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DETERMINING REJECTION REGIONS AND POWER THAT OPTIMIZE THE INSPECTOR'S CHANCES OF DETECTING DIVERSION AND/OR FALSIFICATION*

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ABSTRACT

Hakkila, et al.[1] and Shipley[2] have couched the inspector's verification problem of testing the hypothesis of diversion and falsification in the framework of the general linear statistical model.[3] Three specific models are investigated:

(1) Diversion and k distinct falsifications (k+1 degrees of freedom),
(2) Diversion and the accumulation of all the falsifications (2 degrees of freedom),
(3) Diversion only (1 degree of freedom).

A test statistic has been derived for models (1) and (2) by the likelihood ratio procedure under the hypothesis of zero falsification and zero diversion versus a one-sided alternative of positive diversion, positive falsification, or both. An analogous test has been developed for model (3) for diversion only. A detailed discussion of this development is given by Shipley.[2] Utilizing his notation, the test variables called Inspector's Sufficient Statistics ISS_{k+1}, ISS_2, and ISS_1 are used for testing the above three models (1), (2), and (3) respectively. Note that ISS_1 corresponds to the well known MUF-D statistic, which is currently in vogue. These two tests, for purposes of this discussion, are equivalent.

The objectives of this paper are to review the ISS procedures, develop optimal critical regions for ISS tests, and compare detection probability (power) curves for models (1), (2), and (3). Optimal critical regions depend upon values obtained from one-sided chi-squared tests with more than one degree of freedom. Details of such a procedure are not to be found in the literature, and this strange neglect was one of the motivating factors in this study.

I. INSPECTOR'S SUFFICIENT STATISTIC
GENERAL BACKGROUND

The operator can hide diversion in the measurement uncertainties or through falsification of his reported data; therefore, it is important that the inspector use a test statistic that protects against both possibilities. Consider a single balance period and a unit process accounting area (UPAA) having true initial and final inventories I_0 and I_1, and true input and output transfers T_0 and T_1. The operator measures these quantities, perhaps diverts some goal quantity of material, and then reports to the inspector the possibly falsified inventory and transfer measurements. Let the vector $\bar{Z}$ be defined as

$$\bar{Z} = (\bar{I}_0, \bar{I}_1, \bar{T}_0, \bar{T}_1) \quad ,$$

where $\bar{I}_0$ and $\bar{I}_1$ are the operator's reported inventory measurements, and $\bar{T}_0$ and $\bar{T}_1$ are the operator's reported transfer measurements. For the reference process the initial and final inventory measurements ($\bar{I}_0$ and $\bar{I}_1$) will each be the sum of measurements made on several process vessels, and the input and output transfer measurements ($\bar{T}_0$ and $\bar{T}_1$) each will be the sum of individual transfer measurements.

The inspector can either make his own independent measurements or he can verify the operator's measurements. The inspector's measurements are denoted by a vector Z having four components,

$$Z = (I_0, I_1, T_0, T_1) \quad .$$

where I_0 and I_1 are the inspector's initial and final inventory measurements, and T_0 and T_1 are the inspector's input and output transfer measurements. These inventory and transfer measurements correspond directly with those of the operator.

The statistical analysis is made under the assumption of a model that specifies all measurements are random and follow the normal distribution. This assumption leads to determining the statistics for testing hypotheses of (1) zero diversion, (2) zero falsification, or (3) zero diversion and zero falsification. The procedure involves the likelihood ratio test resulting in the aforementioned ISS, which can be written in general as the sum of distinct, statistically independent chi squared variables.

II. EXPLICIT FORMS OF THE STATISTIC

The ISS in the case where component falsifications of transfer and inventory measurements

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are considered important is labeled ISS_0. For convenience we will assume k = 4 so that ISS_0 can be written as

$$ISS_0 = \frac{\max[0, \bar{I}_0 - \tilde{I}_0](\bar{I}_0 - \tilde{I}_0)}{2[\bar{\sigma}_I^2(0) + \tilde{\sigma}_I^2(0)]}$$

$$+ \frac{\max[0, \bar{I}_1 - \tilde{I}_1](\bar{I}_1 - \tilde{I}_1)}{2[\bar{\sigma}_I^2(1) + \tilde{\sigma}_I^2(1)]}$$

$$+ \frac{\max[0, \bar{T}_0 - \tilde{T}_0](\bar{T}_0 - \tilde{T}_0)}{2[\bar{\sigma}_T^2(0) + \tilde{\sigma}_T^2(0)]}$$

$$+ \frac{\max[0, \bar{T}_1 - \tilde{T}_1](\bar{T}_1 - \tilde{T}_1)}{2[\bar{\sigma}_T^2(1) + \tilde{\sigma}_T^2(1)]}$$

$$+ \frac{\max[0, M_p]M_p}{2\sigma_p^2} \quad . \tag{1}$$

M_p is a weighted sum of operator's and inspector's materials balances given by

$$M_p = \frac{\tilde{\sigma}_I^2(0)\bar{I}(0) + \bar{\sigma}_I^2(0)\tilde{I}(0)}{\bar{\sigma}_I^2(0) + \tilde{\sigma}_I^2(0)}$$

$$- \frac{\tilde{\sigma}_I^2(1)\bar{I}(1) + \bar{\sigma}_I^2(1)\tilde{I}(1)}{\bar{\sigma}_I^2(1) + \tilde{\sigma}_I^2(1)}$$

$$+ \frac{\tilde{\sigma}_T^2(0)\bar{T}(0) + \bar{\sigma}_T^2(0)\tilde{T}(0)}{\bar{\sigma}_T^2(0) + \tilde{\sigma}_T^2(0)}$$

$$- \frac{\tilde{\sigma}_T^2(1)\bar{T}(1) + \bar{\sigma}_T^2(1)\tilde{T}(1)}{\bar{\sigma}_T^2(1) + \tilde{\sigma}_T^2(1)} \quad , \tag{2}$$

and the variance of M_p is

$$\sigma_p^2 = \frac{\bar{\sigma}_I^2(0)\tilde{\sigma}_I^2(0)}{\bar{\sigma}_I^2(0) + \tilde{\sigma}_I^2(0)} + \frac{\bar{\sigma}_I^2(1)\tilde{\sigma}_I^2(1)}{\bar{\sigma}_I^2(1) + \tilde{\sigma}_I^2(1)} \tag{3}$$

$$+ \frac{\bar{\sigma}_T^2(0)\tilde{\sigma}_T^2(0)}{\bar{\sigma}_T^2(0) + \tilde{\sigma}_T^2(0)} + \frac{\bar{\sigma}_T^2(1)\tilde{\sigma}_T^2(1)}{\bar{\sigma}_T^2(1) + \tilde{\sigma}_T^2(1)} \quad ,$$

where $\bar{\sigma}_I^2(i)$ and $\tilde{\sigma}_I^2(i)$ are the operator's and inspector's inventory measurement variances, and $\bar{\sigma}_T^2(i)$ and $\tilde{\sigma}_T^2(i)$ are the operator's and inspector's transfer measurement variances, with i = 0,1. The first four terms of this statistic are sensitive to falsification, and the fifth term is sensitive to missing material. This form of the statistic allows the inspector to test for falsification in individual components of the operator's data as well as for missing material.

If the inspector is not interested in testing for falsification in individual components but only in the total falsification, he employs the statistic ISS_1, which is written as

$$ISS_1 = \frac{\max[0,F]F}{2\sigma_F^2} + \frac{\max[0,M_p]M_p}{2\sigma_p^2} \quad , \tag{4}$$

where F represents total falsification, that is,

$$F = (\bar{I}_0 - \tilde{I}_0) - (\bar{I}_1 - \tilde{I}_1) + (\bar{T}_0 - \tilde{T}_0)$$

$$- (\bar{T}_1 - \tilde{T}_1) \quad ,$$

and σ_F^2 is the variance of F, which is given by

$$\sigma_F^2 = \bar{\sigma}_I^2(0) + \tilde{\sigma}_I^2(0) + \bar{\sigma}_I^2(1) + \tilde{\sigma}_I^2(1)$$

$$+ \bar{\sigma}_T^2(0) + \tilde{\sigma}_T^2(0) + \bar{\sigma}_T^2(1) + \tilde{\sigma}_T^2(1) \quad .$$

If the inspector is not concerned with falsification and wishes to be independent of it, he should use ISS_2, which is written as

$$ISS_2 = \frac{\max[0,M_v]M_v}{2\sigma_v^2} \quad , \tag{5}$$

where M_v is the inspector's materials balance, which is

$$M_v = \tilde{I}(0) - \tilde{I}(1) + \tilde{T}(0) - \tilde{T}(1) \quad ,$$

and the variance of M_v is

$$\sigma_v^2 = \tilde{\sigma}_I^2(0) + \tilde{\sigma}_I^2(1) + \tilde{\sigma}_T^2(0) + \tilde{\sigma}_T^2(1) \quad .$$

Although M_p includes operator's and inspector's measurements, M_v includes only those measurements that the inspector knows to be valid.

These statistics can be applied to test the hypothesis

H_0: falsification = 0 and diversion = 0

versus

H_1: falsification > 0 and/or diversion > 0.

The test is implemented by calculating the value of the relevant statistic ISS_0, ISS_1, or ISS_2, and applying the decision rule

accept H_0 if $ISS_i \le \lambda_i$

accept H_1 if $ISS_i > \lambda_i$,

where the decision thresholds λ_i, i = 0,1,2, is chosen to achieve some false-alarm probability. For a false-alarm probability of 0.05, the values of λ_i are 7.48, 4.23, and 1.35, respectively.

These statistics are sensitive to the total amount of material diverted but are independent of the particular diversion pattern through which the material is diverted. The statistics do depend on how the operator falsifies his reported data. For example, we know the operator's optimal strategy is to set the total falsification equal to the amount diverted.[2]

Another approach to testing the operator's reported measurements for falsification and/or diversion is to employ the statistics MUF and D.[4] The statistic MUF is a materials balance based on the operator's reported data,

$$MUF = \bar{I}(0) - \bar{I}(1) + \bar{T}(0) - \bar{T}(1) \quad ,$$

and the statistic D is the inspector's estimate of falsification in the operator's data,

$$D = \frac{N}{n}\{[\bar{I}(0) - \hat{I}(0)] - [\bar{I}(1) - \hat{I}(1)]$$

$$+ [\bar{T}(0) - \hat{T}(0)] - [\bar{T}(1) - \hat{T}(1)]\} \quad ,$$

where N is the total number of the operator's measurements and n is the number sampled by the inspector.

The D statistic can be used to test for falsification. When the operator's data are found to be unfalsified, the operator's materials balance MUF is used to test for missing material. The statistic ISS_1 is better for this purpose because the components sensitive to falsification and diversion are statistically independent, whereas MUF and D are not independent.

In an alternative testing procedure MUF and D could be combined as MUF − D, which is the operator's materials balance corrected by the inspector's estimate of data falsification. This statistic tests only for missing material. In the case of 100% sampling (n = N), MUF − D is identical to ISS_2, the inspector's materials balance

III. USE OF THE ISS

From the above development, the inspector has at least three options for analyzing the aggregation of his own and the operator's reported data: he can use ISS_0, ISS_1, ISS_2, or combinations of them. The choice of the statistic has two implications regarding (1) overall power of diversion detection and (2) concern with falsification.

It is generally true that if the operator falsifies optimally (equates falsification to diversion) ISS_2 will have the highest detection probability of the three statistics for a specified false-alarm probability and level of diversion. On the other hand, ISS_0 and ISS_1 generally have higher detection probabilities than ISS_2 if the operator has not falsified optimally.[2] However, the optimal falsification severely constrains the operator's flexibility in reporting falsified data to minimize the detection probability. ISS_1 represents a compromise between the characteristics of ISS_0 and ISS_2.

Thus, each of the three statistics has certain advantages that can only be evaluated through careful consideration of the tradeoffs between detection probability and limitations on divertor flexibility. In the following sections, we examine the characteristics of the three statistics, beginning with the simplest, ISS_2.

A. No Data Falsification − ISS_2

The statistic given by Eq. (5) reflects an interest only in detecting positive diversion. Under H_0, the hypothesis of zero diversion, it is possible to obtain negative M_v values; consequently any negative values are meaningless under the alternative hypothesis of positive diversion. As seen by Eq. (5) it is our choice to set any negative estimates equal to zero. The probability of a negative value is 0.5 when H_0 is true. Thus, under H_0 the probability that $M_0^2/\sigma_0^2 = 0$ is equal to 0.5. The probability that M_0^2/σ_0^2 is greater than zero is given by a "one-sided" chi-squared distribution (i.e. $M_v > 0$). This makes the search for significant values in typical chi-squared tables for one degree of freedom (df) somewhat different than usual. The one-df case could have been handled easier by using the normal distribution; however, for purposes of future simplification in higher dimensions (ISS_0 and ISS_1), we find it convenient to use the chi-squared distribution for one df. We will develop the mechanisms for setting up the hypothesis, critical regions, and power in the one-df case.

Let the expected (mean) value of the materials balance be denoted by μ. The hypotheses are given by

$$H_0: \mu = 0, \text{ and}$$

$$H_a: \mu > 0 \quad .$$

For a one-sided test with a false-alarm probability equal to α, we select the corresponding threshold value from the chi-squared tables at 2α and one df.[5] For $\alpha = 0.05$, our critical region is where the test statistic exceeds 2.706/2. (The divisor is a result of the structure of Eq. 5.) Thus, when $ISS_2 > 1.353$, we accept H_1.

To calculate the power of this test, we must specify the alternative hypothesis, that is,

$$H_0: \mu = 0, \text{ and}$$

$$H_1: \mu = d > 0 \quad .$$

The power curve requires the computation of a noncentral chi-squared distribution obtained from designated values of d, the amount of diversion, in the alternative hypothesis. The term "noncentral" is used for chi-squared distributions where the independent normal random variables have a common variance but do not have zero means; therefore, the detection probability for the related sufficient statistics requires the noncentral chi-squared distribution when computing the alternative hypotheses. The noncentral chi-squared distribution is characterized by two parameters: (1) the noncentrality parameter, $\lambda = \mu^2/\sigma^2$, and (2) the degrees of freedom.

The power of the test, or detection probability, for $\alpha = 0.05$ and $d = 1.0\sigma$ is 0.26. We let d denote a particular value of diversion as a function of σ. The power may be found for $\lambda = d^2/\sigma^2$ from a noncentral chi-squared table.[5]

B. Pooled Data Falsification - ISS_1

The test procedure is analogous to ISS_2 so that negative values of either F or M_p are set equal to zero. When the null hypothesis of zero means is true, it then follows: the simultaneous probability of negative values is 1/4, the probability of exactly one being negative is 0.5 times a chi-squared distribution, and the probability of both being positive is the first-quadrant portion of the bivariate normal distribution. Those results will be stated in mathematical terms later.

i. Critical Regions for ISS_1. The null hypothesis is

$$H_0: \mu_M = 0 \text{ and } \mu_F = 0 \quad ,$$

where μ_M and μ_F are the expected values of M_p and F, respectively. We define

$$u = \frac{F}{\sigma\sqrt{2}} \quad , \quad v = \frac{\sqrt{2}M_p}{\sigma} \quad ,$$

which are both standard normal random variables. Using these definitions, distributions under H_0 are given by the following four regions:

1. $u < 0,\ v < 0 \qquad f(u, v) = \frac{1}{4}, \quad u = v = 0$

2. $u < 0,\ 0 < v \qquad f(u, v) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-v^2/2},\ u = 0,$
   $0 < v < \infty$

3. $0 < u,\ v < 0 \qquad f(u, v) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-u^2/2},\ v = 0,$
   $0 < u < \infty$

4. $0 < u,\ 0 < v \qquad f(u, v) = \frac{1}{2\pi}e^{-(u^2+v^2)/2},$
   $0 < u,\ v < \infty$.

Region 1. There is no interest in this case because H_0 will always be accepted.

Region 2: One Degree of Freedom. Along the v axis the critical value is found by solving for x_1 in the equation

$$\frac{1}{2}\int_{x_1}^{\infty} N(0,1) = c_1\alpha \quad ; \qquad (6)$$

where $N(\mu, \sigma^2)$ denotes a normal density function with mean μ and variance σ^2 α is the desired false-alarm probability, and c_1 is a constant to be determined.

Region 3: One Degree of Freedom. Along the u axis the critical value is found by solving for x_2, given by

$$\frac{1}{2}\int_{x_2}^{\infty} N(0,1) = c_2\alpha \qquad (7)$$

where α and c_2 are defined as before.

Region 4: Two Degrees of Freedom. In the first quadrant, a standard type of transformation[6,7] facilitates the determination of the critical region. By setting $w = u^2 + v^2$ in the appropriate formulas above, we obtain

$$g(w) = \frac{e^{-w/2}}{8} \quad .$$

Note that $g(w)$ is 0.25 times the chi-squared probability density function with two degrees of freedom.

The critical value is found by solving for x_3 in this equation

$$\frac{1}{8}\int_{x_3}^{\infty} e^{-w/2}\, dw = \frac{e^{-x_3/2}}{4} = c_3\alpha \quad . \qquad (8)$$

$$x_3 = -2 \ln (4c_3\alpha) \quad ,$$

where c_3 is a constant to be determined.

The restrictions on the c's are

$$c_1 + c_2 + c_3 = 1 \quad \text{and} \quad 0 < c_i < 1$$

for i = 1, 2, 3.

An example of determining the critical region when $x_1 = x_2 = \sqrt{x_3}$ is obtained by solving

$$\int_{\sqrt{x_3}}^{\infty} N(0,1) + \frac{e^{-x_3/2}}{4} = \alpha \qquad (9)$$

for x_3. Equation (9) is the sum of Eqs. (6), (7), and (8). When $\alpha = 0.05$, x_1 and $x_2 \approx 2.06$, and $x_3 = 4.21$. This scheme yields $c_1 = c_2 = 0.2$ and $c_3 = 0.6$, i.e., three times as much of α is contained in the distribution in the first quadrant than distributions along either axis.

Another example for determining the critical or threshold values occurs when the probabilities of false alarm on each axis and in the first quadrant are equal. We set $c_1 = c_2 = c_3$. Thus,

$$\frac{1}{\sqrt{2\pi}}\int_{x_1}^{\infty} N(0,1) = \frac{2\alpha}{3}$$

$$x_2 = x_1, \quad \text{and} \quad x_3 = -2 \ln \left(\frac{4\alpha}{3}\right) \quad .$$

For $\alpha = 0.05$, the boundary of the critical region is

$x_1 \approx x_2 \approx 1.83$ when u or v = 0 and

$x_3 \approx 5.42$ when u and v > 0.

2. Power for IFF$_1$. The alternative hypothesis consists of three parts, namely

$$H_1: \quad \mu_I > 0 \quad , \quad \mu_d = 0 \quad ,$$

$$\mu_d > 0 \quad , \quad \mu_I = 0 \quad ,$$

$$\mu_I > 0 \quad , \quad \mu_d > 0 \quad .$$

Now u and v are distributed as independent, normal random variables with means $\mu_I/c\sqrt{2}$ and $\mu_d\sqrt{2}/c$, and unit variances. The following regions correspond to those given in the description of critical regions.

Region 1. Region 1 does not contribute to the power of the test because both u and v are negative.

Region 2. u < 0, v > 0. Here u is set equal to 0, which corresponds to $\mu_I = 0$, $\mu_d > 0$. Let $h_1(1 - \beta)$ represent the power contributed by the v axis:

$$h_1(1 - \beta) = \int_{-\infty}^{0} N(\mu_u,1) \int_{x_1}^{\infty} N(\mu_v,1)$$

$$= \int_{-\infty}^{-\mu_u} N(0,1) \int_{x_1-\mu_v}^{\infty} N(0,1) \quad .$$

Region 3. u > 0, v < 0. We set v equal to 0, which corresponds to $\mu_d > 0$, $\mu_I = 0$. The power contributed by the u axis is

$$h_2(1 - \beta) = \int_{-\infty}^{-\mu_v} N(0,1) \int_{x_2-\mu_u}^{\infty} N(0,1) \quad .$$

Region 4. u > 0, v > 0. This case corresponds to $\mu_I > 0$, $\mu_d > 0$. The power contributed by this region is

$$h_3(1 - \beta) = \iint_{u^2+v^2 > x_3} N(\mu_u,1)\, N(\mu_v,1)$$

Region 4 prohibits precise computation of the power. The methods of Gaussian quadrature and Legendre Polynomials were used to compute this integration as well as subsequent multiple integration problems.

C. Falsification of Data by Components - I88$_0$.

The case of eight measured quantities, four by the operator and four by the inspector, will now be considered. A more rigorous and detailed treatment of this problem is found in Ref. 2. The following account uses a different approach to arrive at results presented in Ref. 2. Notation is changed to help make this special setting of the model more readily understood for the uninitiated reader. In addition, the likelihood ratio technique of Ref. 2 is replaced by formulating the model, solving for the estimates, and using equations from Ref. 2 in the development to obtain critical regions and power curves.

1. Notation and Model. The notation for this discussion is

- o - operator,
- i - inspector,
- b - beginning of a materials balance period or input to a materials balance area,
- e - ending of a materials balance period or output from a materials balance area,
- I - measured inventory, for example, I(o,b) represents the operator's measured value of initial inventory,
- T - measured transfer, for example, T(i,e) represents the inspector's measurement of material transferred out of the process,
- ξ - the true value of material inventory, for examples $\xi(e)$ represents the unknown precise value of ending inventory,
- τ - the true value of material transfer,
- γ - the true value of falsification, for example $\gamma_I(b)$ represents the unknown value (to the inspector) by which the operator has falsified his report on the beginning inventory,
- ε - a random error, normal, iid (identically, independently distributed) with mean 0 and variance σ^2_ε,
- δ - the unknown value of the amount of diversion,
- d - an estimator of δ,
- f - an estimator of γ, for example $f_T(e)$ is the amount of falsification estimated for T(o,e) and estimates the true value $\gamma_T(e)$, and
- F - total falsification.

The general linear model may be formulated as follows:

$$I(o,b) = \xi(b) + \gamma_I(b) + \varepsilon_I(o,b) \quad ,$$

$$I(o,e) = \xi(e) + \gamma_I(e) + \varepsilon_I(o,e) \quad ,$$

$$I(i,b) = \xi(b) + \varepsilon_I(i,b) \quad ,$$

$$I(i,e) = \xi(e) + \varepsilon_I(i,e) \quad ,$$

$$T(o,b) = \tau(b) + \gamma_T(b) + \varepsilon_T(o,b) \quad ,$$

$$T(o,e) = \tau(e) + \gamma_T(e) + \varepsilon_T(o,e) \quad ,$$

$$T(i,b) = \tau(b) + \varepsilon_T(i,b) \quad , \text{and}$$

$$T(i,e) = \tau(e) + \varepsilon_T(i,e) \quad .$$

2. Finding Estimates of Falsification and Diversion. The problem is to find estimates of ξ, τ, γ, and $\delta = \xi(b)\ \xi(e) + \tau(b) - \tau(e)$. Note that each of the I and T variables are iid and normal and that variances can be readily identified using the above notation (for example, $\sigma_I^2(o,b)$ is the variance of $I(o,b)$, etc.). The solutions for falsification, f (estimates of γ) are readily obtained by letting 0 be the estimate for each ε. Then,

$$f_I(b) = I(o,b) - I(i,b) \quad ,$$

$$f_I(e) = I(o,e) - I(i,e) \quad ,$$

$$f_T(b) = T(o,b) - T(i,b) \quad , \text{ and}$$

$$f_T(e) = T(o,e) - T(i,e).$$

Assuming that the inspector and operator make corresponding measurements equally well, the estimates for the inventories and transfers, if falsification may have occurred, are

$$\hat{\xi}(b) = \frac{1}{2}[I(o,b) - f_I(b) + I(i,b)] = I(i,b) \quad ,$$

$$\hat{\xi}(e) = I(i,e) \quad ,$$

$$\hat{\tau}(b) = T(i,b) \quad , \text{ and}$$

$$\hat{\tau}(e) = T(i,e).$$

The estimator of diversion is given by

$$d = I(i,b) - I(i,e) + T(i,b) - T(i,e) \quad ,$$

and, as expected, the estimate of diversion when falsification takes place is a function of the inspector's measurements alone. If the inspector assumes that no falsification has taken place but wishes to estimate diversion, then δ may be estimated by using

$$d = \frac{1}{2}[I(o,b) - I(o,e) + T(o,b) - T(o,e)$$

$$+ I(i,b) - I(i,e) + T(i,b) - T(i,e)] \quad .$$

We have found a set of normally distributed variables with known variances. The quantity designated by $18H_0$ in Eq. (1) can be expressed as a sum of independent central chi-squared variables under the null hypothesis of zero means:

$$\chi^2 = \frac{f_I(b)^2}{2\sigma_{I0}^2} + \frac{f_I(e)^2}{2\sigma_{I1}^2} + \frac{f_T(b)^2}{2\sigma_{T0}^2} + \frac{f_T(e)^2}{2\sigma_{T1}^2}$$

$$+ \frac{(2d - F)^2}{2(\sigma_{I0}^2 + \sigma_{I1}^2 + \sigma_{T0}^2 + \sigma_{T1}^2)}$$

$$= u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 \quad ,$$

which has five degrees of freedom, and $F = f_I(b) - f_I(e) + f_T(b) - f_T(e)$. We have assumed that the inspector and the operator have equal variances in their measurements, or that

$$\sigma_{I0}^2 = \sigma_I^2(o,b) = \sigma_I^2(i,b) \quad ,$$

$$\sigma_{I1}^2 = \sigma_I^2(o,e) = \sigma_I^2(i,e) \quad ,$$

$$\sigma_{T0}^2 = \sigma_T^2(o,b) = \sigma_T^2(i,b) \quad , \text{ and}$$

$$\sigma_{T1}^2 = \sigma_T^2(o,e) = \sigma_T^2(i,e) \quad .$$

Note that each term can be used as a separate test, and if $F = d$, then each term would test falsification and diversion in respective order.

3. Finding Critical Regions. There are 31 critical regions, which can be classed in five groups having different characteristics, that will be found under

$$H_0: \quad \gamma_I(b) = \gamma_I(e) = \gamma_T(b) = \gamma_T(e) = \delta = 0$$

versus

H_1: at least one parameter in H_0 not 0.

Note that u_i, $i = 1, 2, 3, 4, 5$ as defined in the chi-squared breakdown is a standardized, normal variable, and $-\infty < u_i < \infty$. If all realized $u_i \le 0$, then the hypothesis is automatically accepted, and values of γ and δ may be considered equal to 0. For H_0 true, the probability that any one $u_i < 0$ is equal to 1/2; therefore the chance of all five estimates being negative is 1/32. The other five regions where the hypothesis can be rejected are examined below.

Region 1: One Degree of Freedom. There are four variables whose realizations are 0 and one whose realization is a positive quantity. We wish to find a value of x_1 that satisfies

$$\left(\frac{1}{2}\right)^4 \frac{1}{\sqrt{2\pi}} \int_{\sqrt{x_1}}^{\infty} N(0,1) = c_1\alpha \quad ,$$

where c_1 is a positive constant similar to the c's defined above. There are five possibilities of this event occurring.

Region 2: Two Degrees of Freedom. There are three variables whose realizations are 0 and two whose realizations are positive quantities. We want a value of x_2 such that

$$\left(\frac{1}{2}\right)^3 \frac{1}{4} \int_{x_2}^{\infty} f(\chi^2; 2) = c_2\alpha \quad ,$$

where the integral is the chi-squared distribution with two degrees of freedom. There are ten possibilities where any two of the five variables are positive.

Region 3: Three Degrees of Freedom. Two variables whose realizations are 0 and three whose realizations are positive values give rise to an x_3 such that

$$\left(\frac{1}{2}\right)^2 \frac{1}{8} \int_{x_3}^{\infty} f(\chi^2;3) = c_3\alpha \quad .$$

The integral is the chi-squared distribution with three degrees of freedom. There are ten such regions.

Region 4: Four Degrees of Freedom. One variable whose realization is 0 and four whose realizations are positive values require an x_4 such that

$$\left(\frac{1}{2}\right)\left(\frac{1}{16}\right) \int_{x_4}^{\infty} f(\chi^2;4) = c_4\alpha \quad .$$

The integral is the chi-squared distribution with four degrees of freedom. There are five such regions.

Region 5: Five Degrees of Freedom. All variables take on positive values. Find x_5 such that

$$\frac{1}{32} \int_{x_5}^{\infty} f(\chi^2;5) = c_5\alpha \quad .$$

The integral is the chi-squared distribution with five degrees of freedom.

In all regions, α specifies the size of the rejection area, and $5c_1 + 10c_2 + 10c_3 + 5c_4 + c_5 = 1$. If $\alpha = 0.05$, then $x_1 \sim 1.95$, $x_2 \sim 5.93$, $x_3 \sim 7.74$, $x_4 \sim 9.41$, and $x_5 \sim 10.99$ give rise to equal areas in the 31 different critical regions for H_0 true. On the other hand, equal x_i values (~ 7.48) cause the critical region to be a smooth surface without steps.

4. Power of the Test. The power function is obtained by examining the probability of rejecting the alternative hypothesis. Terms for each integral under the null hypothesis are now replaced in the following manner (μ_{u_j} denotes the value given to the mean of u_j):

Under H_0	Under H_1
$N(0,1)$	$N(\mu_{u_j},1)$
$f(\chi^2;df)$	$f(\chi'^2;df,\lambda)$

where $f(\chi^2;df)$ is a chi-squared distribution, and $f(\chi'^2;df,\lambda)$ is a noncentral chi-squared distribution with $\lambda = \Sigma[\mu^2_{u_j}/\sigma^2_{u_j}]$.

The total power is equal to the sum of the individual contributions and is similar to Eq. (9), but for ISS_0 there are 31 individual critical regions. If we assume that all similar cases (regions) will have identical critical values, the problem simplifies to five different regions.

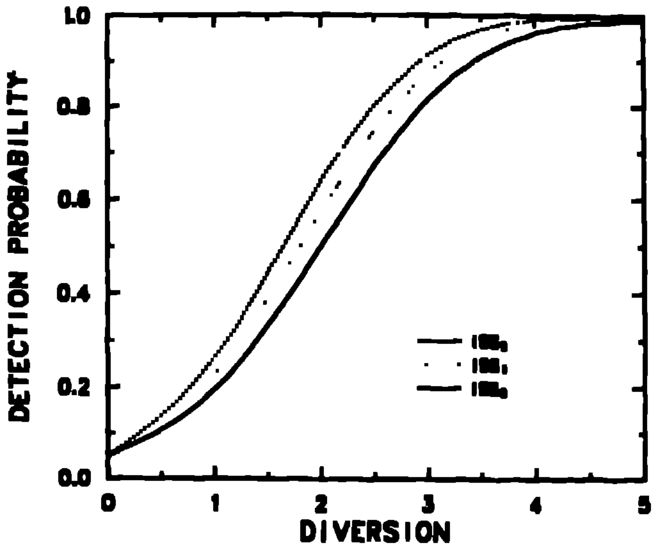

Fig. 1. Power curves for ISS_0, ISS_1, and ISS_2: optimal falsification and 0.05 false-alarm probability.

IV. COMPARING THE THREE STATISTICS

Figure 1 shows power curves for ISS_0, ISS_1, and ISS_2 as functions of the diversion d for optimal falsification in each case. The diversion d is taken as a multiple of either the operator's or inspector's materials balance standard deviation, which are assumed to be equal. The curves were obtained by numerical integration with appropriate, equal critical values for each statistic, and a false-alarm probability of 0.05. The numerical solutions were verified by simulations. Randomly generated samples from a normal distribution were used to compute the power for various diversion and falsification amounts. Typically 20 000 to 50 000 random samples were used to obtain sufficient statistics. The results of the numerical integrations and the simulations were in good agreement.

The results show that ISS_2, the statistic that is independent of operator's measurement falsification, has the highest detection probability, whereas ISS_0, which is sensitive to component falsification, has the lowest for optimal falsification; ISS_1 has intermediate detection probability. However, the differences are not great; at a diversion d of twice the inspector's materials balance standard deviation, the detection probabilities are approximately 0.5, 0.57, and 0.64 for ISS_0, ISS_1, ISS_2, respectively. At d equals three times the inspector's materials balance standard deviation, the respective detection probabilities are about 0.81, 0.87, and 0.91.

In addition, the inspector's use of ISS_0 severely limits the operator's flexibility in falsifying data to hide diversion. If the operator does not falsify properly, then the detection probabilities for ISS_0 and ISS_1 increase, whereas that for ISS_2 remains the same. In particular, Fig. 2 shows power curves for

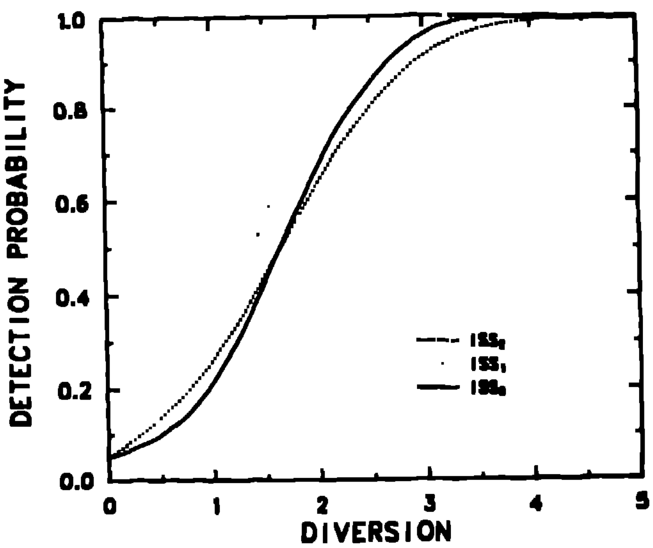

Fig. 2. Power curves for ISS_0, ISS_1, and ISS_2; no falsification and 0.05 false-alarm probability.

ISS_0, ISS_1, and ISS_2 for the case of no falsification at all. Now, ISS_1 is uniformly better than ISS_2, whereas ISS_0 is better than ISS_2 for diversion larger than ∼1.6. These results show that the three statistics are similar in performance, but it appears that ISS_1, which tests for diversion and/or total materials balance falsification, is an effective compromise.

Remark

One important problem is that of finding the critical values that maximizes the detection probability. This study is nearly complete and will be submitted to the literature in the near future.

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