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MASTER

REGULARITY OF SOLUTIONS TO AN INHOMOGENEOUS DIFFERENTIAL EQUATION IN BANACH SPACE

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1. INTRODUCTION

Let $T(t)$, $t \geq 0$, be a strongly continuous semigroup of bounded linear operators in the Banach space X with infinitesimal generator A and let f be a continuous X-valued function on $(0, \infty)$. It is well-known that without some restrictions on the semigroup $T(t)$, $t \geq 0$, or the continuous function f, the weak solution

$$v(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \qquad (1.1)$$

need not be a strong solution of the inhomogeneous linear differential equation

$$\begin{aligned} du(t)/dt &= Au(t) + f(t) \\ u(0) &= x. \end{aligned} \qquad (1.2)$$

R. Phillips [6] has shown that if $x \in D(A)$ and f is continuously differentiable, then the weak solution (1.1) is the unique strong solution of (1.2). The condition that f be continuously differentiable can be weakened by requiring further conditions on the strongly continuous semigroup $T(t)$, $t \geq 0$. For example, it is known [4] that (1.1) is a strong solution of (1.2) for every Hölder continuous function f if $T(t)$, $t \geq 0$, is holomorphic. Webb [8] has established that $T(t)\, X \subset D(A)$ for $t > 0$ is a sufficient condition for a weak solution to be a strong solution for every continuous function of bounded variation. Beirao Da Veiga has recently shown [1] that in a reflexive Banach space X, the condition $T(t)X \subset D(A)$ for $t > 0$ is not needed. That is, if $x \in D(A)$ and f is a continuous function of bounded variation, then the weak solution (1.1) is a strong solution of (1.2).

Our objective in this paper is to characterize the class of strongly continuous semigroups for which a weak solution of equation (1.2) is a strong solution when $f \in C([o,r];X)$. We demonstrate that (1.1) is a strong solution of (1.1) for every continuous function f if and only if the semigroup $T(t)$, $t \geqslant 0$, is of bounded semivariation.

2. OPERATORS OF BOUNDED SEMIVARIATION

Given a closed interval $[a,b]$ of the real line, a subdivision of $[a,b]$ is a finite sequence $d\colon \alpha = d_0 < d_1 < \ldots < d_n = b$. Let $D[a,b]$ denote the set of all subdivisions of $[a,b]$, and X and Y be Banach spaces. For $\alpha\colon [a,b] \to L(X;Y)$ and $d \in D[a,b]$, define

$$SV_d[\alpha] = \sup\left\{ \left\| \sum_{i=1}^{n} [\alpha(d_i) - \alpha(d_{i-1})]x_i \right\| \,\middle|\, x_i \in X,\ \|x_i\| \leqslant 1 \right\}.$$

and

$$SV[\alpha] = \sup\{SV_d[\alpha] \mid d \in D[a,b]\} .$$

We say α is of bounded semivariation (see [3]) if $SV[\alpha] < \infty$.

It follows from

$$SV_d[\alpha] = \sup\left\{ \left\| \sum_{i=1}^{n} [\alpha(d_i) - \alpha(d_{i-1})]x_i \right\| \,\middle|\, x_i \in X,\ \|x_i\| \leqslant 1 \right\}$$

$$\leqslant \sum_{i=1}^{n} \|\alpha(d_i) - \alpha(d_{i-1})\| ,$$

that the concept of bounded semivariation is an extension of the concept of bounded variation. For functions, the concepts of bounded semivariation and bounded variation are equivalent. In fact, if $f\colon [a,b] \to X \subset L(X';C)$ and $\langle \cdot,\cdot \rangle$ is the pairing between X and X', then

$$SV_d[f] = \sup\left\{ \left| \sum_{i=1}^{n} \langle f(d_i) - f(d_{i-1}), x_i \rangle \right| \,\middle|\, x_i \in X',\ \|x_i\| \leq 1 \right\}$$

$$= \sum_{i=1}^{n} \|f(d_i) - f(d_{i-1})\|$$

The following proposition is stated without proof.

Proposition 2.1 (Hönig)

If $\alpha\colon [a,b] \to L(X;Y)$ is of bounded semivariation and $f\colon [a,b] \to X$ is continuous, then the Riemann-Stieltjes integral $\int_a^b d\alpha(s)f(s)$ exists.

Lemma 2.1

If $\alpha\colon [o,r] \to L(X;Y)$ is of bounded semivariation, $\alpha(t)$ is strongly continuous at $t = 0$, and $f\colon[o,r] \to X$ is continuous, then

$$\lim_{t \to 0^+} \int_o^t d\,\alpha(s)\, f(s) = 0 \quad .$$

3. REGULARITY OF SOLUTIONS

When F is the class of Hölder continuous functions, Crandall and Pazy [2] demonstrate that

(i) $T(t)X \subset D(A)$ for $t > 0$

(ii) $\lim_{t \to o^+} t^p \|AT(t)\| < \infty$ for all $p > 1$,

are necessary and sufficient conditions on the semigroup T(t) to assure that the function (1.1) is a strong solution of the Cauchy problem (1.2) for every $f \in F$. Our principal result will be to establish bounded semivariation as a necessary and sufficient condition on the semigroup T(t), $t \geq 0$, to assure that (1.1) is a strong solution of (1.2) when F is the class of continuous functions. We first establish some necessary lemmas.

Lemma 3.1

If f is continuous and $T(\cdot)$ is of bounded semivariation on [o,r], then for $t \in [o,r]$,

$$\int_0^t T(t-s)f(s)ds \in D(A) \tag{3.1}$$

and

$$A \int_0^t T(t-s)f(s)ds = \int_0^t d_s T(t-s)f(s) \tag{3.2}$$

Proof

Under the assumptions on f and $T(\cdot)$, it follows from Proposition 2.1 that the Riemann-Stieltjes integral $\int_0^t d_s T(t-s)f(s)$ exists. For a fixed positive integer n, let $d_i^n = \frac{it}{n}$, where $i = 0, 1, \ldots, n$. Define $g_n(s) = T(t-s)f(d_i^n)$ for $d_{i-1}^n < s \leq d_i^n$ and $g_n(o) = T(t)f(o)$. Since the sequence $\{g_n\}$ is uniformly bounded and converges to $T(t-s)f(s)$ on $[o,t]$,

$$\lim_{n\to\infty} \int_0^t g_n(s)ds = \int_0^t T(t-s)f(s)ds \ . \tag{3.3}$$

Since $\int_0^t g_n(s)ds \in D(A)$ and

$$A \int_0^t g_n(s)ds = \sum_{i=1}^{n} [T(t-d_i^n) - T(t-d_{i-1}^n)]f(d_i^n)$$

$$\to \int_0^t d_s T(t-s)f(s) \ , \tag{3.4}$$

(3.1) and (3.2) follow from the closedness of A.

Lemma 3.2

If f is continuous and $T(\cdot)$ is of bounded semivariation on $[o,r]$, then $\int_0^t d_s T(t-s)f(s)$ is continuous in t on $[o,r]$.

Proof

Assume $0 \leqslant t < r$ and $0 < \Delta t$, and observe that

$$\int_0^{t+\Delta t} d_s T(t+\Delta t-s) f(s) - \int_0^t d_s T(t-s) f(s) = [T(\Delta t) - I] \int_0^t d_s T(t-s) f(s)$$

$$+ \int_0^{\Delta t} d_s T(t-s) f(t+\Delta t-s) .$$

Since it follows from Lemma 2.1 that

$$\lim_{\Delta t \to 0^+} \int_0^{\Delta t} d_s T(t-s) f(t+\Delta t-s) = 0,$$

continuity of $\int_0^t d_s T(t-s) f(s)$ from the right is a consequence of the fact that $\lim_{h \to 0^+} [T(h) - I]x = 0$ for all $x \varepsilon X$.

To show continuity from the left, let $0 < t \leqslant r$ and $0 < \Delta t$. Then

$$\int_0^t d_s T(t-s) f(s) - \int_0^{t-\Delta t} d_s T(t-\Delta t-s) f(s)$$

$$= \int_0^{t-\Delta t} d_s T(t-\Delta t-s)(T(\Delta t) - I) f(s) + \int_{t-\Delta t}^t d_s T(t-s) f(s)$$

$$= \int_0^{t-\Delta t} d_s T(s)(T(\Delta t) - I) f(t-\Delta t-s) + \int_0^{\Delta t} d_s T(s) f(t-s) .$$

Since

$$\left\| \int_0^{t-\Delta t} d_s T(s)(T(\Delta t) - I) f(t-\Delta t-s) \right\| \leq SV[T(\cdot)] \sup_{o \leq s \leq t-\Delta t} \| (T(\Delta t) - I) f(s) \|$$

and

$$\lim_{\Delta t \to o^+} \int_o^{\Delta t} d_s T(s) f(t-s) = 0 \quad ,$$

continuity from the left follows from the uniform continuity of f on [o,r] and the fact that $\lim_{h \to o^+} (T(h) - I)z = 0$ uniformly for z in a compact subset of X.

Proposition 3.1

The function (1.1) is a strong solution of the Cauchy problem (1.2) for every continuous function f if and only if $T(\cdot)$ is of bounded semivariation on [o,r].

Proof

It is easily shown (Pazy [5]) that the weak solution v defined by (1.1) is a strong solution of (1.2) if and only if $v(t) \in D(A)$ and $Av(t)$ is continuous on [o,r]. If f is continuous and $T(\cdot)$ of bounded semivariation on [o,r], then these conditions are satisfied by virtue of Lemmas 3.1 and 3.2.

To demonstrate the necessity of the assumption that $T(\cdot)$ be of bounded semivariation on [o,r], assume $SV[T(\cdot)] = \infty$ on [o,r] and define the bounded linear operator $L: C([o,r]; X) \to X$ by $L(f) = \int_0^r T(r-s) f(s) ds$.

Since every weak solution is a strong solution, $L(f) \in D(A)$ for every continuous f. It follows from the closedness of A and the boundedness of L that $AL: C([o,r]; X) \to X$ is bounded. Since $SV[T(\cdot)] = \infty$, there exists a sequence $d_n \in D$ such that $SV_{d_n}[T(\cdot)] > N$ and $\Delta d_n \leq \frac{1}{N}$. Define $f_{d_n}(t) = f(d_i^n)$ if $d_{i-1}^n < t \leq d_i^n$ and define

$$L_n(f) = \int_0^r T(r-s)f_{d_n}(s)ds \ .$$

Since $AL_n(f) \to AL(f)$ for fixed $f \in C([o,r]; X)$, it follows from the Banach-Steinhaus theorem that there exists M such that $\|AL_n\| \leq M$ for all $n \geq 1$. But

$$\|AL_n(f)\| = \|A\int_0^r T(r-s)f_{d_n}(s)ds\|$$

$$= \|\sum_{i=1}^{n} A\int_{d_{i-1}^n}^{d_i^n} T(r-s)f(d_i^n)ds\|$$

$$= \|\sum_{i=1}^{n} [T(r-d_i^n) - T(r-d_{i-1}^n)]f(d_i^n)\| \ .$$

Thus

$$\|AL_n\| = \sup\left\{\|\sum_{i=1}^{n} [T(r-d_i^n) - T(r-d_{i-1}^n)]f(d_i^n)\| \,\middle|\, f\in C([o,r]; X);\ \|f\| \leq 1\right\} \ .$$

The proof will be complete if we show that for fixed $d = \{s_i\} \in D$,

$$\sup\left\{\left\| \sum_{i=1}^{n} [T(r-s_i) - T(r-s_{i-1})]f(s_i)\right\| \,\middle|\, f\varepsilon C([o,r];\ X);\ \|f\| \leqslant 1\right\}$$

$$= \sup\left\{\left\| \sum_{i=1}^{n} [T(r-s_i) - T(r-s_{i-1})]x_i\right\| \,\middle|\, x_i\varepsilon X;\ \|x_i\| \leqslant 1\right\} = SV_d[T(\cdot)]\ .$$

That

$$\sup\left\{\left\| \sum_{i=1}^{n} [T(t-s_i) - T(t-s_{i-1})]f(s_i)\right\| \,\middle|\, f\varepsilon C([o,r];\ X),\ \|f\| \leqslant 1\right\}$$

$$\leqslant \sup\left\{\left\| \sum_{i=1}^{n} [T(t-s_i) - T(t-s_{i-1})]x_i\right\| \,\middle|\, x_i\varepsilon X,\ \|x_i\| \leqslant 1\right\}$$

follows easily. To see the converse, for fixed but arbitrary $x_i \varepsilon X$, $i = 1, \ldots, n$, $\|x_i\| \leqslant 1$, define $f(s) = (1-\lambda)x_{i-1} + \lambda x_i$ for $s = (1-\lambda)s_{i-1} + \lambda s_i$, $0 \leqslant \lambda \leqslant 1$. Then

$$\left\| \sum_{i=1}^{n} [T(t-s_i) - T(t-s_{i-1})]x_i\right\| = \left\| \sum_{i=1}^{n} [T(t-s_i) - T(t-s_{i-1})]f(s_i)\right\|$$

$$\leqslant \sup\left\{\left\| \sum_{i=1}^{n} [T(t-s_i) - T(t-s_{i-1})]f(s_i)\right\| \,\middle|\, f\varepsilon C([o,r];\ X),\ \|f\| \leqslant 1\right\}$$

The result now follows.

The characterization of strongly continuous semigroups of bounded semivariation remains an open question. However, we can establish some partial results in this direction.

Lemma 3.3

A necessary condition for a strongly continuous semigroup $T(\cdot)$ to be of bounded semivariation on $[o,r]$ is that $T(t)X \subset D(A)$ for $t > 0$.

Proof

It suffices to prove that $T(t)X \subset D(A)$ for $t \in (o,r)$ since $T(t_0)X \subset D(A)$ implies $T(t)X \subset D(A)$ for $t \geq t_0$. Suppose there exist $x_0 \in X$ and $t_0 \in (o,r)$ such that $T(t_0)x_0 \notin D(A)$ and consider the Cauchy problem $u'(t) = Au(t) + f(t)$, $u(o) = 0$, where $f(s) = T(s)x_0$. The unique weak solution of this equation given by $u(t) = tT(t)x_0$ is not a strong solution since it is not differentiable at $t = t_0$. This is in contraction to Proposition 3.1 since $f \in C([o,r]; X)$.

Lemma 3.4

If a strongly continuous semigroup is of bounded semivariation on $[o,r]$, $r > 0$, then it is of bounded semivariation on $[o,a]$ for any $a > 0$.

Proof

If $a > r$, then it is easily established that

$$SV_{[o,a]}[T(\cdot)] \leq SV_{[o,r]}[T(\cdot)] + SV_{[r,a]}[T(\cdot)] .$$

As a consequence of Lemma 3.3, $T(t)$, $t \geq 0$, is continuously differentiable in the uniformly operator topology for $t > 0$, and, consequently, since $r > 0$, there exists $M > 0$ such that $\|T'(s)\| \leq M$ for $s \in [r,a]$. Thus if $\{d_i\}$ is a subdivision of $[r,a]$ and

$\|x_i\| \leq 1$,

then

$$\left\| \sum_{i=1}^{n} [T(d_i) - T(d_{i-1})]x_i \right\| = \left\| \sum_{i=1}^{n} \int_{d_{i-1}}^{d_i} T'(s)x_i ds \right\| \leq M(a-r) .$$

It follows that $SV_{[r,a]}[T(\cdot)] < \infty$, and validity of the lemma is established.

We can thus define a strongly continuous semigroup to be of bounded semivariation if it is of bounded semivariation on some, and hence all, intervals of the form $[o,r]$, $r > 0$. Our next proposition characterizes strongly continuous groups of bounded semivariation.

Proposition 3.2

A strongly continuous group is of bounded semivariation if and only if its infinitesimal generator is bounded.

Proof

Let A denote the infinitesimal generator of a strongly continuous group $T(t)$, $t \varepsilon R$, and assume A is bounded. Then for arbitrary $d \varepsilon D[-r,r]$,

$$\left\| \sum_{i=1}^{n} [T(s_i) - T(s_{i-1})]x_i \right\| = \left\| \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} T'(s)x_i ds \right\|$$

$$= \left\| \sum_{i=1}^{n} A \int_{s_{i-1}}^{s_i} T(s)x_i ds \right\| \leq \|A\| \; M \int_{-r}^{r} e^{\omega s} ds ,$$

implying that $T(\cdot)$ is of bounded semivariation on $[-r,r]$.

To prove the converse, suppose $T(t)$, $t \in R$, is of bounded semivariation. By Lemma 3.3, $T(t)X \subset D(A)$ for $t > 0$. Thus for $x \in X$ and $t_0 \neq 0$, $x = T(o)x = T(t_0)T(-t_0)x \in D(A)$, implying that $D(A) = X$. The boundedness of A now follows from the closed graph theorem.

Corollary 3.1

Suppose the operator A in equation (1.1) is the infinitesimal generator of a strongly continuous group. Then a weak solution of (1.1) is a strong solution for every continuous function f if and only if A is bounded.

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