(\frac{2n}{n+1}\right)\left(\frac{4n+5}{n+2}\right)\right]\eta^{n-1} \tag{3.14}$$

$$\equiv \frac{2}{3}\alpha f(\eta) \quad . \tag{3.14}$$

In writing Eq.(3.14), I have factored out the root $\beta = 0$. This corresponds to motion at constant velocity, and is the expected result for a free electron. The question is whether there are other solutions to Eq.(3.14), corresponding to runaways or other unphysical motions.

The series (3.14) converges for $|\eta| < 1/2$. Inside its radius of convergence, the series can be summed and one obtains

$$f(\eta) = -\left(\frac{4i}{3}\right)\left[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}\right]$$

$$-\left(\frac{7}{3\eta}\right)\left[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2} - \frac{2}{7}\right]$$

$$+\left(\frac{2i}{3\eta^2}\right)\left[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2}\right] \quad . \tag{3.15}$$

Thus, to examine the question of runaways, one must determine the roots of the equation

$$1 = \frac{2}{3}\,\alpha f(\eta) \quad , \tag{3.16}$$

where α is the fine structure constant and $f(\eta)$ is given by Eq. (3.15). Here is what you find:[1,2]

i) For <u>physical</u> values of the fine structure constant, in fact for all $\alpha \lesssim 1$, Eq. (3.16) has <u>no</u> roots inside the radius of convergence $|\eta| = 1/2$;

ii) For large α, interpreted either as a strong coupling limit or a semi-classical limit, one does obtain a real root of Eq. (3.16) for $|\eta| < 1/2$. In fact, what one finds in this case is a small $\hbar$ expansion about the classical runaway solution

$$\beta \sim (1/\tau)[1 + (\text{numerical coefficient})\,\hbar^2 + \ldots]$$
$$= (1/\tau)[1 + (\text{numerical coefficient})'\,(1/\alpha)^2 + \ldots] \quad , \tag{3.17}$$

where $\tau = (2e^2/3mc^3)$;

iii) The large and small α regimes are separated in that there is a critical value of α, $\alpha_{crit} > 1$, such that

$$\left.\frac{d\eta}{d\alpha}\right|_{\alpha \to \alpha_{crit}} \to \infty \quad .$$

This behavior is like that of a first order phase transition, and it means that the radius of convergence of Eq. (3.17) cannot include the physical value of α .

Thus the analysis of the roots of Eq. (3.16) shows that runaway solutions are not present in nonrelativistic quantum electrodynamics, for physical values of the fine structure constant. Our understanding of this result is that the interaction of a charged point particle with its quantized self-field generates an effective charge distribution spread out over a Compton wavelength (Eq. (3.12)). This structure is reflected in the particle's equation of motion, resulting in a form for the quantum mechnanical equation of motion for a point charge (Eqs. (3.7) and (3.13)) which is similar to that of an extended classical charge (Eq.(2.1)). Several analyses[2,7,8] indicate that a classical charged particle of sufficient size (charge radius > classical electron radius) does not exhibit runaway behavior.

It is also interesting to inquire about the significance of the condition $|\eta| < 1/2$. Recalling the various definitions, we see that it says that

$$E_{mm} < \frac{1}{2} mc^2 \quad . \tag{3.18}$$

This condition represents a restriction on the energy eigenstates of particle plus field between which one can consistently evaluate matrix elements of Eq. (3.17), and expresses the fact that our results are limited to the nonrelativistic domain. It is remarkable that the criterion (3.18) is generated by the dynamical equations themselves.

b) External Forces

Now let us consider how the electron moves in response to a time dependent external force $\vec{F}(t)$. If we again neglect the in-fields, the equation of motion can be solved by Fourier transformation and the solution can be written in the form

$$m_0 \ddot{\vec{R}}(t) = \int_{-\infty}^{\infty} dt' G(t - t') \vec{F}(t') \quad , \tag{3.19}$$

where the response function $G(t - t')$ is given by [1,2]

$$G(t - t') = \frac{1}{2\pi} \int_{-c/2\lambda}^{c/2\lambda} d\omega \frac{e^{i\omega(t - t')}}{1 - \frac{2}{3} \alpha f(i\omega\lambda/c)} \quad . \tag{3.20}$$

In Eq. (3.20), the function $f(i\omega\lambda/c)$ is defined by Eq. (3.15) .

To derive Eqs. (3.19) - (3.20), we have had to require that $\vec{F}(\omega)$ vanish for $|\omega| > c/2\lambda$. This is because the series defining $f(i\omega\lambda/c)$ does not converge unless $|\omega| < c/2\lambda$. This condition is of course closely related to that derived above, $E_{mn} < \frac{1}{2} mc^2$, as a requirement of non-relativistic motion, and it also follows directly from the condition that the applied force change by a small amount in the time required for light to cross a Compton wavelength.

It is not difficult to see that the response function (3.20) does not allow for observable preacceleration if $\alpha << 1$. An approximate evaluation of (3.20) shows that the quantum response function is spread about the origin $(t = t')$ with a minimum width given by the characteristic time $\Delta T \sim 2\lambda/c$. This time is determined jointly by the uncertainty principle and the equation of motion (through the condition $\omega < c/2\lambda$), and the time at which the particle starts to move in response to the applied force cannot be determined

more accurately than ΔT. The time scale for preacceleration, on the other hand, is

$$\tau \sim (e^2/mc^3) = \alpha(\lambda/c) << \Delta T \quad \text{if} \quad \alpha << 1 \quad ,$$

so there can be no observable violation of causality. Note, however, that this conclusion would not follow if $\alpha \sim 1$.

If the force is cut off at a frequency small compared to c/λ, the correspondence limit of Eq. (3.19) can be obtained by expanding the denominator of the response function (3.20). One finds that

$$m_0 \ddot{\vec{R}}(t) = \vec{F}(t) + \tau \dot{\vec{F}}(t) + \ldots$$

$$= \int_0^\infty ds e^{-s} \vec{F}(t + s\tau) \quad . \tag{3.21}$$

Thus, in the classical regime, Eqs. (3.19)-(3.20) give the same result as Eq. (1.3) - the solution to the classical Lorentz-Dirac equation which results when the runaway solution is eliminated by fiat. The interesting point is that Eqs. (3.19)-(3.20) place a limit, originating from quantum theory, on the applicability of the classical solution (1.3).

IV. DISCUSSION OF THE CORRESPONDENCE BETWEEN QUANTUM AND CLASSICAL ELECTRODYNAMICS

We have studied the quantum theory of a nonrelativistic charged particle coupled to the quantized electromagnetic field. This model, which we have been calling "nonrelativistic quantum electrodynamics," is defined by the Hamiltonian (3.1).

We have mentioned that nonrelativistic classical electrodynamics appears to be internally consistent in describing the motion of extended charged particles. However, taking the point charge limit of the theory of a classical extended charge results in a set of equations whose solutions display runaway behavior and preacceleration. In quantum mechanics, on the other hand, the point charge theory is consistent, displaying neither runaway behavior nor observable acausality. Furthermore, the correspondence limit of the solutions of the quantum mechanical equation of motion reproduce those properties, and only those properties, of the solutions of the classical Lorentz-Dirac equation which are physically reasonable. Thus, a consistent picture of a classical point electron emerges as the correspondence limit of a quantum mechanical point electron, but not as the point limit of a classical extended charge.[14]

V. CONCLUSION

In conclusion, I want to mention some questions which I believe this work raises. The model studied here, based on the Hamiltonian (3.1), lacks relativistic invariance. Consequently, no overall consistency for this model can be claimed. Moreover, we do not know which of our conclusions about the consistency of nonrelativistic quantum electrodynamics would continue to hold in relativistic quantum electrodynamics. Thus, an attempt to extend our calculations to the domain of fully relativistic field theory is clearly important.

Another point concerns the interesting fact that, according to our calculation, nonrelativistic quantum electrodynamics apparantly would display runaway behavior and preacceleration if the fine structure constant were greater than about one. This suggests an upper bound on α. Is this bound real, or would it disapperar if more physics, such as pair creation, were included in the model? If the bound is real, what general property of the theory is being revealed?

Finally, as has recently been stressed by Grotch and Kazes,[15] it would be quite instructive to understand more clearly the relationship between our method of calculation and standard perturbation theory. Some progress has been made on this question, and in fact it turns out that quite different assumptions and approximations underlie the two methods. There is, therefore, no particular reason why the results obtained from these two methods should agree in any given order of approximation. I will not go into detail about this work here, since it is discussed elsewhere.[16,17]

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