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Application of Theta Functions for Numerical Evaluation of Complete Elliptic Integrals of the First and Second Kinds

D. K. Lee

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APPLICATION OF THETA FUNCTIONS FOR NUMERICAL EVALUATION OF COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

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ABSTRACT

An approximation method based on the use of theta functions is shown to be efficient and useful in numerical evaluation of complete elliptic integrals of the first and second kinds, $K(k)$ and $E(k)$, respectively. The integrals are expressed in terms of power series of the form $\sum a_n q^{n^2}$, $0 \leq q < 1$, where q is the nome determined uniquely from a given value of the argument k. The series converge very rapidly, except for small domains near $|k| = 1$, where they either converge slowly or fail to converge. When applied on Cray 2 computers for $0 \leq k^2 \leq 0.9955$, the procedure is found to be more efficient than both the Chebyshev approximations of the Hastings form and the standard Gauss arithmetic-geometric mean process. Numerical results that demonstrate the accuracy and efficiency of the approximation method are presented.

The complete elliptic integrals of the first and second kinds are defined, respectively, by [1–3]

$$\begin{aligned} K(k) &= \int_0^{\pi/2} \frac{d\phi}{\left(1-k^2\sin^2\phi\right)^{1/2}} \\ &= \int_0^1 \frac{dx}{\left[(1-x^2)(1-k^2x^2)\right]^{1/2}} \\ &= \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) , \qquad |k|<1 , \end{aligned} \tag{1}$$

$$\begin{aligned} E(k) &= \int_0^{\pi/2} \left(1-k^2\sin^2\phi\right)^{1/2} d\phi \\ &= \int_0^1 \left(\frac{1-k^2x^2}{1-x^2}\right)^{1/2} dx \\ &= \frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;k^2\right) , \qquad |k|\le 1 , \end{aligned} \tag{2}$$

where ${}_2F_1\,(a,b;c;z)$ is the Gauss hypergeometric series.

The functions $K(k)$ and $E(k)$ are useful in the calculation and analysis of various types of problems in many branches of physics. An example involving both $K(k)$ and $E(k)$ is the problem of calculating the magnetic field $\mathbf{B}$ and vector potential $\mathbf{A}$ due to a circular current loop. Their expressions in the cylindrical coordinates are given by [4,5]

$$\mathbf{B}(\rho,z) = B_\rho\widehat{\rho} + B_z\widehat{z} , \tag{3}$$

$$B_\rho = \frac{\mu_0 I}{2\pi}\frac{z}{\rho\left[(a+\rho)^2+z^2\right]^{1/2}}\left[\frac{a^2+\rho^2+z^2}{(a-\rho)^2+z^2}E(k)-K(k)\right] , \tag{4}$$

$$B_z = \frac{\mu_0 I}{2\pi}\frac{1}{\left[(a+\rho)^2+z^2\right]^{1/2}}\left[\frac{a^2-\rho^2-z^2}{(a-\rho)^2+z^2}E(k)+K(k)\right] , \tag{5}$$

$$\mathbf{A}(\rho,z) = A_\phi\widehat{\phi} , \tag{6}$$

$$A_\phi = \frac{\mu_0 I a}{\pi}\frac{1}{\left[(a+\rho)^2+z^2\right]^{1/2}}\frac{(2-k^2)K(k)-2E(k)}{k^2} , \tag{7}$$

where

$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2} , \tag{8}$$

and I and a are the current and radius of the loop, respectively.

A useful numerical method for evaluating the complete elliptic integrals $K(k)$ and $E(k)$ is the method of the arithmetic-geometric mean described in Ref. [1]. This method has the following advantages: (1) the numerical accuracy of the calculation can easily be specified by a single parameter, and (2) the algorithm is so simple that it is quite portable. This procedure involves evaluation of a square root (geometric mean) in each loop of an iteration process which continues until the specified accuracy is attained. On the other hand, the method of Chebyshev approximations of the Hastings form is based on the truncated modified Legendre form [1,2,6]:

$$K(k) = \sum_{n=0}^{N'} a_n \eta^n + \ln\left(\frac{1}{\eta}\right) \sum_{n=0}^{N'} b_n \eta^n , \tag{9}$$

$$E(k) = \sum_{n=0}^{N'} c_n \eta^n + \ln\left(\frac{1}{\eta}\right) \sum_{n=1}^{N'} d_n \eta^n , \tag{10}$$

where

$$\eta = 1 - k^2 = k'^2$$

is the complementary parameter. A useful discussion and extensive compilation of numerical values of a_n, b_n, c_n, and d_n for $2 \le N' \le 10$ can be found in Ref. [2].

It is often necessary to evaluate, with high precision, the difference between $K(k)$ and $E(k)$:

$$D(k) = K(k) - E(k) . \tag{11}$$

For example, near the axis of the circular current loop ($\rho = 0$), both B_ρ and A_ϕ become proportional to $D(k)$. Since $K(0) = E(0) = \pi/2$, accurate calculation of $D(k)$ for small $|k|$ cannot rely on Eqs. (9) and (10). One can either use the method of the arithmetic-geometric mean or the power series expansion obtained from Eqs. (1) and (2),

$$D(k) = \frac{\pi}{2}\left[\frac{1}{2}k^2 + \left(\frac{1}{2}\right)^2 \frac{3}{4}k^4 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{5}{6}k^6 + \cdots\right] . \tag{12}$$

Unfortunately, neither of these procedures is very efficient.

An alternative approach for computing $K(k)$, $D(k)$, and $E(k)$ near $k = 0$ is to express them in terms of the nome q given by [1,7,8]

$$q = \exp[-\pi K(k')/K(k)]$$
$$= \frac{k^2}{16} + 8\left(\frac{k^2}{16}\right)^2 + 84\left(\frac{k^2}{16}\right)^3 + 992\left(\frac{k^2}{16}\right)^4 + \cdots . \tag{13}$$

The derivation of such expressions is based on the relationships between the complete elliptic integrals and two of the theta functions, defined by [7]

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz , \tag{14}$$

$$\theta_4(z,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz . \tag{15}$$

The results given in Refs. [7,8] are

$$K(k) = \frac{\pi}{2}\left[\theta_3(0,q)\right]^2$$
$$= 2\pi\left(\frac{1}{2} + \sum_{n=1}^{\infty} q^{n^2}\right)^2 , \tag{16}$$

$$D(k) = \frac{\pi^2}{4K(k)} \frac{\theta_4''(0,q)}{\theta_4(0,q)}$$
$$= -\frac{\pi^2}{K(k)} \frac{\sum_{n=1}^{\infty}(-1)^n n^2 q^{n^2}}{\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n q^{n^2}} , \tag{17}$$

where

$$\theta_4''(0,q) = \left.\frac{d^2\theta_4(z,q)}{dz^2}\right|_{z=0} .$$

The q-series of $\theta_3(0,q)$, $\theta_4(0,q)$, and $\theta_4''(0,q)$ converge extremely fast except near $q = 1$, since they contain powers only of the form q^{n^2}. Therefore, the main problem is to find q for a given k. The series given by Eq. (13) can be used, but it converges slowly unless $|k| \ll 1$. A more convenient method is to express q in terms of

$$\lambda \equiv \frac{1}{2}\frac{1-\left(1-k^2\right)^{1/4}}{1+\left(1-k^2\right)^{1/4}} \tag{18}$$

by using the relation [7]

$$\lambda\left(1 + 2\sum_{n=1}^{\infty} q^{4n^2}\right) = \sum_{n=0}^{\infty} q^{(2n+1)^2} . \tag{19}$$

It can be shown that the solution of Eq. (19) for q in terms of λ has the form

$$q = \lambda \left(1 + \sum_{m=1}^{\infty} \alpha_m \lambda^{4m} \right) . \tag{20}$$

Note that the domain of $0 \le k \le 1$ corresponds to $0 \le \lambda \le 1/2$ and $0 \le q \le 1$. By substituting Eq. (20) into Eq. (19) and equating coefficients of the same powers of λ, one can obtain values of α_m, but this process becomes extremely complicated as m becomes large. More useful procedures are not known at the present time. The numerical values of α_m are given in Ref. [8] for $1 \le m \le 4$, and an extension to $m = 12$ appears in Table I.

It may be remarked that $\lambda(k)$ and $q(k)$ are highly nonlinear functions of k, and the power series in terms of q or λ in Eqs. (16), (17), and (20) are useful not only for $|k| \ll 1$, but also for a much wider domain that excludes only very small portions near $|k| = 1$. Table II lists numerical values of λ, λ^4, q, q^9, q^{16}, q^{25}, q^{36}, $K(k)$, and $E(k)$ for many values of k. In the table, it is seen that, if $k = 0.9995$, for example, then $\lambda = 0.3490$ and $q = 0.3607$, while $k = 1$ gives $\lambda = \frac{1}{2}$ and $q = 1$. Tables III and IV show absolute values of relative errors, $|K(k) - K^*(k)|/K(k)$ and $|E(k) - E^*(k)|/E(k)$, respectively, where

$$K^*(k) = 2\pi \left(\frac{1}{2} + \sum_{n=1}^{N} (q^*)^{n^2} \right)^2 , \tag{21}$$

$$E^*(k) = K^*(k) + \frac{\pi^2}{K^*(k)} \frac{\sum_{n=1}^{N} (-1)^n n^2 (q^*)^{n^2}}{\frac{1}{2} + \sum_{n=1}^{N} (-1)^n (q^*)^{n^2}} , \tag{22}$$

$$q^* = \lambda \left(1 + \sum_{m=1}^{M} \alpha_m \lambda^{4m} \right) . \tag{23}$$

These tables give results of double-precision computations performed on a Cray 2 with $N = 6$ for $M = 1, 2, \cdots, 12$. Entries of ** indicate errors of less than 2×10^{-28}, which is approximately the limit of accuracy of the calculation. Errors obtained with $N = 7$ are identical (at least to three significant figures) to those in Tables III and IV; results based on $N = 5$ differ so little from those based on $N = 6$ that for actual applications for $0 \le k^2 \le 0.9999$, $N = 5$ is sufficient. Since the values of N and M required for a specified accuracy of computation depend on the value of k, an efficient program must treat N and M as functions of k. Such relationships are listed in Tables V and VI for machine precisions of 7.11×10^{-14}

$$q = \exp[-\pi K(k')/K(k)]$$
$$= \frac{k^2}{16} + 8\left(\frac{k^2}{16}\right)^2 + 84\left(\frac{k^2}{16}\right)^3 + 992\left(\frac{k^2}{16}\right)^4 + \cdots \ . \tag{13}$$

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$$K(k) = \frac{\pi}{2}\left[\theta_3(0,q)\right]^2$$
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The q-series of $\theta_3(0,q)$, $\theta_4(0,q)$, and $\theta_4''(0,q)$ converge extremely fast except near $q = 1$, since they contain powers only of the form q^{n^2}. Therefore, the main problem is to find q for a given k. The series given by Eq. (13) can be used, but it converges slowly unless $|k| \ll 1$. A more convenient method is to express q in terms of

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$$E^*(k) = K^*(k) + \frac{\pi^2}{K^*(k)} \frac{\sum_{n=1}^{N} (-1)^n n^2 (q^*)^{n^2}}{\frac{1}{2} + \sum_{n=1}^{N} (-1)^n (q^*)^{n^2}} , \tag{22}$$

$$q^* = \lambda \left(1 + \sum_{m=1}^{M} \alpha_m \lambda^{4m} \right) . \tag{23}$$

These tables give results of double-precision computations performed on a Cray 2 with $N = 6$ for $M = 1, 2, \cdots, 12$. Entries of ** indicate errors of less than 2×10^{-28}, which is approximately the limit of accuracy of the calculation. Errors obtained with $N = 7$ are identical (at least to three significant figures) to those in Tables III and IV; results based on $N = 5$ differ so little from those based on $N = 6$ that for actual applications for $0 \leq k^2 \leq 0.9999$, $N = 5$ is sufficient. Since the values of N and M required for a specified accuracy of computation depend on the value of k, an efficient program must treat N and M as functions of k. Such relationships are listed in Tables V and VI for machine precisions of 7.11×10^{-14}

Table I

Numerical Values of α_m

m	α_m
1	2
2	15
3	150
4	1,707
5	20,910
6	268,616
7	3,567,400
8	48,555,069
9	673,458,874
10	9,481,557,398
11	135,119,529,972
12	1,944,997,539,623

Table II

Numerical Values of λ, λ^4, q, q^9, q^{16}, q^{25}, q^{36}, $K(k)$, and $E(k)$

k^2	$\lvert k\rvert$	λ	λ^4	q	q^9	q^{16}	q^{25}	q^{36}	K	E
0.1000	0.31623	0.0066	1.88E-09	0.0066	2.33E-20	1.25E-35	2.91E-55	2.93E-79	1.6124	1.5308
0.2000	0.44721	0.0139	3.78E-08	0.0139	1.99E-17	2.04E-30	4.06E-47	1.57E-67	1.6596	1.4890
0.3000	0.54772	0.0223	2.46E-07	0.0223	1.35E-15	3.68E-27	4.97E-42	3.34E-60	1.7139	1.4454
0.4000	0.63246	0.0319	1.03E-06	0.0319	3.40E-14	1.14E-24	3.88E-38	1.34E-54	1.7775	1.3994
0.5000	0.70711	0.0432	3.49E-06	0.0432	5.26E-13	1.48E-22	7.77E-35	7.63E-50	1.8541	1.3506
0.6000	0.77460	0.0570	1.06E-05	0.0570	6.37E-12	1.25E-20	7.96E-32	1.65E-45	1.9496	1.2984
0.7000	0.83666	0.0747	3.11E-05	0.0747	7.23E-11	9.38E-19	6.79E-29	2.74E-41	2.0754	1.2417
0.8000	0.89443	0.0993	9.71E-05	0.0993	9.36E-10	8.90E-17	8.33E-26	7.69E-37	2.2572	1.1785
0.9000	0.94868	0.1401	3.85E-04	0.1402	2.09E-08	2.22E-14	4.64E-22	1.91E-31	2.5781	1.1048
0.9500	0.97468	0.1789	1.03E-03	0.1793	1.92E-07	1.14E-12	2.19E-19	1.35E-27	2.9083	1.0605
0.9800	0.98995	0.2267	2.64E-03	0.2279	1.66E-06	5.31E-11	8.82E-17	7.61E-24	3.3541	1.0286
0.9900	0.99499	0.2597	4.55E-03	0.2622	5.86E-06	4.99E-10	2.92E-15	1.18E-21	3.6956	1.0160
0.9950	0.99750	0.2899	7.07E-03	0.2943	1.65E-05	3.16E-09	5.23E-14	7.50E-20	4.0393	1.0089
0.9980	0.99900	0.3254	1.12E-02	0.3334	5.09E-05	2.33E-08	1.19E-12	6.74E-18	4.4953	1.0040
0.9990	0.99950	0.3490	1.48E-02	0.3607	1.03E-04	8.22E-08	8.51E-12	1.15E-16	4.8411	1.0022
0.9995	0.99975	0.3699	1.87E-02	0.3862	1.91E-04	2.45E-07	4.67E-11	1.33E-15	5.1873	1.0012
0.9998	0.99990	0.3937	2.40E-02	0.4172	3.83E-04	8.42E-07	3.22E-10	2.15E-14	5.6451	1.0005
0.9999	0.99995	0.4091	2.80E-02	0.4388	6.03E-04	1.89E-06	1.14E-09	1.33E-13	5.9916	1.0003

Table III

Absolute Values of the Relative Error of the Approximation Form $K^*(k)$ Given by Eqs. (21) and (23)

$k^2 \backslash M$	1	2	3	4	5	6	7	8	9	10	11	12
0.1000	1.38E-18	2.59E-26	***[a]	***	***	***	***	***	***	***	***	***
0.2000	1.16E-15	4.39E-22	2.43E-28	***	***	***	***	***	***	***	***	***
0.3000	7.76E-14	1.91E-19	5.36E-25	***	***	***	***	***	***	***	***	***
0.4000	1.92E-12	1.98E-17	2.33E-22	3.05E-27	***	***	***	***	***	***	***	***
0.5000	2.90E-11	1.01E-15	4.02E-20	1.72E-24	***	***	***	***	***	***	***	***
0.6000	3.43E-10	3.63E-14	4.37E-18	5.65E-22	7.69E-26	***	***	***	***	***	***	***
0.7000	3.78E-09	1.18E-12	4.17E-16	1.59E-19	6.35E-23	2.64E-26	***	***	***	***	***	***
0.8000	4.70E-08	4.56E-11	5.04E-14	5.99E-17	7.47E-20	9.63E-23	1.27E-25	2.24E-28	***	***	***	***
0.9000	9.86E-07	3.80E-09	1.66E-11	7.85E-14	3.88E-16	1.98E-18	1.04E-20	5.55E-23	3.01E-25	1.74E-27	***	***
0.9500	8.58E-06	8.81E-08	1.03E-09	1.29E-11	1.70E-13	2.32E-15	3.24E-17	4.61E-19	6.65E-21	9.73E-23	1.44E-24	2.14E-26
0.9800	6.99E-05	1.85E-06	5.59E-08	1.81E-09	6.15E-11	2.16E-12	7.77E-14	2.85E-15	1.06E-16	3.99E-18	1.52E-19	5.83E-21
0.9900	2.37E-04	1.08E-05	5.64E-07	3.15E-08	1.85E-09	1.12E-10	6.94E-12	4.39E-13	2.81E-14	1.83E-15	1.20E-16	7.92E-18
0.9950	6.43E-04	4.59E-05	3.72E-06	3.23E-07	2.95E-08	2.77E-09	2.67E-10	2.62E-11	2.61E-12	2.63E-13	2.68E-14	2.75E-15
0.9980	1.88E-03	2.15E-04	2.78E-05	3.84E-06	5.57E-07	8.33E-08	1.28E-08	1.99E-09	3.15E-10	5.05E-11	8.16E-12	1.33E-12
0.9990	3.68E-03	5.60E-04	9.60E-05	1.76E-05	3.39E-06	6.71E-07	1.36E-07	2.81E-08	5.90E-09	1.25E-09	2.68E-10	5.78E-11
0.9995	6.53E-03	1.26E-03	2.75E-04	6.39E-05	1.55E-05	3.89E-06	9.98E-07	2.61E-07	6.90E-08	1.85E-08	4.99E-09	1.36E-09
0.9998	1.24E-02	3.12E-03	8.77E-04	2.63E-04	8.24E-05	2.66E-05	8.77E-06	2.94E-06	1.00E-06	3.45E-07	1.20E-07	4.19E-08
0.9999	1.87E-02	5.54E-03	1.83E-03	6.43E-04	2.35E-04	8.88E-05	3.42E-05	1.34E-05	5.33E-06	2.14E-06	8.67E-07	3.54E-07

[a] Indicates error $< 2 \times 10^{-28}$.

Table IV

Absolute Values of the Relative Error of the Approximation Form $E^*(k)$ Given by Eqs. (21)–(23)

$k^2 \backslash M$	1	2	3	4	5	6	7	8	9	10	11	12
0.1000	1.34E-18	2.52E-26	***[a]	***	***	***	***	***	***	***	***	***
0.2000	1.10E-15	4.14E-22	***	***	***	***	***	***	***	***	***	***
0.3000	7.04E-14	1.73E-19	4.86E-25	***	***	***	***	***	***	***	***	***
0.4000	1.66E-12	1.72E-17	2.02E-22	2.56E-27	***	***	***	***	***	***	***	***
0.5000	2.37E-11	8.26E-16	3.28E-20	1.40E-24	***	***	***	***	***	***	***	***
0.6000	2.59E-10	2.74E-14	3.29E-18	4.26E-22	5.78E-26	***	***	***	***	***	***	***
0.7000	2.55E-09	7.94E-13	2.81E-16	1.07E-19	4.29E-23	1.77E-26	***	***	***	***	***	***
0.8000	2.67E-08	2.59E-11	2.87E-14	3.41E-17	4.25E-20	5.47E-23	7.22E-26	***	***	***	***	***
0.9000	4.00E-07	1.54E-09	6.75E-12	3.19E-14	1.58E-16	8.05E-19	4.22E-21	2.25E-23	1.22E-25	5.48E-28	2.28E-28	***
0.9500	2.37E-06	2.44E-08	2.85E-10	3.58E-12	4.72E-14	6.43E-16	8.97E-18	1.28E-19	1.84E-21	2.69E-23	3.97E-25	5.90E-27
0.9800	1.10E-05	2.92E-07	8.81E-09	2.86E-10	9.70E-12	3.41E-13	1.23E-14	4.50E-16	1.67E-17	6.30E-19	2.40E-20	9.19E-22
0.9900	2.36E-05	1.08E-06	5.61E-08	3.14E-09	1.84E-10	1.11E-11	6.91E-13	4.37E-14	2.80E-15	1.82E-16	1.19E-17	7.88E-19
0.9950	3.95E-05	2.82E-06	2.28E-07	1.98E-08	1.81E-09	1.70E-10	1.64E-11	1.61E-12	1.60E-13	1.62E-14	1.65E-15	1.69E-16
0.9980	5.96E-05	6.76E-06	8.72E-07	1.21E-07	1.75E-08	2.62E-09	4.01E-10	6.25E-11	9.90E-12	1.59E-12	2.56E-13	4.18E-14
0.9990	6.95E-05	1.04E-05	1.79E-06	3.28E-07	6.30E-08	1.25E-08	2.53E-09	5.23E-10	1.10E-10	2.32E-11	4.98E-12	1.07E-12
0.9995	7.31E-05	1.38E-05	2.99E-06	6.94E-07	1.69E-07	4.23E-08	1.08E-08	2.83E-09	7.49E-10	2.01E-10	5.42E-11	1.48E-11
0.9998	6.92E-05	1.66E-05	4.62E-06	1.38E-06	4.32E-07	1.39E-07	4.60E-08	1.54E-08	5.25E-09	1.81E-09	6.28E-10	2.20E-10
0.9999	6.18E-05	1.71E-05	5.53E-06	1.93E-06	7.05E-07	2.66E-07	1.02E-07	4.01E-08	1.59E-08	6.40E-09	2.59E-09	1.06E-09

[a]Indicates error $< 2 \times 10^{-28}$.

Table V

Maximum Values of k^2 for Which the Relative Errors of $K^*(k)$ and $E^*(k)$ Do Not Exceed the Machine Precision (7.11×10^{-15}) of Cray 1 and Cray 2 for Given Values of M and N

M	N	$k^2_{\max}$
1	2	0.2217
2	3	0.5533
3	3	0.7600
4	4	0.8698
5	4	0.9262
6	4	0.95606
7	4	0.97255
8	5	0.98214
9	5	0.98794
10	5	0.99160
11	5	0.993984
12	5	0.995586

Table VI

Maximum Values of k^2 for Which the Relative Errors of $K^*(k)$ and $E^*(k)$ Do Not Exceed the Machine Precision (1.19×10^{-7}) of VAX 8600 and VAX 8700 for Given Values of M and N

M	N	$k^2_{\max}$
1	2	0.8071
2	3	0.9538
3	3	0.98378
4	4	0.99315
5	4	0.99667
6	4	0.998207

(Cray 1 and Cray 2) and 1.19×10^{-7} (VAX 8600 and VAX 8700), respectively. The approximations (21)–(23) for $k^2 \leq k^2_{\max}$ with values of N and M given in the tables will yield the relative errors of both $K^*(k)$ and $E^*(k)$ within the machine precision.

Tables VII and VIII show CPU times (in seconds) required to compute 10^6 values of both $K(k)$ and $E(k)$, using FORTRAN programs of three approximation procedures: Chebyshev approximations of the Hastings form given by Eqs. (9) and (10) (method A), the standard Gauss arithmetic-geometric mean process (method B), and the θ function expansions given by Eqs. (21)–(23) (method C). The timing results given in Table VII were obtained by running the programs (in non-vectorized modes) on five Cray computers: Cray 1A (serial 6), Cray 1S (serial 33), Cray X-MP/22 (serial 119), Cray 2/64 (serial 2001), and Cray 2/128 (serial 2018). The table shows that the two compilers CFT77 and CIVIC give quite different CPU times. The results from the Cray 1A and the Cray 1S are practically the same and hence are listed under the single heading of Cray 1. The times given in Table VIII are results obtained from running the programs on a VAX 8600 and a VAX 8700.

The evaluation of the fourth root for the complementary parameter in Eq. (18) was carried out by taking square roots twice and consumed a considerable portion of the total computing time. The efficiency of method C, therefore, can be improved greatly if a better method of determining the fourth root becomes available. A slight further improvement is possible if the method of Chebyshev approximation is applied, as in the case of Eqs. (9) and (10), to the series of q/λ in powers of λ^4 in Eq. (20). A rather interesting conclusion that can be made from Tables VII and VIII concerning the relative speeds of the three approximation schemes is that on Cray 2's method C is most efficient and method A is least efficient, whereas on VAX 8600 and VAX 8700 method A is most efficient and method B is least efficient. The advantages of the new method based on the θ functions are (1) accurate and efficient evaluation of $D(k)$ for small $|k|$, (2) efficiency on Cray 2 computers, (3) portability, (4) potential room for improvement, and (5) relatively low additional cost for higher accuracies. The obvious major defect is that the small regions near $|k| = 1$ must be excluded.

Table VII

Cray CPU Times (sec) Required to Compute 10^6 Values of $K(k)$ and $E(k)$ Each for $0 \leq k^2 \leq 0.9955$

Approximation Method	Precision	CFT77 Cray 2/128	CFT77 Cray 2/64	CFT77 Cray X-MP/22	CFT77 Cray 1	CIVIC Cray 2/128	CIVIC Cray 2/64	CIVIC Cray X-MP/22	CIVIC Cray 1
Chebyshev; Eqs. (9) and (10), $N' = 8$	5.56×10^{-15}	5.25	5.49	3.24	5.18	6.96	9.42	6.55	8.71
Gauss arithmetic-geometric mean	7.11×10^{-15}	3.42	3.52	7.98	10.42	5.66	6.36	9.68	12.09
q-series; Eqs. (21)–(23), Table V	7.11×10^{-15}	2.86	2.90	5.23	7.33	3.89	4.28	5.92	8.10

Table VIII

VAX CPU Times (sec) Required to Compute 10^6 Values of $K(k)$ and $E(k)$ Each for $0 \leq k^2 \leq 0.9982$

Approximation Method	Precision	VAX 8600	VAX 8700
Chebyshev; Eqs. (9) and (10), $N' = 4$	1.57×10^{-8}	23.6	24.9
Gauss arithmetic-geometric mean	1.19×10^{-7}	56.7	43.8
q-series; Eqs. (21)–(23), Table VI	1.19×10^{-7}	40.8	32.3

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