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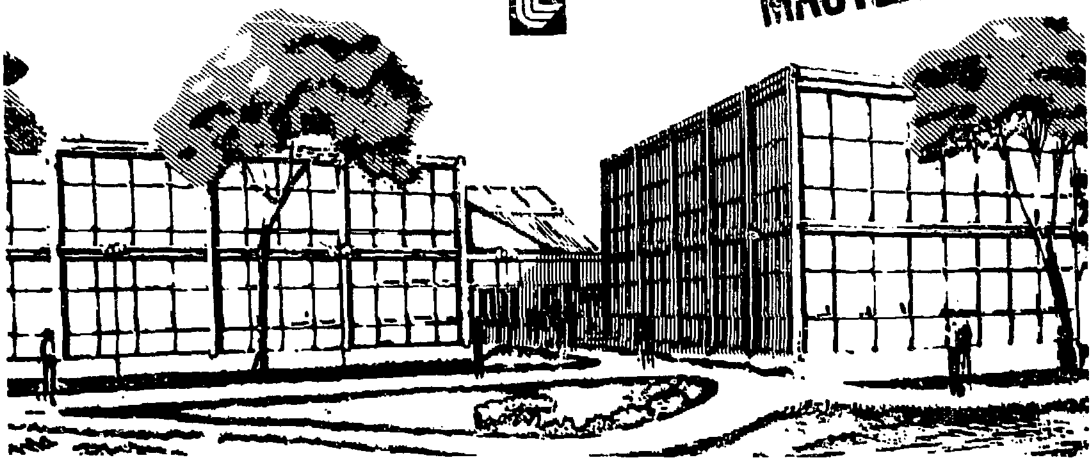

Wave Propagation in Viscoelastic Media

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SUMMARY

The mathematical formulations of the wave propagation problem in a linear viscoelastic solid are reviewed from the point of view of constitutive equations and the theory of linear physical systems. Various general results from the theory of propagating singular surfaces and from the mathematical theory of hyperbolic equations are applied to the analysis of the wave propagation process. The impulse responses of three viscoelastic media are analyzed using asymptotic methods. The three material models are the "standard" linear solid, the standard linear solid with a continuous spectrum of relaxation times and the power law solid. The standard linear solid with a continuous spectrum of relaxation times and the power law solid have a nearly constant quality factor Q over the seismic frequency band. The impulse responses of these two viscoelastic solids are compared. The results show significant and discernible features in the wave profile. This leads to the conclusion that differentiation of the models can be made by comparing wave shapes and that a complete knowledge of Q over the entire frequency range is required to determine the wave propagation problem when initiated by an impulsive process.

INTRODUCTION

With the current revival of interest in the effects of anelastic attenuation on seismic wave propagation, a number of linear frequency dependent attenuating or viscoelastic media have been proposed to account for the observed dispersive-attenuation phenomena. All of these models have the property of nearly constant seismic quality factor (specific dissipative function) Q over the seismic frequency range. Outside the seismic frequency band, the models differ. For example, Liu, Anderson, and Kanamori (1976) show that a standard linear solid with a continuous spectrum of relaxation times has a nearly constant Q in the seismic frequency band. Strick (1970) uses a power law solid with a slowly varying Q over most of the frequency range.

Associated with attenuation, there is physical dispersion, namely, the speed of propagation of waves depends on frequency. In this light, Kanamori and Anderson (1977) have shown that the values of Q outside the seismic frequency band affect mainly the magnitude of the phase velocity but do not affect significantly the relative dispersion within the seismic frequency band. Therefore, they conclude that lack of knowledge about Q outside the seismic band does not alter the dispersion relation used for surface wave and free oscillation problems. These are low frequency phenomena, however. It is natural to ask whether the results of Kanamori and Anderson are applicable to wave propagation problems having a large high frequency content.

In the case of the impulse response, most of the linear, frequency dependent material models in seismology yield wave forms with grossly similar structure (e.g., Futterman, 1962; Lamb, 1962; Savage, 1965;

Azimi et al., 1968; Strick, 1970). Thus, we want to know whether there are distinguishing features in the wave profiles to differentiate the various frequency dependent models.

In the seismology literature, there are two equivalent mathematical formulations of the wave propagation problem in a dispersive-attenuating medium. In the usual formulation, the constitutive equation is prescribed in the form of a differential operator law or of an integral law. The constitutive equation and the field equations form a complete system, and together with initial and boundary conditions, they yield a well-posed mathematical problem. We may alternately, use the theory of linear physical systems to formulate a wave propagation problem. In this approach, it is assumed that the propagating wave packet satisfies causality and, therefore, the material functions are related by the Kramers-Kronig integral relations. Integrating these two points of view, we gain additional insight about the wave propagation process.

It is the intent of this paper to review in some detail the two formulations of the wave propagation problem in a linear, frequency dependent dispersive-attenuating medium, and to examine the influences of the material response outside of the nearly constant Q region on the evolution of an impulse.

The models studied are that of the "standard" linear solid, the standard linear solid with a continuous spectrum of relaxation times, and the power law solid. The "standard" linear solid is analyzed because it has a differential operator law essential to the use of perturbation methods in extracting pertinent results on the wave propagation process.

Organization of this paper is as follows: In Part I, we discuss the linear theory of viscoelasticity, in particular, the characterization of a viscoelastic medium and present some models of current geophysical interest.

In Part II, the wave propagation problem is discussed. We begin with the usual formulation and collect relevent results from the theory of propagating singular surfaces and the mathematical theory of hyperbolic equations. Next, we discuss the formulation of the wave propagation problem in terms of the theory of linear physical systems and introduce the power law solid.

In Part III, we analyze the impulse response in a "standard" linear solid, a standard linear solid with a continuous spectrum of relaxation times, and a power law solid. Asymptotic methods are applied to the integral representation of the solutions. The asymptotic solutions of the standard linear solid with a spectrum of relaxation times and of the power law solid are compared to show the influences of the material response outside the nearly constant Q region.

I. Linear Viscoelasticity

The constitutive equation for a linear viscoelastic medium can be described either by linear hereditary (integral) laws or by differential operator laws.

Leitman and Fisher (1973) give a complete and detailed review on the theory of linear viscoelasticity. Hence, only the scalar case will be discussed. In the following discussions, the notations follow Leitman and Fisher.

I.1 Integral Laws

We begin with the hereditary law, in particular, the Boltzmann law. Let σ and ε be the stress and strain fields associated with some deformation process. Then the Boltzmann law is given by

$$\sigma(t) = L_G[\varepsilon(t)] = G(t)*\varepsilon(t) \quad \text{for} \quad t \geq 0$$

$$= G(o)\varepsilon(t) + \int_0^t \dot{G}(t-\tau)\varepsilon(\tau)d\tau \quad , \tag{1}$$

where G(t) is the relaxation function and $*$ denotes convolution. Moreover, the relaxation function G(t) is represented by

$$G(t) = G(o) + \int_0^t \dot{G}(\tau)d\tau \quad , \quad t \geq 0 \tag{2}$$

where G(o) is called the instantaneous (initial) elasticity. It is related to the elastic response of the medium. If G(o) = 0, then the medium has no elastic response.

If

$$G(\infty) = \lim_{t \to \infty} G(t)$$

exists, $G(\infty)$ is called the equilibrium elasticity or the equilibrium modulus.

Alternatively, there is a linear hereditary law such that the strain process $\varepsilon(t)$ is determined by the stress process through

$$\varepsilon(t) = J(t)*\sigma(t) \quad . \tag{3}$$

Here, J(t) is called the creep compliance. J(t) has a characterization of:

$$J(t) = J(o) + \int_0^t \dot{J}(s)ds \quad . \tag{4}$$

where J(o) is the initial elastic compliance. If J(t) is the creep compliance corresponding to the relaxation function G(t), then,

$$G(t)*[J(t)*\sigma(t)] = \sigma(t)$$

and

$$J(t)*[G(t)*\varepsilon(t)] = \varepsilon(t) \quad .$$

This gives immediately that

$$G(o)J(o) = 1$$

and

$$G(o)\dot{J}(s) + \dot{G}(s)J(o) + (\dot{G}*\dot{J})(s) = 0 \text{ for } o < s < \infty.$$

Clearly, G(o) and J(o) are required to be nonvanishing.

For most purposes, the constitutive equations defining the visco-elastic properties are cumbersome mathematical expressions involving convolutions. By introducing integral transforms, the equations become algebraic in their respective transformed values of stress and strain.

Applying Laplace transform to (1) and (3), we obtain immediately that

$$[G(o) + \tilde{G}][J(o) + \tilde{J}] = 1 \tag{5}$$

where $\tilde{\phi}$ is the Laplace transform of ϕ. This relation is of some importance in correlating the creep and relaxation behavior in a viscoelastic solid.

The use of Fourier transform leads directly to the "complex modulus" description of the mechanical properties of a viscoelastic solid. Taking the Fourier transform of (1), we obtain

$$\hat{\sigma} = G(o)\hat{\varepsilon} + \hat{\dot{G}}\hat{\varepsilon}$$

where $\hat{\phi}$ is the Fourier transform of ϕ. Denoting the "complex modulus" by $M(\omega)$, we have

$$M(\omega) = G(o) + \hat{\dot{G}}$$

where $$\hat{\dot{G}} = \int_{-\infty}^{\infty} \dot{G} \exp(i\omega t)\, dt \quad .$$

Since $\dot{G}(t) = 0$ for $t < 0$, then

$$\hat{\dot{G}} = \int_0^{\infty} \dot{G} \cos \omega t\, dt + i \int_0^{\infty} \dot{G} \sin \omega t\, dt$$

and

$$\widetilde{M}_r = G(o) + \int_0^{\infty} \dot{G} \cos \omega t\, dt \tag{6a}$$

$$\widetilde{M}_i = \int_0^{\infty} \dot{G} \sin \omega t\, dt \quad . \tag{6b}$$

$\widetilde{M}_r$ and $\widetilde{M}_i$ are respectively the real and imaginary parts of the "complex modulus" M.

If we assume that $\dot{G}(t)$ is an integrable function, it follows from Riemann-Lebesque lemma that

$$\lim_{|\omega| \to \infty} [G(o) + \hat{\dot{G}}_c(\omega)] = G(o)$$

and

$$\lim_{|\omega| \to \infty} \hat{\dot{G}}_s(\omega) = 0$$

where the half range Fourier cosine and sine transform of $\dot{G}$ defined for real ω are

$$\hat{G}_c(\omega) = \int_0^\infty \dot{G}(t) \cos \omega t \, dt$$

and

$$\hat{G}_s(\omega) = \int_0^\infty \dot{G}(t) \sin \omega t \, dt \quad .$$

Hence, in the high frequency limit, the stress response is mainly elastic. Moreover, the phase lag or the "loss angle" $\phi(\omega)$ for each frequency ω:

$$\tan \phi(\omega) = \frac{\tilde{M}_i}{\tilde{M}_r}$$

$$= \frac{\hat{G}_s}{G(o) + \hat{G}_c}$$

vanishes as $\omega \to \infty$ provided there is an elastic response, i.e., $G(o) \neq 0$. For a viscoelastic material with $G(o) = 0$, then

$$\lim_{|\omega| \to \infty} \tan \phi(\omega) = \lim_{|\omega| \to \infty} \left(\hat{G}_s / \hat{G}_c\right) \quad .$$

It is clear from equations (6a,b) that $\tilde{M}_r$ and $\tilde{M}_i$ are not independent quantities as both are generated from $\dot{G}(t)$. In fact, they are conjugate integrals, i.e.,

$$\tilde{M}_r(\omega) - G(o) = \frac{2}{\pi} \int_0^\infty \frac{\tilde{M}_i(\omega')\omega^2}{\omega'(\omega^2 - \omega'^2)} \, d\omega' \qquad (7a)$$

$$\tilde{M}_i(\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{\tilde{M}_r(\omega')}{\omega'^2 - \omega^2} \, d' \qquad (7b)$$

or Hilbert transform pairs (Titchmarsh (1937)). These relations are also called the Kramers-Kroning integral relations.

Similar results are obtained when the creep law Eq. (3) is Fourier analyzed. The real and imaginary parts of the "complex creep compliance" are also Hilbert transforms of each other. The precise form of these relations may be found in Gross (1968).

Although the linear viscoelastic theory is purely mechanical as developed thus far, thermodynamics must restrict the relaxation function. Such restrictions have been studied by Gurtin and Herrera (1965). They postulate that to deform a viscoelastic solid from its virgin state, work must be done; i.e.,

$$\int_0^t \sigma\dot{\varepsilon}d\tau \geq 0 \qquad\qquad 0 \leq t < \infty$$

for all smooth strain histories satisfying $\varepsilon(o) = 0$. Material laws having this property are called dissipative. Furthermore, the constitutive relation is strongly dissipative of and only of its dissipative and the only strain history that yields zero work is the zero-strain history.

Gurtin and Herrera (1965) prove that for a dissipative scalar integral law of linear viscoelasticity:

- $G(t)$ is continuous for $0 \leq t < \infty$
- $|G(t)| \leq G(o)$ for $0 < t < \infty$
- G is of positive type
- G is a characteristic function
- if $G(\infty)$ exists, then $0 \leq G(\infty) \leq G(o)$.

The statement of primary interest is that a dissipative integral law has its instantaneous elasticity $G(o)$ larger than its equilibrium modulus $G(\infty)$.

This concludes our general description of integral constitutive laws. We turn now to differential operator laws.

I.2 Differential Operator Laws

A differential operator law constitutive relation is specified by

$$\sum_{k=0}^{N} p_k D^k \sigma = \sum_{k=0}^{N} q_k D^k \varepsilon$$

where p_k and q_k $k = 0, 1, 2,..N$ are scalar constants and $D^k\phi$ denotes kth derivative of ϕ. The differential operator law has an obvious mechanical interpretation in terms of springs and dashpots. It can be shown that every differential operator law is a Boltzmann law but not conversely. We illustrate the results discussed in this section with examples of current geophysical interest.

I.3 "Standard" Linear Solid (SLS)

Liu, Anderson, and Kanamori (1976) have considered the use of a "standard" linear viscoelastic solid as a model for a medium with a single relaxation time. This solid is described by a differential operator law:

$$\dot{\sigma} + \frac{1}{\tau_\sigma}\sigma = M_I\,[\dot{\varepsilon} + \frac{1}{\tau_\varepsilon}\varepsilon] \qquad (8)$$

where M_I is called the instantaneous modulus and τ_σ and τ_σ are respectively, stress and strain relaxation times. Two mechanical analogs of the "standard" linear solid differential operator law are possible, see Figs. (1a,b).

$C_1 > 0$ and $C_2 > 0$ are spring constants and $\nu > 0$ is the viscosity associated with the dashpot.

Associated with Fig. (1a), the following differential operator law is obtained

$$\dot{\sigma} + \frac{C_1 + C_2}{\nu}\sigma = C_1\left[\dot{\varepsilon} + \frac{C_2}{\nu}\varepsilon\right] \quad .$$

For this mechanical hookup, we have

$$\tau_\sigma = \frac{\nu}{C_1 + C_2} \quad , \quad \tau_\varepsilon = \frac{\nu}{C_2} \quad \text{and} \quad M_I = C_1 \quad .$$

Associated with Fig. (1b), we have

$$\dot{\sigma} + \frac{C_2}{\nu}\sigma = (C_1 + C_2)\left[\dot{\varepsilon} + \frac{C_1 C_2}{\nu(C_1 + C_2)}\varepsilon\right] \quad .$$

The corresponding definitions of τ_σ, τ_ε and M_I are

$$\tau_\sigma = \frac{\nu}{C_2} \quad , \quad \tau_\varepsilon = \frac{\nu(C_1 + C_2)}{C_1 C_2} \quad \text{and} \quad M_I = C_1 + C_2.$$

The integral law associated with Eq. (8) is

$$\sigma(t = M_I\varepsilon - \frac{M_I}{\tau_\sigma}\left(1 - \frac{\tau_\sigma}{\tau_\varepsilon}\right)\int_0^t \varepsilon(\tau)\exp\left[-\frac{t-\tau}{\tau_\sigma}\right] d\tau. \qquad (9)$$

Thus,

$$\dot{G}(t) = M_I - \frac{M_I}{\tau_\sigma}\left(1 - \frac{\tau_\sigma}{\tau_\varepsilon}\right)\int_0^t \exp(-\tau/\tau_\sigma)\ d\tau$$

and

$$\dot{G}(t) = -\frac{M_I}{\tau_\sigma}\left(1 - \frac{\tau_\sigma}{\tau_\varepsilon}\right)\exp[-\tau/\tau_\sigma] \quad .$$

Clearly, $G(o) = M_I$ and $G(\infty) = M_I\ \tau_\sigma/\tau_\varepsilon$. For Eq. (9) to be a dissipative integral law, it is necessary that $G(\infty) < G(o)$, implying that $\tau_\sigma < \tau_\varepsilon$. In view of the mechanical analog, we see that for Fig. (1a)

$$\frac{\tau_\sigma}{\tau_\varepsilon} = \frac{C_2}{C_1 + C_2} < 1$$

and for Fig. (1b)

$$\frac{\tau_\sigma}{\tau_\varepsilon} = \frac{C_1}{C_1 + C_2} < 1.$$

The "complex modulus" is simply given by

$$M(\omega) = \widetilde{M}_r + i\widetilde{M}_i = \frac{i\omega + 1/\tau_\sigma}{i\omega + 1/\tau_\sigma} M_I \quad . \tag{10}$$

Equation (10) may be viewed as a bilinear mapping from ω plane to the M plane. From Kober (1957) it is known that the real ω axis is mapped onto a circle centered at $\widetilde{M}_r = M_I/2\ (1 + \tau_\sigma/\tau_\varepsilon)$ with radius $M_I/2\ (1 - \tau_\sigma/\tau_\varepsilon)$. The positive real axis corresponds to the semi circle lying in the upper half plane (Fig. 2). $\widetilde{M}_r$ and $\widetilde{M}_i$ have the following explicit representation:

$$\widetilde{M}_r = \frac{\omega^2 + \frac{1}{\tau_\sigma \tau_\varepsilon}}{\omega^2 + \frac{1}{\tau_\sigma^2}} M_I$$

and

$$\widetilde{M}_i = \frac{\omega\left[\frac{1}{\tau_\sigma} + \frac{1}{\tau_\varepsilon}\right]}{\omega^2 + \left(\frac{1}{\tau_\sigma}\right)^2} M_I$$

and, therefore, the phase lag or the "loss-angle" is

$$\tan \phi = \frac{\widetilde{M}_i}{\widetilde{M}_r} = \frac{\omega\left[\frac{1}{\tau_\sigma} - \frac{1}{\tau_\varepsilon}\right]}{\omega^2 + 1/\tau_\sigma\tau_\varepsilon}$$

The magnitude of the "complex modulus" is then

$$|M| = M_I\left[\frac{\omega^2 + 1/\tau_\varepsilon{}^2}{\omega^2 + 1/\tau_\sigma{}^2}\right]^{\frac{1}{2}} \quad .$$

O'Connell and Budiansky (1978) have proposed a standard definition of the intrinsic quality factor Q as the ratio of the real and imaginary parts of the "complex modulus". This definition is a property of the material only and is independent of any deformation process. It is adopted for use in later discussions. From the definition of Q, it is seen that

$$Q^{-1} = \widetilde{M}_i/\widetilde{M}_r = \tan \phi \quad .$$

I.4 Standard Linear Solid with a Continuous Spectrum of Relaxation Times (SLSCS)

For the case of a linear viscoelastic solid with a continuous spectrum of relaxation times, Eq. (9) can be generalized to yield

$$\sigma(t) = M_I\varepsilon(t) - M_I \int_0^t \varepsilon(\tau)d\tau \int_0^\infty \frac{1}{\tau_\sigma}(1 - \tau_\sigma/\tau_\varepsilon) \exp[-(t-\tau)/\tau_\sigma] \times D(\tau_\sigma)\, d\tau_\sigma \qquad (11)$$

where $D(\tau_\sigma)$ is the distribution function. Following Liu, Anderson, and Kanamori (1976), it is assumed that

$$D(\tau_\sigma) = \begin{cases} D/\tau_\sigma & \tau_{min} < \tau_\sigma < \tau_{max} \\ 0 \quad , & \text{otherwise} \end{cases}$$

and

$$1 - \tau_\sigma/\tau_\varepsilon = \kappa \quad , \quad 0 < \kappa < 1 \quad .$$

Substituting these two expressions into Eq. (11) and performing the necessary integrations, we obtain

$$\sigma(t) = M_I \varepsilon(t) + M_I \mathscr{C} \int_0^t \frac{\exp[-(t-\tau)/\tau_{min}] - \exp[-(t-\tau)/\tau_{max}]}{t-\tau} \varepsilon(t) d\tau \ . \tag{12}$$

By definition,

$$G(t) = M_I + M_I \mathscr{C} \int_0^t \frac{\exp(-\tau/\tau_{min}) - \exp(-\tau/\tau_{max})}{\tau} d\tau$$

where $\mathscr{C} = \kappa D$. Therefore,

$$G(o) = M_I$$

and

$$\dot{G}(t) = M_I \mathscr{C} \frac{\exp(-t/\tau_{min}) - \exp(-t/\tau_{max})}{t} \ .$$

Performing the integration and using the definition of the exponential integral $E_1(z) = \int_z^\infty t^{-1} \exp(-t) dt$ (Abramowitz and Stegun, 1964), we have

$$G(t) = M_I + M_I \mathscr{C}[\ln(\tau_{min}/\tau_{max}) - E_1(t/\tau_{min}) + E_1(t/\tau_{max})] .$$

It is clearly seen that

$$G(\infty) = M_I[1 + \mathscr{C} \ln(\tau_{min}/\tau_{max})] \quad .$$

For (12) to be a dissipative constitutive law, it is necessary that

$$0 < \mathscr{C} < 1/\ln(\tau_{max}/\tau_{min}) \quad .$$

The Fourier transform of $\dot{G}(t)$ is given by

$$\hat{G}(\omega) = M_I \mathscr{C} \ln\left(\frac{i\omega + 1/\tau_{max}}{i\omega + 1/\tau_{min}}\right)$$

and, therefore,

$$\hat{\dot{G}}_c = \frac{M_I \mathscr{C}}{2} \ln\left[\frac{(\omega\tau_{min})^2 + (\tau_{min}/\tau_{max})^2}{(\omega\tau_{min})^2 + 1}\right]$$

$$\hat{\dot{G}}_c = M_I \mathscr{C} \tan^{-1}\left[\frac{\omega(\tau_{max} - \tau_{min})}{\omega^2 \tau_{min}\tau_{max} + 1}\right] \quad .$$

The phase lag ϕ is

$$\tan \phi = \mathscr{C} \tan^{-1}\left[\frac{\omega(\tau_{max} - \tau_{min})}{\omega^2 \tau_{min}\tau_{max} + 1}\right]$$

$$\div \left\{1 + \frac{\mathscr{C}}{2} \ln\left[\frac{(\omega\tau_{min})^2 + (\tau_{min}/\tau_{max})^2}{(\omega\tau_{min})^2 + 1}\right]\right\}$$

The constitutive equation (12) for SLSCS is a four parameter family integral law: M_I, τ_{min}, τ_{max}, and $\mathscr{C}$. For

$$1/\tau_{max} << \omega << 1/\tau_{min} \quad \text{and } \mathscr{C}\,\ell n(\tau_{min}/\tau_{max}) << 1$$

then

$$Q^{-1} = \tan\phi \simeq \mathscr{C}\tan^{-1}\frac{\omega(\tau_{max} - \tau_{min})}{1 + \omega^2\tau_{min}\tau_{max}} \simeq \pi\mathscr{C}/2 \quad .$$

Under these circumstances, we may replace $\mathscr{C}$ by Q^{-1} and consider the set

$$M_I,\ \tau_{min},\ \tau_{max},\ \text{and } Q_m^{-1} = \pi\mathscr{C}/2 \quad .$$

A typical plot of Q^{-1} and the magnitude of the complex modulus is given in Fig. 3.

This concludes the discussion on the characterization of a linear viscoelastic solid. The mathematical formulation of a wave propagation problem is examined next.

II. The Wave Propagation Problem

In this part, we consider the wave propagation problem. The usual mathematical formulation is discussed first.

II.1 The usual formulation

We consider the impact or signaling problem in a linear viscoelastic medium. We seek a mathematical solution for the velocity distribution $u(x,t)$, the stress profile $\sigma(x,t)$, and the strain field $\varepsilon(x,t)$ in the region $x > o$ and $t > o$ satisfying the usual field equations:

equation of motion,

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial x} \qquad , \tag{13}$$

the kinematic equation,

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial u}{\partial x} \qquad , \tag{14}$$

and a linear constitutive relation

$$\sigma(x,t) = F[\varepsilon(x,t)] \qquad . \tag{15}$$

The initial conditions are homogeneous, i.e.,

$$u(x,o) = \sigma(x,o) = \varepsilon(x,o) = 0 \quad \text{for } x > o \qquad .$$

The boundary conditions are

$$u(o,t) = f(t)$$

and

$$\lim_{x \to \infty} u(x,t) = 0 \qquad \text{for } t \geq 0 \qquad .$$

Here f(t) is the signaling data.

The impact problem with the constitutive equation given by (1), namely,

$$\sigma(x,t) = G(o)\varepsilon(x,t) + \int_0^t \dot{G}(x,t-\tau)\varepsilon(x,\tau)d\tau$$

may be solved using integral transforms. Let

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

be the Laplace transform of f(t), then we have the following representation of the solution to the impact problem

$$u(x,t) = \frac{1}{2\pi i} \int_B \bar{f}(s) \exp\left\{st - xs\sqrt{\rho/[G(o) + \bar{\dot{G}}(s)]}\right\} ds$$

where B is the Bromwich path.

We note that Eq. (16) is a representation of the solution if: (1) it satisfies the governing equations and the associated initial and boundary conditions; (2) U(x,t) $\sigma(x,t)$, $\varepsilon(x,t)$, and their partial derivatives with respect to x as well as f(t) are functions of bounded variation and of exponential order, i.e., $U(x,t) = (e^{\alpha t})$ for $\alpha > 0$ when $t > 0$; and (3) the Laplace inversion integral is uniformly convergent for $-\infty < t < \infty$. The last two conditions are associated with the Laplace transform (Widder 1946). We will assume that f(t) satisfies the necessary requirements. However, from modern transform theory (Gel'fand and Shilov 1966), f(t) can be a generalized function. Before proceeding, we collect some useful results from the theory of propagating singular surfaces in a linear viscoelastic solid and from the theory of mathematical theory of hyperbolic equations. These results are essential to understanding the wave propagation problem.

II.2 Results from the Theory of Propagating Singular Surfaces and the Theory of Hyperbolic Equations

An important and relevant result (Herrera and Gurtin, 1965) from the theory of propagating singular surfaces is that the viscoelastic wave-speeds are dicated by the initial response of the material, i.e., the instantaneous modulus, G(o). In the case of a "standard" linear solid or the family of distributed "standard" linear solids, this asserts that discontinuous motion propagates with wave speed corresponding to the instantaneous modulus.

In the mathematical theory of hyperbolic equation with lower order terms (Cole, 1968; Whitham, 1959, 1974), the curves along with distribances or singularities propagate are called characteristics. For the standard linear solid, they are

$$r = t - x/c \quad \text{and} \quad s = t + x/c$$

with $$c = \sqrt{G(o)/\rho} \quad .$$

On the other hand, signals can propagate with speed c_∞ with $c_\infty < c$. The curves

$$\tilde{r} = t - x/c_\infty \quad \text{and} \quad \tilde{s} = t + x/c_\infty$$

with $$c_\infty < c$$

are called subcharacteristics since they lie inside the domain of dependence — a region in x-t plane bounded by the characteristics and the data curve (see Fig. 4).

Since singularities propagate along characteristics, waves propagating along the subcharacteristics directions must be smooth. This phenomenon will be made clear when we discuss the impulse response.

Another pertinent result from the theory of propagating singular surfaces is that the magnitude of the propagating jump discontinuity changes at a rate given by

$$q(t) = \exp\left[\frac{t}{2\rho c^2}\ \dot{G}(o)\right] q(o) \quad . \tag{17}$$

This relation is independent of the order of the propagating wave; the order is defined as that of the lowest order derivative of the particle motion with a finite jump across the propagating wave (Fisher (1965)). Note that the relaxation function $G(t)$ is a monotonically decreasing function and, therefore, $\dot{G}(o) < o$. Thus, the magnitude of the jump discontinuity is exponentially decaying. From (5), an integral equation relating the creep function $J(t)$ to the relaxation function $G(t)$, we have

$$G(o)\dot{J}(o) = -\ \dot{G}(o)J(o) \quad .$$

As $J(o) = 1/G(o)$, it follows that

$$\dot{G}(o) = -\ G^2(o)\dot{J}(o)$$

and Eq. (17) may be written in terms of the initial slope of the creep function:

$$q(t) = \exp\left[-\ \frac{tG(o)}{2}\ \dot{J}(o)\right] q(o) \quad . \tag{18}$$

In summary, for a viscoelastic medium whose creep law or relaxation function has a non-zero initial slope, then, there exists a precursor wave propagating with a speed $\sqrt{G(o)/\rho}$ and decaying exponentially with time. Moreover, if the equilibrium modulus $G(\infty)$ exists then waves propagating with a speed $C_\infty = \sqrt{G(\infty)/\rho}$ must have smooth profiles.

The above general results concerning the wave propagation process will become evident as the impulse response solutions are discussed.

II.3 Theory of Linear Physical Systems and Wave Propagation

Consider a plane wave propagating in an infinite homogeneous dispersive-attenuating medium (Futterman, 1962; Lamb, 1962; Strick, 1970):

$$u_\omega(x,t) = A(\omega)\ \exp\left\{-\alpha(\omega)x + i[\omega t - x\theta(\omega)]\right\}/2\pi \quad ,$$

for $\qquad x > o$

where $\alpha(\omega)$ is the attenuation coefficient and $\theta(\omega)$ is the phase lag function.

Associated with the plane wave, there is a spatial seismic quality factor $Q_x(\omega)$ (Strick, 1970):

$$Q_x^{-1}(\omega) = 2\alpha(\omega)/\theta(\omega) \qquad o < \omega < \infty \tag{19}$$

expressing the decay of the successive wave peaks by a factor of exp $(-\pi/Q_x)$.

For the propagation of a wave-packet, we superpose over all frequencies to obtain

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega)\exp\left\{-\alpha(\omega)x + i[\omega t - x\theta(\omega)]\right\}d\omega \tag{20}$$

Suppose the signaling data at $x = o$ is $f(t)$, a bounded and continuous function with $\int_o^t |f|^2 dt < \infty$ for $t > o$, then

$$u(o,t) = f(t) \quad , \qquad t > o \quad .$$

Evaluating (20) and x = o, we have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp[i\omega t] d\omega \quad .$$

Therefore, $A(\omega)$ is the Fourier transform of $f(t)$. For a physically reasonable solution, $u(x,t)$ must be a square-integrable function, i.e.,

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(\omega)|^2 \exp[-2x\alpha(\omega)] d\omega < \infty \quad (21)$$

for $\qquad x > o \quad .$

It follows from Eq. (21) that

$$\alpha(\omega) \geq o \qquad \text{for} \quad -\infty < \omega < \infty \quad .$$

Moreover, the requirement that $u(x,t)$ be a causal function, i.e.,

$$u(x,t) = o \qquad t < o$$

provides a relation between $\alpha(\omega)$ and $\theta(\omega)$, namely, that

$$\theta(\omega) = \omega/c + \tilde{\theta}(\omega) \quad , \tag{22}$$

$$\tilde{\theta}(\omega) = \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\omega')}{\omega^2 - \omega'^2} d\omega' \quad , \tag{23a}$$

and

$$\alpha(\omega) = \alpha(o) - \frac{\omega^2}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{\theta}(\omega')}{\omega^2 - \omega'^2} d\omega \tag{23b}$$

where c is a constant. The integral relations (23a) and (23b) are identified as the Hilbert transform pair or the Kramers-Kronig relations. Moreover, it is noted that $\tilde{\theta}(\omega)$ is an odd function while $\alpha(\omega)$ is an even function.

Derivation of Eq. (20) by Papoulis (1962) makes use of Cauchy's integral theorem and of theorem V of Paley and Wiener (1934) on the Fourier transform of an entire function of the exponential type. During the application of the Cauchy's integral formula, it is necessary to impose

$$p\left[\frac{\alpha(p/i) + i\vartheta(p/i)}{p^2 + \omega^2}\right] \to o \quad \text{as} \quad p \to \infty$$

an asymptotic condition on the high frequency behavior of the material function. Using theorem XII of Paley and Wiener (1934), we may restate the above asymptotic condition on $\alpha(\omega)$ as

$$\int_0^\infty \frac{\alpha(\omega)}{1 + \omega^2}\, d\omega < \infty \quad .$$

If $\alpha(\omega) \sim |\omega|^s$, then $s < 1$ (Guillemin (1963)).

In the terminology of net-work theory, Eq. (22) is a sum of an <u>all-pass</u> function ω/c and a minimum-phase-shift function, $\alpha + i\vartheta$. Substituting (22) into (20), i.e.,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega)\exp\left\{ - x[\alpha(\omega) + i\vartheta(\omega)] + i\omega(t - x/c)\right\} d\omega \qquad (24)$$

we note that

$$u(x,t) = o \qquad \text{for} \qquad t - x/c < o.$$

This clearly shows that $x/c = t$ is the wave front and $c < \infty$ is its speed of propagation. As $c \to \infty$, we have $u(x,t) = o$ for $t < o$. Since the theory of propagating singular surfaces for a linear viscoelastic medium asserts that the wave speeds are dictated by the instantaneous modulus

$$c = \sqrt{G(o)/\rho} \quad .$$

We have thus demonstrated that the all-pass function is associated with the elastic response of the medium. The minimum-phase-shift function is assigned to the anelastic behavior.

To relate the attenuation coefficient and the phase lag function to the "complex modulus", we substitute $s = i\omega$ into (16) and compare with (20) to yield

$$\alpha(\omega) = -\omega \text{ Im } [\rho/M(\omega)]^{\frac{1}{2}}$$

and

$$\theta(\omega) = \omega \text{ Re } [\rho/M(\omega)]^{\frac{1}{2}} \quad . \tag{25}$$

Instead of using the attenuation coefficient and the phase lag function in Eq. (20), we may write it in terms of the complex refractive index $n(\omega)$ (Lamb, 1962), i.e.,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp\left\{ i\omega[t - xn(\omega)/c_o] \right\} d\omega$$

where

$$n(\omega) = n_r(\omega) - i\, n_i(\omega) \quad .$$

The real and imaginary parts of the complex refractive index are related to the attenuation coefficient and the phase lag function as follows

$$n_i(\omega)/c_o = \alpha(\omega)/\omega$$

and

$$n_r(\omega)/c_0 = 1/c + \gamma/\omega \quad .$$

Since $n_i(\omega)$ and $n_r(\omega)$ are Hilbert transform pairs, therefore,

$$c_0 = c = \sqrt{G(o)/\rho} \quad .$$

The identification of the elastic response in a dispersive-attenuating medium and consequently the value of c_0 has been a source of confusion and misunderstanding. For example, Futterman (1962) chooses his "non-dispersive" behavior at zero frequency and, therefore, his impulse response arrives earlier than the "non-dispersive" signal. Gladwin and Stacy (1974) and Stacy, et al. (1975) have interpreted Futterman's "non-dispersive" behavior as elastic thus conclude that Futterman's linear theory at attenuation is acausal.

Savage (1976) in commenting on the interpretation of Gladwin and Stacy and Stacy and et al. suggests that c_0 must correspond to the immediate or high frequency response of the medium by considerating a "standard" linear solid. With such a choice, with c_0 corresponding to the high frequency non-dispersive behavior, Savage shows that Futterman's apparent paradox is resolved.

Indeed, from considerations of the theory of propagating singular surfaces and of the mathematical theory of hyperbolic equations, the only choice possible is that suggested by Savage. The choice of any reference speed other than $\sqrt{G(o)/\rho}$ would result in the arrival at that travel time of a pulse with a smooth profile, thereby creating an apparent paradox. Moreover, if the equilibrium response $G(\infty)$ exists, then the smooth wave is propagating along a subcharacteristic.

II.4 Power Law Solid (PLS)

Stick (1970) introduces a dispersive-attenuating medium defined by the following equation

$$\alpha(\omega) + i\vartheta(\omega) = (i\omega)^s K \tag{26}$$

with $s < 1$ so that the material law satisfies the asymptotic material behavior as imposed by the causality condition.

Corresponding to the constitutive Eq. (26), there is a creep law given by

$$\rho J(t) = \frac{1}{c^2} + \frac{2K}{c}\frac{t^{1-s}}{\Gamma(2-s)} + \frac{K^2}{\Gamma(3-2s)}\, t^{2(1-s)} \tag{27}$$

and

$$\rho\,\frac{dJ}{dt} + \frac{2K}{c}\frac{t^{-s}}{\Gamma(1-s)} + \frac{K^2}{\Gamma(2-2s)}\, t^{1-2s} \tag{28}$$

where $\Gamma(z)$ is the gamma function. The derivation of the creep law is straight forward using (5), (22), (25), and inverting a Laplace transform.

Accordingly,

$$\frac{dJ}{dt} \to \infty \quad \text{as} \quad t \to o \ ,$$

and the precursor wave will have a zero amplitude at the wave front $x = ct$. This result follows from the theory of propagating singular surfaces.

As $t \to \infty$, Eq. (28) gives

$$\rho\,\frac{dJ}{dt} \to \frac{K^2}{T(2-2s)}\, t^{1-2s} \quad .$$

If $s = \frac{1}{2}$, then the power law solid behaves as a fluid with a viscosity

$$\rho/K^2 \quad .$$

Another feature of the power solid is that the spatial seismic quality factor Q_x is almost constant for $s \to 1$

$$Q_x^{-1}(\omega) = 2\alpha(\omega)/\theta(\omega)$$

$$= 2[\tan(s\pi/2) + \omega^{1-s}/CK \cos(s\pi/2)]^{-1} \quad . \qquad (29)$$

Indeed for $\frac{1}{c} = o$ giving a medium without any elastic response, then

$$Q_x^{-1}(\omega) = 2 \cot (s\pi/2)$$

a truly constant Q_x material for $o < s < 1$. This model has been applied to polymeric materials by Kolsky (1956). Kjartansson (1978) suggests its use for modeling geological materials. The relation between the intrinsic quality factor $Q(\omega)$ and the spatial quality factor Q_x is given by

$$Q = [\theta/\alpha - \alpha/\theta]/2$$

or

$$Q^{-1} = 2/[\theta/\alpha - \alpha/\theta] \quad . \qquad (30)$$

From (29), we have

$$\theta/\alpha = 2Q_x = \tan(s\pi/2) + \omega^{1-s}/Kc \cos(s\pi/2)$$

and

$$Q^{-1} = 2/[2Q_x - 1/(2Q_x)]$$

$$0 < \omega < \infty \qquad\qquad s < 1$$

The limiting behaviors of Q^{-1} for $\omega \to o$ and $\omega \to \infty$ are

$$Q^{-1}(\omega) = \frac{2}{\tan(s\pi/2) - \cot(\pi s/2)}\left[1 - \frac{\omega^{1-s}}{Kc\ \sin(s\pi/2)}\,\frac{1 + \cot^2(s\pi/2)}{1 - \cot^2(s\pi/2)} + 0\left(\omega^{2(1-s)}\right)\right]$$

$$\text{as} \quad \omega \to o$$

and

$$Q^{-1}(\omega) = \frac{2Kc\ \cos(s\pi/2)}{\omega^{1-s}}\left[1 + 0\left(\frac{1}{\omega^{1-s}}\right)\right]$$

$$\text{as} \quad \omega \to \infty \quad .$$

III. Impulse Response

In the previous parts we discussed the intrinsic properties of a "standard" linear solid, a "standard" linear solid with a continuous spectrum of relaxation times, and a power law solid. Now, we discuss the wave propagation properties.

The "standard" linear solid with a continuous spectrum of relaxation times (SLSCS) and the power law solid (PLS) are constructed so that Q^{-1} is nearly constant over the seismic frequency range. Outside the seismic band, these two materials behave differently. Their Q^{-1} behaviors are summarized in Table I.

We present the far field or long time asymptotic solutions of the impulse response in the three above mentioned linear viscoelastic solids Detail construction of the asymptotic solution for the "standard" linear solid and the standard linear solid with a continuous spectrum of relaxation time may be found in Chin and Thigpen (1978) and Minster (1978 a,b). However, Chin and Thigpen's analysis is reported here. The asymptotic solution of the power law solid, on the other hand, is new. This analysis clarifies and extends Strick's (1970) construction.

Asymptotic expansion of the impulse response for the "standard" linear solid is discussed first, followed by the solution of the standard linear solid with a continuous spectrum of relaxation times. The essential structure of the wave propagation problem in these two solids is similar with slight differences to account for the subtle changes in the properties of the medium. Finally, we present the asymptotic solution for the power law solid. The response of the standard linear solid with a continuous spectrum of relaxation times will be compared to that of the power law solid to show the influences of the material response outside the seismic band.

III.1 "Standard" Linear Solid

For the "standard" linear solid, the integral representation of the solution of the impulse response is by (16),

$$u(x,t) = \frac{1}{2\pi i} \int_B \exp\left\{st - \frac{x}{c}\, s\sqrt{\frac{s + 1/\tau_\sigma}{s + 1/\tau_\varepsilon}}\right\} ds$$

where

$$c = \sqrt{\frac{M_I}{\rho}}\,,$$ and B is the Bromwich path.

Defining the following dimensionless variables

$$\tau = t/\tau_\sigma, \quad a = \tau_\sigma/\tau_\varepsilon, \text{ and } \beta = x/ct$$

we have

$$\tau_\sigma u(x,t) = \frac{1}{2\pi i} \int_B \exp\left\{\tau\left[s - \beta s\sqrt{\frac{s+1}{s+a}}\right]\right\} ds \quad . \qquad (31)$$

The branch cut associated with the radical is taken along $-1 < x < -a$. Asymptotic expansion of the integral is performed with respect to the parameter $\tau >> 1$. This implies that

$$t >> \tau_\sigma \quad .$$

that is, we are studying the behavior of the solution at times large compared to the stress-relaxation time. We may equivalently perform the asymptotic expansion at large distance from the source. In this case the integrand can be rewritten as

$$\frac{x}{c\tau_\sigma}\left\{\tilde{\beta}s - s\sqrt{\frac{s+1}{s+a}}\right\}$$

where $\tilde{\beta} = ct/x$. Here the interpretation for the far field solution is $x/c\tau_\sigma >> 1$ or $x >> \tau_\sigma\sqrt{\frac{M_I}{\rho}}$. In either case, β or $\tilde{\beta}$ represents a ray emanating from the origin in the $x - t$ plane.

Becasue of the decay of the integrand for $\beta > 1$ ($\tilde{\beta} < 1$), the Bromwich path may be closed in the right-half plane by a semi-circle with radius $R \to \infty$ to yield

$$u(x,t) = o \qquad \text{for} \qquad x > ct \quad .$$

This means that there is no signal prior to $x = ct$, in accordance with the predications of the theory of propagating singular surfaces. Thus, we need only to examine

$$o < \beta < 1$$

or

$$1 < \tilde{\beta} < \infty \quad .$$

In Chin and Thigpen (1978), the long time asymptotic development is chosen since $o < \beta < 1$.

In developing the asymptotic expansion for (31), Chin and Thigpen (1978) found that the steepest-descent method breaks down in the neighborhood of $\beta = o$ and $\beta = 1$ and uniformly valid asymptotic expansion techniques (Oliver, 1974; Bleistein, 1967; Bleistein and Handelsman, 1975) or modified steepest-descent methods (Whitham, 1974) are required. Examples of difficulties encountered in the steepest-descent method are confluencing saddle points and essential-singularity/saddle-point interactions. In each of these instances, the appearance of a second parameter (in addition to the primary variable with which the asymptotics are performed) induces the nonuniformity. An asymptotic expansion with respect to λ is said to be uniformly valid with respect to the parameters $\{\alpha\}$ if it is valid for all pertinent values of $\{\alpha\}$ (Erdelyi, 1956).

In the neighborhood of $\beta = 1$, the nonuniformity of the asymptotic expansion is due to two saddle points confluencing at infinity to produce a saddle point of infinite order. This gives rise to a wave that propagates with a speed $c = \sqrt{M_I/\rho}$ and decays exponentially with a rate given by $[1/\tau_\sigma - 1/\tau_\varepsilon]/(2c)$. The leading term of this expansion is just that given by the theory of propagating singular surfaces. In particular, the following asymptotic expansion is obtained:

$$\tau_\sigma u(x,t) \sim \exp\left[-\frac{1-a}{2}\beta\tau\right]\left\{\delta(1-\beta) + \left[\frac{\beta(1-a)(1+3a)}{8(1-\beta)}\right]^{\frac{1}{2}}\left[1-\frac{1+2a+5a^2}{2(1+3a)}\tau(1-\beta)\right] \times I_1\left\{\left[\frac{\beta(1-\beta)(1-a)(1+3a)}{2}\right]^{\frac{1}{2}}\tau\right\} + \dots\right\} \qquad (32)$$

where $I_n(z)$ is the nth order modified Bessel function of the first kind. This expansion is valid for $(\beta-1)\tau << 1$ and for any τ.

In the neighborhood of $\beta = o$, it is found that the steepest-descent analysis is invalidated by the coincidence of the saddle point with the $1/\sqrt{s+a}$ signularity. The manifestation of this interaction is a boundary layer serving to connect the main diffusive wave to the boundary condition. The asymptotic solution in the neighborhood of $\beta = o$ is given by

$$\tau_\sigma u(x,t) \sim (1-a)e^{-a\tau}\left[\frac{\beta a}{1-a}\right]^{2/3}\left\{F_0(\gamma) - \left(\frac{1}{a}-\frac{3}{2}\right)\frac{(\beta\tau a)^{4/3}}{(1-a)\tau^{1/3}}F_1(\gamma) + \dots\right\} \qquad (33)$$

where

$$\gamma = (\beta\tau a)^{2/3}\,[(1-a)\tau]^{1/3}$$

$$F_0(\gamma) = \frac{1}{\gamma\sqrt{\pi}}\int_o^\infty (2u^2-1)e^{-u^2}\, I_0\left[(4\gamma)^{3/4}\sqrt{u}\right]\, du$$

and

$$F_1(\gamma) = \frac{1}{\gamma^{3/2}\sqrt{\pi}}\int_o^\infty u(2u^2-3)e^{-u^2}\, I_0\left[(4\gamma)^{3/4}\sqrt{u}\right] du\ .$$

For $o < \beta < 1$, steepest-descent method applies. There are two real solutions to the saddle point equation

$$\frac{1}{\beta} = \left(\frac{s+1}{s+a}\right)^{\frac{1}{2}} \left[1 - \frac{1-a}{2} \frac{s}{(s+1)(s+a)}\right] \quad .$$

The primary saddle point $\tilde{s}$ has the following properties

$$- a < \tilde{s} < \infty \quad ,$$

$$\tilde{s} = \left[\frac{(1-a)(1+3a)}{8(1- \ )}\right]^{\frac{1}{2}} + O(1) \qquad \text{as} \quad \beta \to 1$$

$$\tilde{s} = \frac{\sqrt{a}}{1-a}(\beta - \sqrt{a}) + O[(\beta - \sqrt{a})^2] \qquad \text{as} \quad \beta \to \sqrt{a}$$

and

$$\tilde{s} = - a + \beta^{2/3} (a/2)^{2/3} (1-a)^{1/3} + O(\beta^{4/3}) \quad \text{as} \quad \beta \to 0 \quad .$$

Figure 5 is a plot of $\tilde{s}$ vs β for some values of a. The secondary saddle point $\hat{s}$ is given by

$$- \infty < \hat{s} < -1 \quad ,$$

$$\hat{s} = - \left[\frac{(1-a)(1+3a)}{8(1-\beta)}\right]^{\frac{1}{2}} + O(1) \qquad \text{as} \quad \beta \to 1$$

and

$$\hat{s} = - [1 + \beta^2/4(1-a) + O(\beta^4)] \qquad \text{as} \quad \beta \to 0 \quad .$$

The secondary saddle point $\hat{s}$ as a function of β and some values of a is shown in Fig. 6. The elevation of the primary saddle point $\hat{s}$ is higher than that of the secondary saddle point $\hat{s}$ and attains equal value as $\tilde{s} \to \infty$ and $\hat{s} \to -\infty$. Thus, the primary saddle point $\tilde{s}$ contributes predominantly to the integral except in the neighborhood of infinity where the saddle points coalesce to form a saddle point of infinite order. A typical steepest descent path for $o < \beta < 1$ passing though the primary saddle point is depicted in Fig. 7.

A one-term asymptotic representation of the solution using steepest descent method is

$$\tau_\sigma u(x,t) = \left[\frac{1}{2\pi\tau F''(\tilde{s}(\beta))}\right]^{\frac{1}{2}} \exp\left[\tau F(\tilde{s}(\beta))\right] \tag{34}$$

where

$$F(z) = z\left[1 - \beta\left(\frac{z+1}{z+a}\right)^{\frac{1}{2}}\right]$$

and

$$F''(z) = \beta\,\frac{(1-a)}{4}\left(\frac{z+1}{z+a}\right)^{\frac{1}{2}} \frac{(1+3a)z + 4a}{(z+1)^2(z+a)^2} \quad .$$

The functions $F[\tilde{s}(\beta)]$ and $F''[\tilde{s}(\beta)]$ are plotted as a function of β in Figs. 8 and 9 respectively.

From these figures and (34), it is readily seen that

$$\tau_\sigma u(x - c\sqrt{a}\ t,t) \sim [\tau F''(\sqrt{a})]^{-\frac{1}{2}}$$

and

$$u(x,t) \leq u(x - c\sqrt{a}\ t,t) \quad .$$

To see the physical significance of the above observation, we expand $u(x,t)$ about $\beta = \sqrt{a}$ to obtain

$$\tau_\sigma u(x,t) \sim \left[\frac{a}{2\pi(1-a)\tau}\right]^{\frac{1}{2}} \exp\left[-\frac{\tau(\beta-\sqrt{a})^2}{2(1-a)}\right]\left\{1 + O(\beta-\sqrt{a})\right\} \quad .$$

In terms of the physical variables, the leading term gives

$$\left[\frac{\tau_\sigma^2}{2\pi(\tau_\varepsilon-\tau_\sigma)t}\right]^{\frac{1}{2}} \exp\left[-\frac{(x-c\sqrt{a}\ t)^2}{2\tau_\sigma c^2(1-\tau_\sigma/\tau_\varepsilon)t}\right] \quad . \tag{35}$$

This clearly shows that the main wave is a propagating diffusive wave as (35) is a solution of the diffusion equation with diffusivity

$$\tfrac{1}{2}c^2\tau_\sigma(1-\tau_\sigma/\tau_\varepsilon) \quad .$$

This is also an intrinsic property of a Kelvin-Voigt solid. Therefore, the low frequency response of the "standard" linear solid behaves as a Kelvin-Voigt solid with a relaxation time of

$$\tau_\sigma(1-\tau_\sigma/\tau_\varepsilon) \quad .$$

This diffusive behavior may alternatively be obtained using perturbation methods discussed by Cole (1968) and Whitham (1959, 1974). Following Cole and Whitham, we extract the approximating equation in the neighborhood of $x = c\sqrt{a}\ t$. This is done by transforming to the wave coordinates

$$(t,x) \rightarrow (t,\ \xi = x - c\sqrt{a}\ t) \quad .$$

The effect of this transformation is to make the derivatives with respect to t small, so that the approximate form of the Eqs. (8), (13), and (14) is

$$2\,\frac{\partial^2 u}{\partial\xi\partial t} = c^2\tau_\sigma\left(1 - \tau_\sigma/\tau_\varepsilon\right)\frac{\partial^3 u}{\partial\xi^3} \quad .$$

In the derivation, it is noted that the diffusive term is directly related to the high frequency response of the material medium and manifests itself solwly and cumulatively. The wave propagation process in a "standard" linear viscoelastic solid may be summarized as follows:

(1) a precursor wave propagates with a speed corresponding to the instantaneous modulus G(o) and attenuates with distance (Eq. 32);

(2) a main diffusive wave propagates with a speed corresponding to the equilibrium modulus $G(\infty)$ and spreads as $1/\sqrt{t}$ (Eq. 34);

(3) a boundary layer is formed connecting the main wave to the boundary data (Eq. 33). A graphic representation of these results is shown in Fig. 10.

III.2 Standard Linear Solid with a Continuous Distribution of Relaxation Times

For this case, the integral representation of the solution is given by

$$\tau_{min} u(x,t) = \frac{1}{2\pi i} \int_B \exp\left[s\tau \left\{ 1 - \beta/\left[1 + \mathscr{C}\ln\left(\frac{s+a}{s+1}\right)\right]^{\frac{1}{2}} \right\}\right] ds \tag{36}$$

where $\tau = t/t_{min}$

$\beta = x/ct$

$a = \tau_{min}/\tau_{max}$.

Associated with the integrand are the branch points

$$s = -a, \quad s = -1, \text{ and } s = -\frac{a - e^{-1/\mathscr{C}}}{1 - e^{-1/\mathscr{C}}} = -\tilde{a} \quad .$$

The branch cuts are taken along the negative real axis $-1 < x < -\tilde{a}$.

The saddle point structure for (36) is similar to that of the "standard" linear solid: There are two real solutions to the saddle point equation. The primary saddle point $-\tilde{a} < \hat{s} < \infty$ contributes predominantly to the asymptotic expansion of the integral. The secondary saddle point $-\infty < \hat{s} < -1$ contributes significantly only in the neighborhood of infinity where the two saddle points coalesce to produce a saddle point of infinite order.

The primary saddle point $\tilde{s}$ is a function of β, a and $\mathscr{C}$. The local behaviors of $\tilde{s}$ as a function of β about $\beta = 1$, $\beta = \sqrt{1 + \mathscr{C}\,\ell n\, a}$ and $\beta = 0$ are respectively

$$\tilde{s} = \left\{\frac{\mathscr{C}(1-a^2)}{4}\left[1 - \frac{3}{2}\,\frac{\mathscr{C}(1-a)}{1+a}\right]\right\}^{\frac{1}{2}} (1-\beta)^{-\frac{1}{2}} + 0\left[(1-\beta)^{-1}\right] \quad ,$$

$$\tilde{s} = \frac{a(1+\mathscr{C}\ell n\ a)^{\frac{1}{2}}}{\mathscr{C}(1-a)}\left[\beta - (1+\mathscr{C}\ell n\ a)^{\frac{1}{2}}\right]$$

$$+ 0\left[\left[\beta - (1+\mathscr{C}\ell n\ a)^{\frac{1}{2}}\right]^2\right]$$

and

$$\tilde{s} = -\tilde{a} + \left[\left(\frac{\tilde{a}}{2}\right)^2 \frac{(1-\tilde{a})(a-\tilde{a})}{\mathscr{C}(1-a)}\right]^{1/3} \beta^{2/3} + 0(\beta^{4/3}) \quad .$$

Because of the similarities of the saddle-point structure, it is expected that the asymptotic solution of the standard linear solid with a continuous distribution of relaxation times has the same characteristics as that of the "standard" linear solid with modifications accounting for the subtle changes in material response: The precursor wave has a decay rate of

$$- \tfrac{1}{2}\,(1/\tau_{min} - 1/\tau_{max}) \quad .$$

The main wave has a form

$$u \sim \left[\frac{1 + \mathscr{C}\ell n\ a}{2\pi\mathscr{C}} \left(\frac{a}{1 - a}\right) \frac{1}{\tau}\right]^{\frac{1}{2}} \exp\left\{- \frac{a}{2\mathscr{C}(1-a)}\ \tau\left[\beta - (1 + \mathscr{C}\ell n\ a)^{\frac{1}{2}}\right]^2\right\} \quad (37)$$

as $\qquad \beta \to (1 + \mathscr{C}\ell n\ a)^{\frac{1}{2}}$

Once again, the main wave is a propagating diffusive wave that propagates with a speed corresponding to the equlibrium modulus $G(\infty) = M_I(1 + \mathscr{C}\ell n\ a)$ and diffuses in time or space with a diffusivity

$$\mathscr{C}M_I(\tau_{max} - \tau_{min})/2\rho \quad .$$

It should be emphasized that the cumulative dispersive-attenuating phenomenon is pervasive and it requires time to develop. Analogous to the "standard" linear solid, the low frequency response of the material is, once again, Kelvin-Voigt like with a relaxation time of $\mathscr{C}(\tau_{max} - \tau_{min})$.

III.3 Power Law Solid

The integral representation of the solution for the impulse response of a power law solid as given by

$$\alpha(\omega) + i\ (\omega) = i\omega/c + K(i\omega)^s \qquad s < 1$$

is

$$u(x,t) = \frac{1}{2\pi i} \int_B \exp\left\{pt - \frac{x}{c}\,p - xKp^s\right\} dp \qquad (38)$$

with the branch cut taken along the entire negative real axis. Note that for $t - x/c < o$, the integrand is a decaying analytic function in the right half plane. Closing the Bromwich path in the right half plane by a semi-circle with radius $R \to \infty$ and using Jordan's lemma, we obtain

$$u(x,t) = 0$$

for $x/c > t$ confirming that $u(x,t)$ is indeed a causal function. In fact, for $c < \infty$, we have causality relative to the reduced time $t - x/c$. As $c \to \infty$, $u(x,t) = 0$ for $t < 0$.

Before proceeding to the asymptotic expansion of (38), we introduce the following change of variables,

$$p = \frac{xK}{t - x/c}^{\;1/(1-s)} \zeta$$

and

$$z = xK\left(\frac{xK}{t - x/c}\right)^{s/(1-s)}$$

to obtain

$$u(x,t) = \left(\frac{xK}{t - x/c}\right)^{1/(1-s)} \frac{1}{2\pi i} \int_{B'} \exp\left\{z(\zeta - \zeta^2)\right\} d\zeta \qquad (39)$$

where B' is the image of B in s-plane. By rewriting z the similarity variable in the form of

$$(t - x/c)\left(\frac{kK}{t - x/c}\right)^{1/(1-s)}$$

(39) becomes

$$u(x,t) = \frac{1}{(t - x/c)}\; zF(z;s) = \left(\frac{z}{xK}\right)^{1/s} F(z;s) \qquad (40)$$

where

$$F(z;s) = \frac{1}{2\pi i} \int_{B'} e^{z(\zeta-\zeta^s)} \, d\zeta$$

Note that (40) states that the pulse shape (in time) is the same at all stations or the pulse shape (in space) is the same at all reduced times. This obervation has been noted by Pipkin (1973) and Kjartanssen (1978).

For $s = \frac{1}{2}$, a Table of Integral Transforms (Erdelyi (1954)) gives

$$F(z;\tfrac{1}{2}) = \frac{1}{2\sqrt{\pi z}} \exp\,(-\,z/4) \quad .$$

In other values of $s(o < s < 1)$, there are no known inverse transforms, and asymptotic methods will be utilized to carry out the integration.

Using theorem (7.1) of Olver (1974), we obtain for $z >> 1$ the following asymptotic expansion of the integral $F(z;s)$

$$F(z;s) = \frac{\exp\left\{-\frac{1-s}{s}\tilde{z}\right\}}{[2\pi(1-s)\tilde{z}]^{\frac{1}{2}}} \left\{1 + \frac{(2-s)(2s-1)}{24(1-s)}\frac{1}{\tilde{z}} + 0\,(1/\tilde{z}^2)\right\} \qquad (41)$$

where $\tilde{z} = s^{1/(1-s)}\, z$.

Note that $\tilde{z}$ = constant are curves in x - t plane satisfying the relation

$$t = x/c + \left[sKx/\tilde{z}(1-s)\right]^{1/s} \qquad (42)$$

$\tilde{z} = o$ corresponds to $x = o$ and $t > o$. $\tilde{z} = \infty$ corresponds to the wave-front $t = x/c$. For convenience, with $c < \infty$ let

$$t = \tau/(sKc)^{1/(1-s)} \quad ,$$

$$x = \xi/(sKc^s)^{1/(1-s)}$$

then we have

$$\tau = \zeta + \left[\xi/\tilde{z}(1-s)\right]^{1/s} \tag{43}$$

and inversely

$$\tilde{z} = (\tau-\xi)\left(\frac{\xi}{\tau-\xi}\right)^{1/(1-s)}$$

Figure 11 is a plot of (43) with Strick's value of $s = .9227$ and $\tilde{z} =$ 1, 10, 100. Note that as $\xi \to \infty$, the second term dominates signifying the response of a polymeric medium ($1/c \to 0$).

The scaled impulse response

$$\tilde{u}(x,t) = u(x,t)/(sKc)^{1/(1-s)}$$

is given by

$$\tilde{u}(\xi,\tau) = \frac{\exp\left\{-\frac{1-s}{s}\tilde{z}\right\}}{\tau-\xi}\left[\frac{\tilde{z}}{2\pi(1-s)}\right]^{\frac{1}{2}}\left\{1 + \frac{(2-s)(2s-1)}{24(1-s)}\frac{1}{\tilde{z}} + O(1/\tilde{z}^2)\right\} \tag{44}$$

The leading term of the expansion called a one-term asymptotic approximation is just Strick's Eq. (20). The one-term asymptotic approximation

$$\tilde{u}_1 = \frac{\exp\left[-\frac{1-s}{s}\tilde{z}\right]}{\tau-\xi}\left[\frac{\tilde{z}}{2\pi(1-s)}\right]^{\frac{1}{2}} \tag{45}$$

with ξ fixed has a maximum at

$$\tilde{z}_m = \frac{2-s}{2(1-s)} \quad . \tag{46}$$

At this value of $\tilde{z}$, the $O(1/\tilde{z})$ term has a value of .07045 for $s = .9227$, that is, Eq. (45) underestimates the value of u at $\tilde{z} = \tilde{z}_m$ by 7%. This is confirmed by FFT calculations of Kjartensson (1978), which show that the one-term asymptotic approximation underestimates the low frequency amplitudes in the later part of the pulse. Equation (44) having a two-term expansion can provide a sufficiently accurate approximation to the solution of the impulse response between the wave front and the pulse maximum. Thereafter, the asymptotic approximation degrades and becomes invalid as $\tilde{z} \to o$.

In the region where the asymptotic expansion is valid, it is seen that the pulse peak propagates with a variable velocity and attenuates with $x^{-1/s}$. The wave front propagates with zero amplitude. This result is predicted by the theory of propagating singular surfaces. Moreover as the pulse propagates into the material medium, the response is more polymer-like.

III.4 Comparison of Impulse Responses for SLSCS and PLS

By comparing the asymptotic expansions of the impulse response of a standard linear solid with a continuous spectrum of relaxation times (SLSCS) and of a power law solid (PLS), we see significant and discernible differences in the wave profile. The differences are symptomatic of their respective material behaviors.

For $x > ct$, there is no signal. At the wave front $x = ct$, SLSCS has a precursor with an exponentially decaying amplitude. The rate of attenuation is related to the initial slope of the relaxation function. The PLS, on the other hand, has an infinite creep rate initially giving rise to a vanishing amplitude.

The main wave for SLSCS propagates with a constant speed $\sqrt{M_I(1 + \mathscr{C}\ln a)/\rho}$ and spreads diffusively with a diffusivity of

$M_I(\tau_{max} - \tau_{min})/2\rho$. For this case, the main signal behaves as in a Kelvin-Voigt solid with a relaxation time of $\mathscr{O}(\tau_{max} - \tau_{min})$. Correspondingly, the main wave in a PLS propagates with a variable speed. The pulse maximum traces out a curve in x - t plane approximated by

$$t = x/c + \left[\frac{2(1-s)}{2-s}\right]^{(1-s)/s} (sKx)^{1/s}$$

and attenuates as

$$x^{-1/s} \quad .$$

Moreover as $x \to \infty$, $t \sim x^{1/s}$ and the attenuation rate is proportional to t^{-1}. This describes a polymeric material.

Although both of these models are constructed so that the seismic quality factor Q is very nearly frequency independent over the seismic frequency range, their impulse responses are indeed characteristic of their respective material behaviros. This implies that in studying impulse response of linear dispersive-attenuating medium the material response functions must be known for all frequencies.

CONCLUSIONS

Comparing the impulse response of two linear viscoelastic models having a nearly constant Q over the seismic frequency range but differ otherwise, we conclude that

1. there are significant and discernible features in the wave profiles to permit differentiation of the material models,
2. a complete knowledge of Q over the entire frequency range is necessary to determine the wave propagation problem

when initiated by a rapidly varying process such as an explosion or an earth quake.

Conclusion (2) is contrary to the situation in surface wave and free oscillation problems in which the dispersive properties within the seismic frequency band are insignificantly influenced by the knowledge of Q outside the seismic frequency range.

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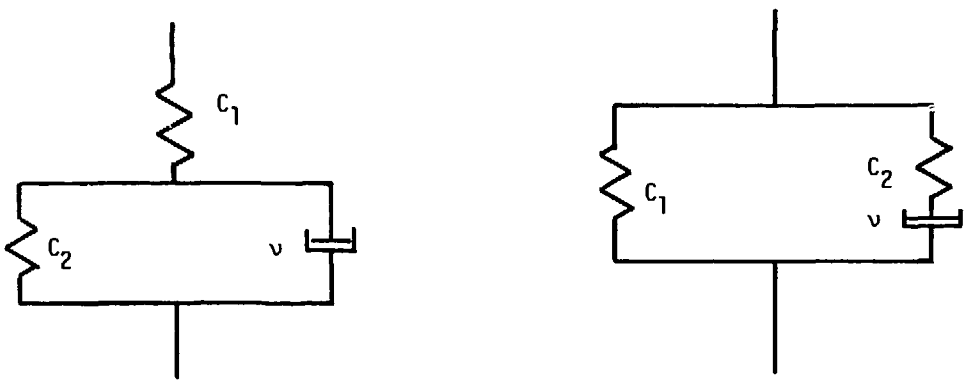
	SLSCS	PLS
$\omega \to 0$	$\sim \omega$	constant
$\omega \to \infty$	$\sim \omega^{-1}$	$\sim \omega^{-(1-s)}$ $s < 1$

Table I. Q^{-1} behavior for SLSCS and PLS.



(a) (b)

Fig. 1

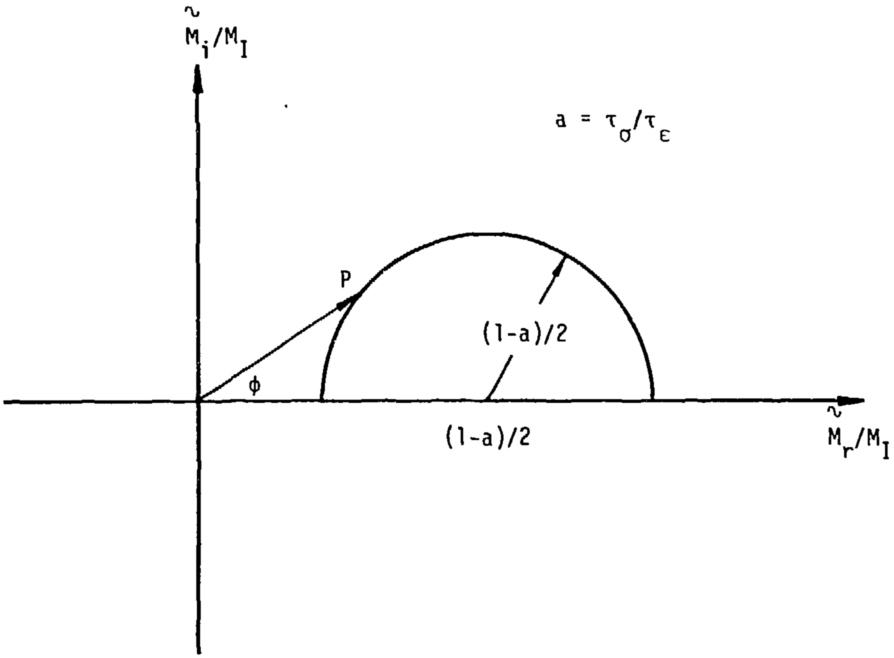
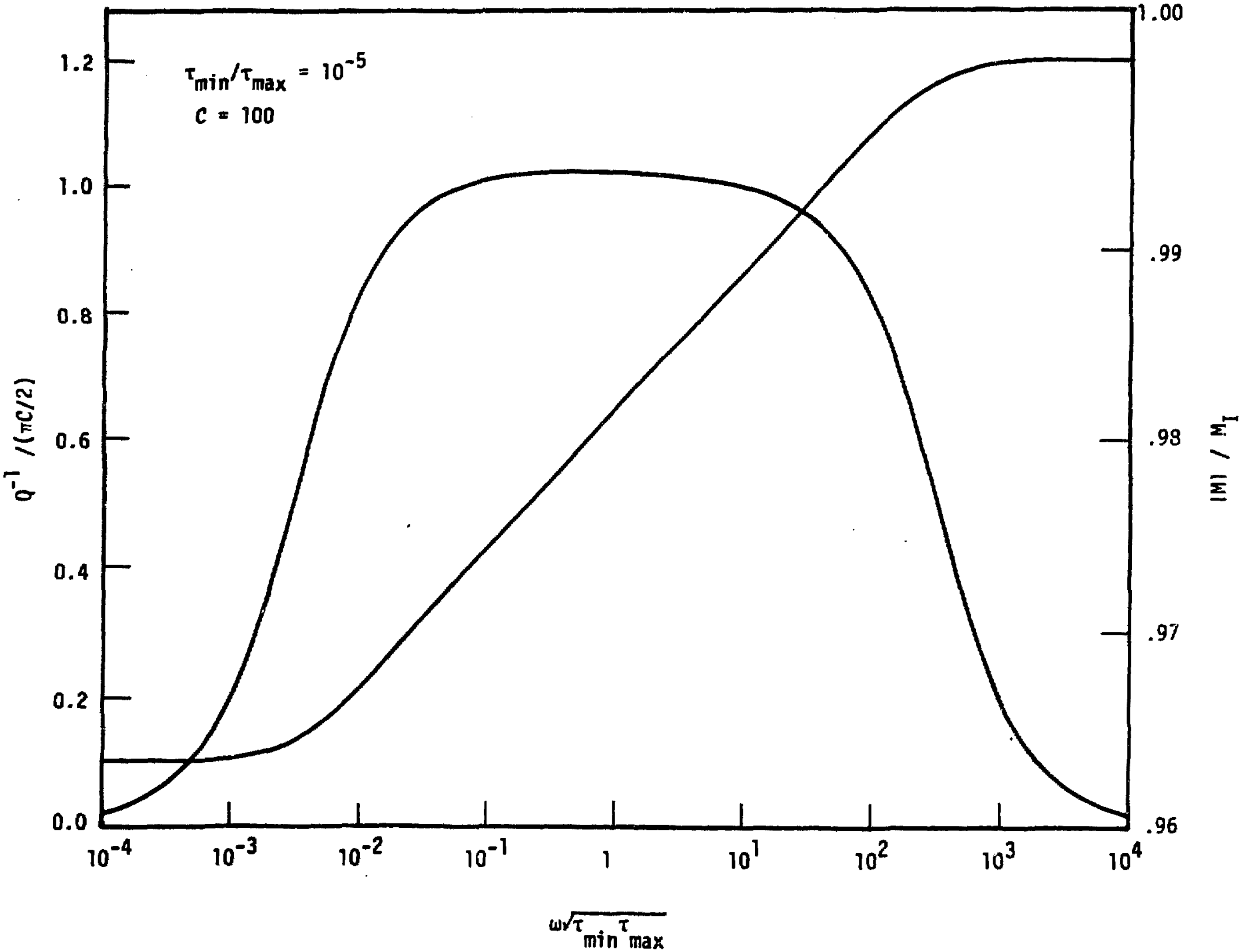

$\tilde{M}_i/M_I$
$a = \tau_\sigma/\tau_\varepsilon$
P
(1-a)/2
ϕ
(1-a)/2
$\tilde{M}_r/M_I$

Fig. 2



$\tau_{min}/\tau_{max} = 10^{-5}$
$C = 100$
$Q^{-1}/(\pi C/2)$
$|M|/M_I$
$\omega\sqrt{\tau_{min}\tau_{max}}$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
.96
.97
.98
.99
1.00
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10^{1}
10^{2}
10^{3}
10^{4}

Fig. 3

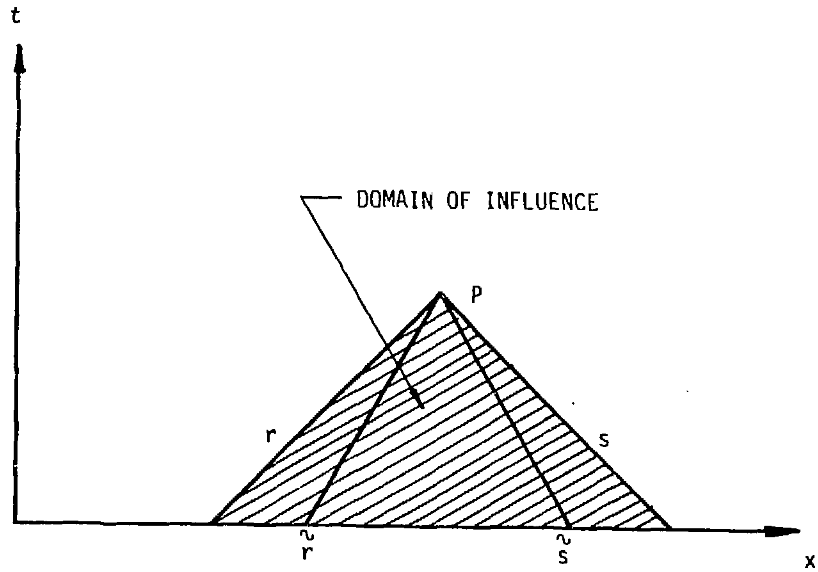


Fig. 4

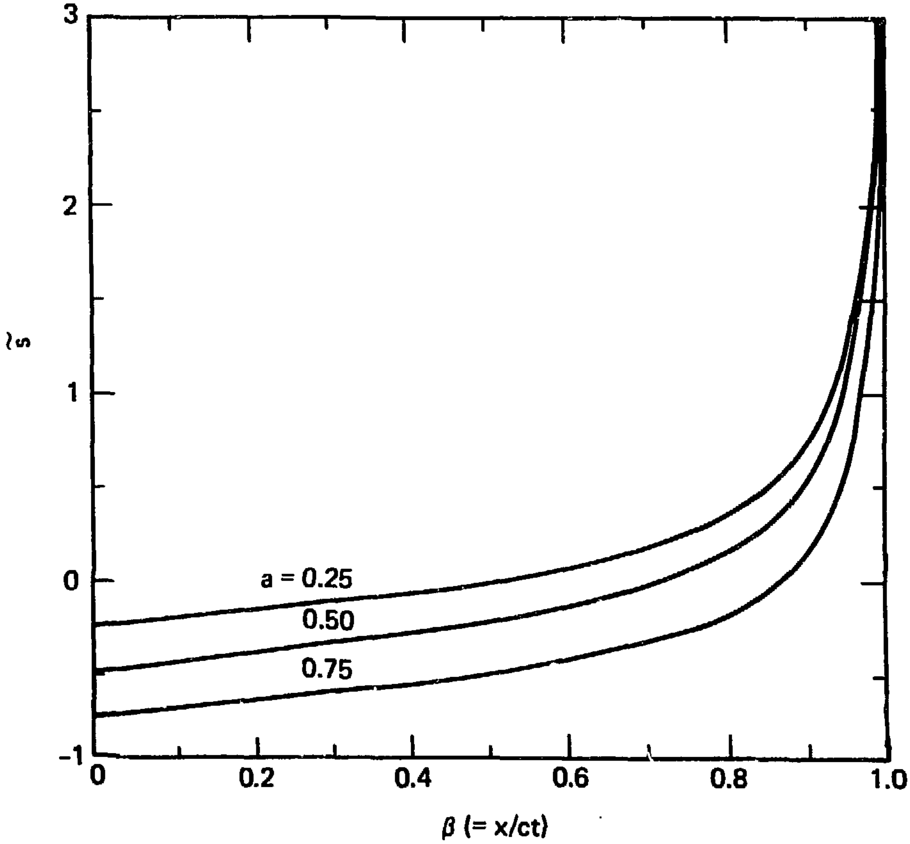


Fig. 5

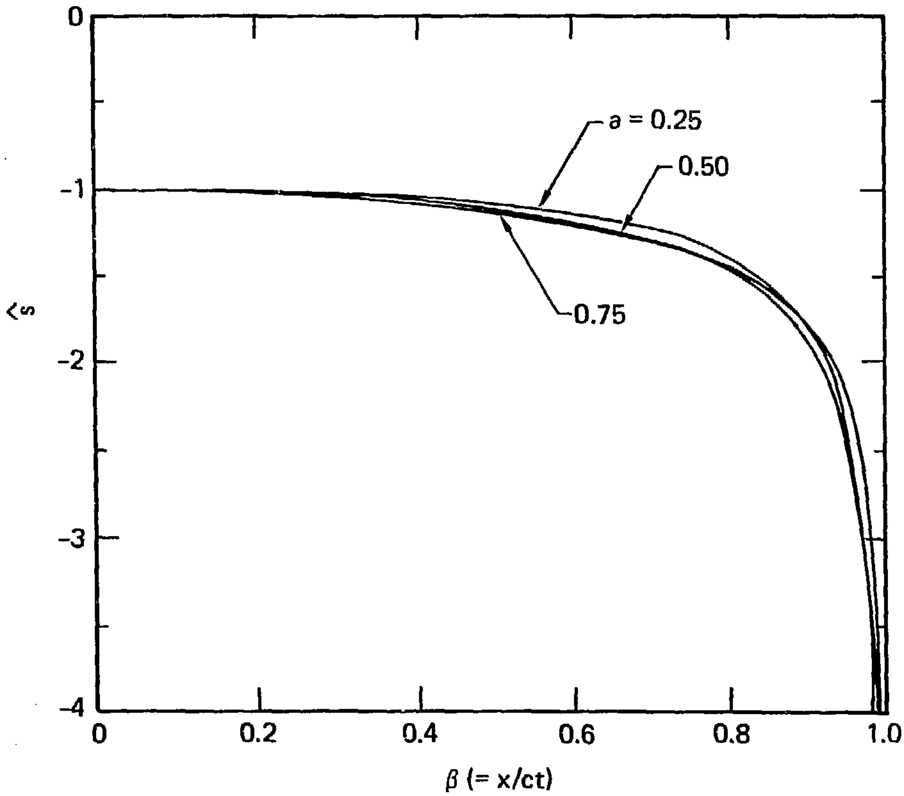


Fig. 6

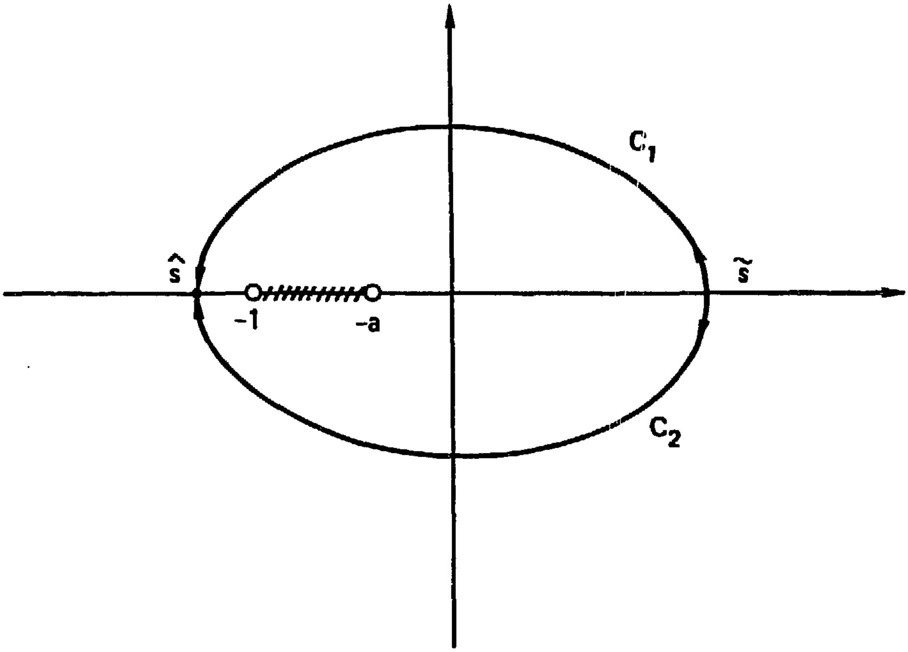


Fig. 7

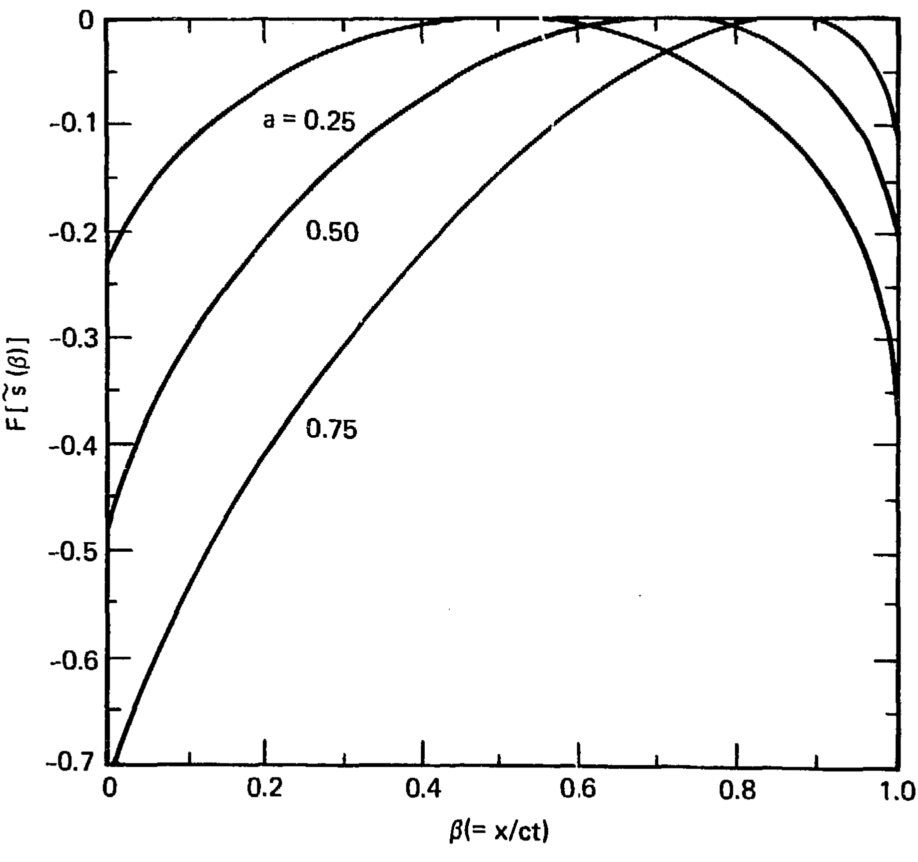


Fig. 8

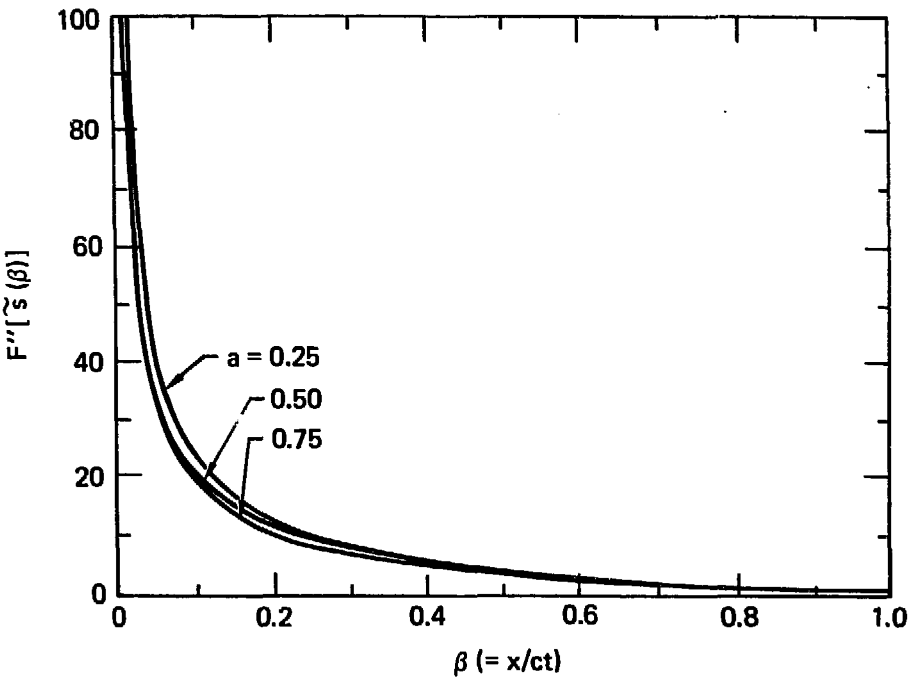


Fig. 9

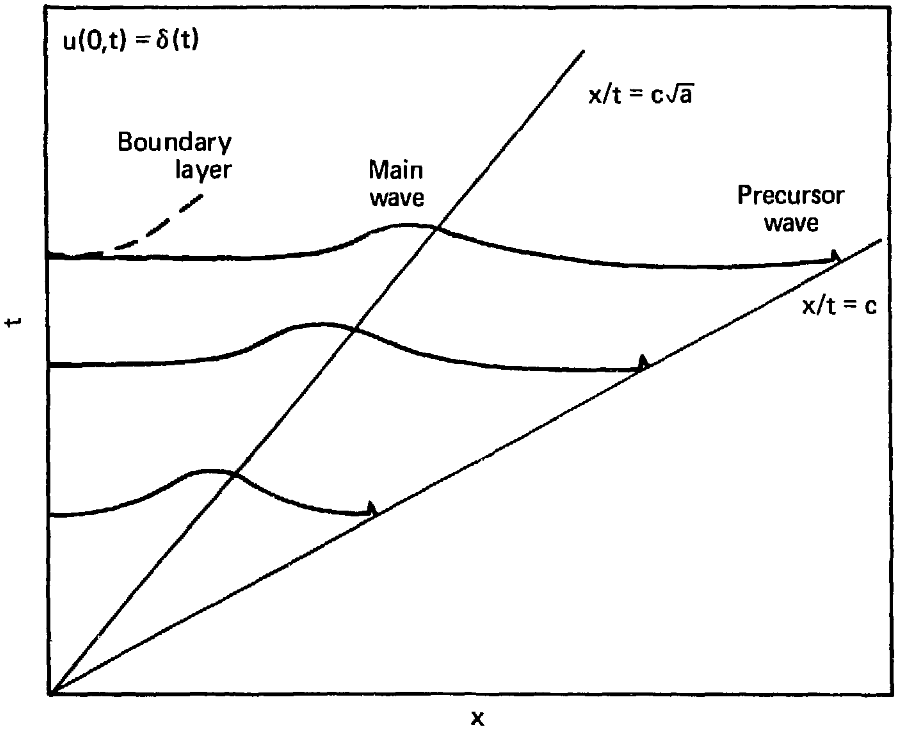


Fig. 10

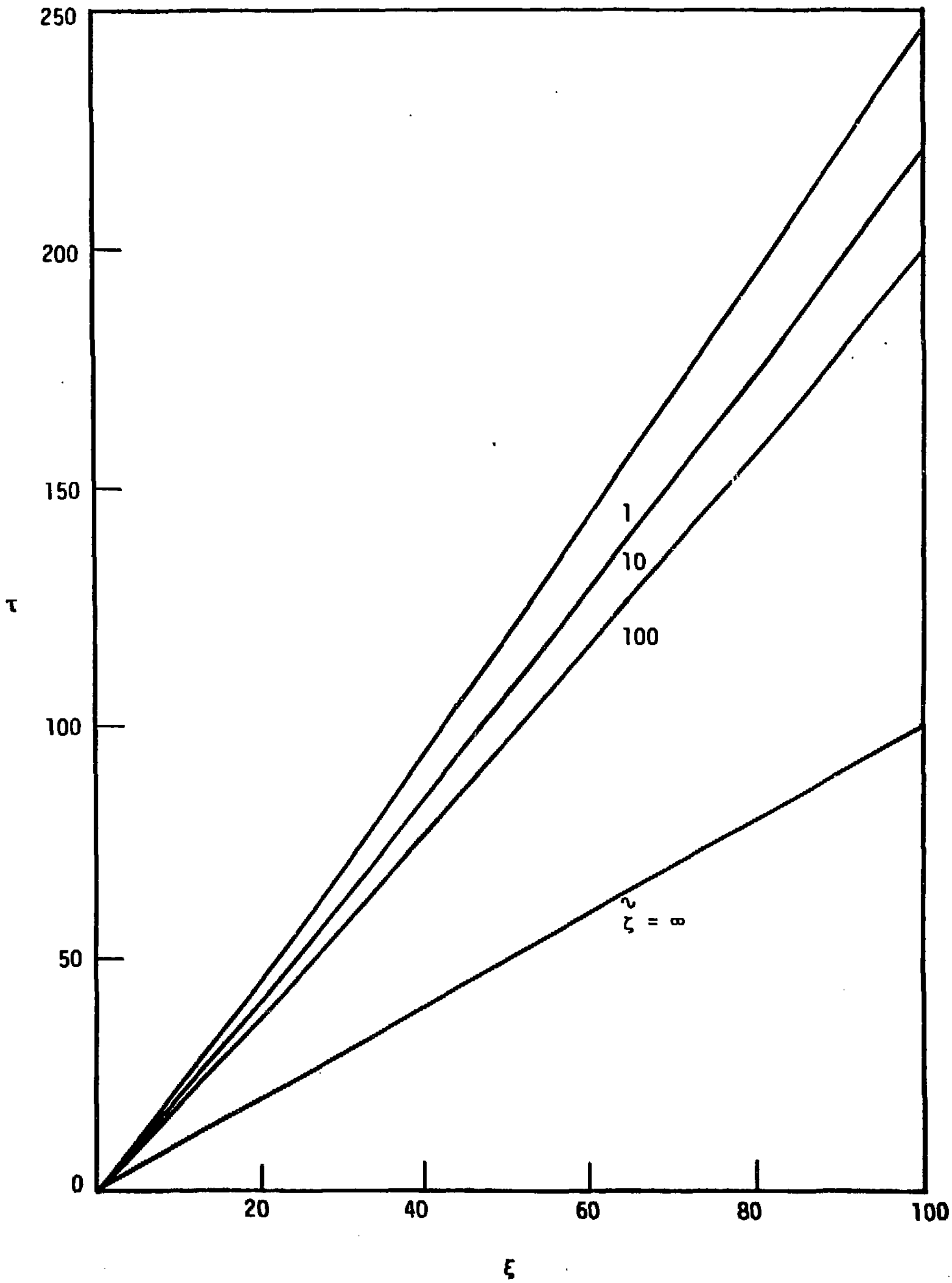


Fig. 11

FIGURE CAPTIONS

Fig. 1 Mechanical analogs of the "standard" linear solid.

Fig. 2 Mapping in the "complex modulus" plane.

Fig. 3 $Q^{-1}/(\pi\mathcal{C}/2)$ and $|M|/M_I$ as functions of $\omega\sqrt{\tau_{min}\tau_{max}}$.

Fig. 4 Characteristic and subcharacteristic curves.

Fig. 5 Primary saddle point $\tilde{s}$ as a function of x/ct.

Fig. 6 Secondary saddle point $\hat{s}$ as a function of x/ct.

Fig. 7 Steepest descent curve.

Fig. 8 Values of $F[\tilde{s}(\beta),a]$.

Fig. 9 Values of $F''[\tilde{s}(\beta),a]$.

Fig. 10 Sketch of the asymptotic solution for the impulse-function case.

Fig. 11 Trajectories of $\tilde{z}$ = constant.