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AN EFFICIENT EXACT ALGORITHM FOR THE "LEAST-SQUARES" IMAGE REGISTRATION PROBLEM

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Abstract

Image registration involves estimating how one set of n-dimensional points is rotated, scaled, and translated into a second set of n-dimensional points. In practice, n is usually 2 or 3. We give an exact algorithm to solve the "least-squares" formulation of the two-dimensional registration problem. The algorithm, which is based on parametric linear programming, can be viewed as a refinement of the $O(k^3)$ approximation method proposed by Zikan and Silberberg [13]. The approach can be extended to handle registration of images of different cardinalities.

1. Background of the Problem

The basic image registration problem is stated in ZIKAN and SILBERBERG [13]: *"Assume that a collection of n-dimensional points undergoes an affine transformation made up of a rotation, a scale change, and a translation; moreover, assume that noise (random and/or systematic) is added to each transformed point. Knowing the positions of the original and transformed points, but not their identities, can the transformation, the noise, and the point-to-point matching be recovered?"*

The *image registration problem* is fundamental in robotics, as the ability to "register" images is a necessary prerequisite for "on-board" automated visual reasoning of mobile agents. Many research papers have been written on the subject. Henry S. Baird [4] (*ACM Distinguished Thesis Series, 1984*) outlines some of the older, mostly heuristic approaches. Many of the methods employ the so-called *pruned tree* approach; see e.g., GENNERY [6] and WONG and SALAY [12], where the *tree of partial matchings* is searched for the desired solution. All permutations of the image points are connected into a tree via the partial matches. The full matches (permutations) form the *leaves* of the tree. The hope is that, although the tree contains $k!$ leaves, most of the branches can be pruned (removed from further consideration) early. The elegant approach of Baird also employs the tree-pruning technique and claims to be computationally superior to the other methods. The inherent problems associated with the tree-pruning methods are explained in [13].

Image registration is often formulated as a "least-squares" problem. Partial results toward the solution of the general problem have been discovered and rediscovered several times. Perhaps the oldest paper on the subject is GREEN [8], 1952. Long before various computer vision problems became the most pressing open problems of the robotics-computer science of today, B. Green gave a solution to the *optimal rotation subproblem*. Green's result was later improved by SHONEMANN [11]. Both papers appeared in *Psychometrica* as the research focused on *factor analysis* rather then on *image registration*. Clearly unaware of the classical results, FAUGERAS and HEBERT [5] (in a more general work) gave solutions to the two- and three-dimensional rotation subproblems by a method different from that of Green and Shonemann. Still later, ARUN, HUANG, and BLOSTEIN [3] rediscovered the original method in the image registration setting. The latter two papers also give partial results on the *optimal translation subproblem*. All these results implicitly or explicitly assume that the one-to-one matching of points is either fixed or known. These and other aspects of the general least-squares problem are treated in ZIKAN and SILBERBERG [13]. An $O(k^3)$ approximation solution method to the general problem is also there given.

For other formulations of the image registration problem, see BAIRD [4], ALT, MEHLHORN, WAGENER, and WELZL [1], and ARKIN, MITCHELL, and ZIKAN [2]. A strongly polynomial algorithm which solves the Baird's formulation can be (essentially) found in [1]. In [2] a strongly polynomial algorithm to a more general problem is given. Most natural formulations of the image registration problem can be

at least approximately cast within this framework; the "least-squares" and Baird formulations are no exceptions. If we can extrapolate from the experience of other branches of applied mathematics, however, then we can see that a good specialized algorithm for the "least-squares" registration formulation is likely to be computationally superior to other serious image registration approaches. We utilize *parametric linear programming* to provide such an algorithm here.

The following notions (and the associated notation) from complex plane geometry, mathematical programming, and metric space theory provide a convenient formal framework for the two-dimensional image registration problem.

If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a two-dimensional vector, then let $\mathbf{x} = x_1 + ix_2 = [||\mathbf{x}||, \theta_x]$ be the corresponding complex number, where $||\mathbf{x}|| = ||x||_2$ is the magnitude (two-norm), θ_x is the argument, and $[||\mathbf{x}||, \theta_x]$ is the polar representation. Let $\overline{\mathbf{x}}$ denote the complex conjugate of $\mathbf{x}$ and recall that in polar coordinates $\mathbf{xy} = [||\mathbf{x}|| \cdot ||\mathbf{y}||, \theta_x + \theta_y]$ and $\mathbf{x}\overline{\mathbf{y}} = [||\mathbf{x}|| \cdot ||\mathbf{y}||, \theta_x - \theta_y)]$. The scalar product of $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} \circ \mathbf{y} = x_1 y_1 + x_2 y_2 = ||\mathbf{x}|| ||\mathbf{y}|| \cos(\theta_x - \theta_y)$, the "cross" product is $\mathbf{x} \times \mathbf{y} = ||\mathbf{x}|| ||\mathbf{y}|| \sin(\theta_x - \theta_y)$. Recall that $\mathbf{x}\overline{\mathbf{y}} = \mathbf{x} \circ \mathbf{y} + i\mathbf{x} \times \mathbf{y}$ and that $x^T y = \mathbf{x} \circ \mathbf{y}$. The scalar product is also called the "inner" or the "dot" product. To prevent possible ambiguity in mathematical formulas, let the notation $\langle \cdot, \cdot \rangle$ denote the inner product. Therefore $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \circ \mathbf{y}$ for complex numbers, $\langle x, y \rangle = x^T y$ for real k-dimensional vectors, and if A and B are real $n \times k$ matrices, then let $\langle A, B \rangle = \text{trace}(AB^T)$.

Assume that $\mathbf{A}$ and $\mathbf{B}$ are k-dimensional vectors of *complex* numbers and define

$$(\mathbf{A}, \mathbf{B}) = \sum_{j=1}^{k} \mathbf{a}_j \overline{\mathbf{b}}_j, \tag{1-1}$$

where $\mathbf{a}_j$ and $\mathbf{b}_j$ are the j-th components of the vectors $\mathbf{A}$ and $\mathbf{B}$, respectively. This is the *complex inner product*.

Consider the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Counterclockwise rotation of two-dimensional vectors by the angle θ can be represented by multiplication of the corresponding complex numbers by $e^{i\theta}$. If $\mathbf{r} = [s_r, \theta_r]$, then the multiplication by $\mathbf{r}$ corresponds to the counterclockwise rotation by θ_r and scaling by $s_r \geq 0$. Let α_i denote the argument of the complex number $\mathbf{a}_i$ and β_i be the argument of $\mathbf{b}_i$. Define

$$\begin{aligned} d_{ij}(\mathbf{r}) &= ||\mathbf{a}_i - \mathbf{r}\mathbf{b}_j||_2^2 = ||\mathbf{a}_i||^2 + ||\mathbf{r}||^2||\mathbf{b}_j||^2 - 2\langle\mathbf{a}_i, \mathbf{r}\mathbf{b}_j\rangle \\ &= ||\mathbf{a}_i||^2 + s_r^2||\mathbf{b}_j||^2 - 2s_r||\mathbf{a}_i|| ||\mathbf{b}_j|| \cos(\alpha_i - \beta_j - \theta_r) \end{aligned} \tag{1-2}$$

for all k^2 distinct $\{i, j\}$ pairs.

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Consider the parametric linear programs

$$\min_{r \in C^1} \min_{x} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{ij}(\mathbf{r}) \cdot x_{ij}$$

$$\text{subject to } \sum_{i=1}^{k} x_{ij} = 1, \text{ for all } j = 1, 2, \ldots, k$$

$$\sum_{j=1}^{k} x_{ij} = 1, \text{ for all } i = 1, 2, \ldots, k$$

$$x_{ij} \geq 0, \quad \text{for all } i, j = 1, 2, \ldots, k, \quad (1-3)$$

and

$$\min_{\|r\|=1} \min_{x} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{ij}(\mathbf{r}) \cdot x_{ij}$$

$$\text{subject to } \sum_{i=1}^{k} x_{ij} = 1, \text{ for all } j = 1, 2, \ldots, k$$

$$\sum_{j=1}^{k} x_{ij} = 1, \text{ for all } i = 1, 2, \ldots, k$$

$$x_{ij} \geq 0, \quad \text{for all } i, j = 1, 2, \ldots, k. \quad (1-4)$$

These problems arise naturally in the context of "least-squares" image registration, ZIKAN and SILBERBERG [13]. If $(\mathbf{r}^*, x^*)$ is an optimal solution of (1-3), then point-to-point matching can be recovered from x^*, and rotation and scaling from $\mathbf{r}^*$

It is natural to view an n-dimensional image consisting of k points as a $k \times n$ matrix with rows corresponding to the appropriate image points. If the image is two-dimensional, then it is also convenient to associate with the image a k-dimensional complex vector $\mathbf{A}$ where each coordinate a_i corresponds to one image point. In this paper we choose the latter formalism. The association of a vector (matrix) to an image is not unique; it depends on the ordering of the points. In general, consider the complex k-dimensional vector space C^k. Assume that r and t are complex numbers. The image associated with the vector $\mathbf{A} \in C^k$ can be translated by $\mathbf{A} \oplus \mathbf{t} = \{a_1 + t, a_2 + t, \ldots, a_k + t\}$ and rotated and scaled by $r\mathbf{A}$. Affine transformation of image $\mathbf{A}$ can then be written as:

$$(\mathbf{r}, \mathbf{t})(\mathbf{A}) = \mathbf{r}(\mathbf{A} \oplus \mathbf{t}). \quad (1-5)$$

A relabeling of image points by permutation π acts on the image by $\pi(\mathbf{A}) = P_\pi \mathbf{A}$, where P_π is the permutation matrix corresponding to π. Two images are to be considered as being *equivalent* if and only if they differ by a registration and relabeling only,

$$\mathbf{A} \sim \mathbf{B} \longleftrightarrow \mathbf{A} = (\mathbf{r}, \pi, \mathbf{t})\mathbf{B} = \mathbf{r}P_\pi(\mathbf{B} \oplus \mathbf{t}). \quad (1-6)$$

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Recall that $\|\mathbf{A}\| = (\mathbf{A}, \mathbf{A})^{\frac{1}{2}}$ is the euclidean (Frobenius) norm of the vector $\mathbf{A}$.† Assume that vector $\mathbf{A}$ corresponds to an image which has been translated rotated and scaled. Assume also that noise has been added to each point. Denote the original image by $\mathbf{B}$. If we predict the unknown transformation from the optimal solution of the problem

$$\min_{(\mathbf{r}, \pi, \mathbf{t})} \|\mathbf{A} - (\mathbf{r}, \pi, \mathbf{t})\mathbf{B}\|, \qquad (1-7)$$

then we have the *"least-squares"* estimate, in spirit analogous to least-squares estimates in other branches of science. The matching and transformation give the least sum of the squared distances between the matched points. In [13] we show that (1-7) essentially reduces to (1-3) or (1-4). Thus these parametric programs are important from the practical standpoint.

2. On Frobenius Norm Image Registration

Many practical aspects of using the Frobenius norm based criteria for n-dimensional image registration were addressed in ZIKAN and SILBERBERG [13]. Most importantly, the issue of missing and spurious points was discussed.‡ In this paper we restrict our discussion to the case where no missing or spurious points occur. The extension of our results to the unequal cardinality case would be done analogously to the development in [13].

On the theoretical side, it is shown in [13] that the optimal translation, independent of rotation, permutation or scaling, always translates the center of mass (centroid) of $\mathbf{B}$ to the center of mass of $\mathbf{A}$. Let A and B be $n \times k$ real or complex matrices, and α and β be the respective centers of mass (centroids) of their row vectors. (For us, $n = 1$ and the coordinates are viewed as image points.)

Theorem 1. *If A and B are $n \times k$ matrices over the real or complex field V, then $\alpha - \beta$ solves the translation problem*

$$\min_{t \in V^n} \|A - (B \oplus t)\|_F. \qquad (2-1)$$

Proof: A simple proof from first principles can be found in [13, Section 3.1]. □

The centroids of the images can be superimposed and conveniently identified with the origin of the cartesian coordinate system.

It is also argued in [13] that the scaling parameter can be, and perhaps should be, estimated before one begins the computation of the optimal rotation and matching. This result is enhanced later in

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† In the case of n-dimensional images, where the $k \times n$ real matrices take the place of vector $\mathbf{A}$, we use the *Frobenius matrix norm* $\|M\|_F = (\text{trace}(MM^T))^{\frac{1}{2}} = \langle M, M \rangle^{\frac{1}{2}}$. The Frobenius metric is the same as the euclidean norm when M is viewed as a kn-dimensional vector. If M is a complex matrix, then $\|M\|_F^2 = \text{trace}(MM^*)$, where $M^* = \overline{M}^T$.

‡ The presence of (missing and) spurious points leads to the problem of registering images of different cardinalities.

this paper, when it is shown that positive scaling has no effect on optimal rotation and matching. If the matching is fixed, then the optimal rotation and scaling of a two-dimensional registration problem can be recovered with the help of the following theorem.

Theorem 2. *Let* **A** *and* **B** *be k-dimensional complex vectors, such that* $(\mathbf{A}, \mathbf{B}) \neq 0$. *If*

$$e^{i\theta} = \frac{(\mathbf{A}, \mathbf{B})}{\|(\mathbf{A}, \mathbf{B})\|} \qquad (2-2)$$

and

$$s = \frac{\|(\mathbf{A}, \mathbf{B})\|}{\|\mathbf{B}\|^2}, \qquad (2-3)$$

then $e^{i\theta}$ *solves the problem* $\min_{e^{i\theta}} \|\mathbf{A} - e^{i\theta}\mathbf{B}\|$, *and*

$$\mathbf{r} = se^{i\theta} = \frac{(\mathbf{A}, \mathbf{B})}{\|\mathbf{B}\|^2} \qquad (2-4)$$

solves $\min_{\mathbf{r}} \|\mathbf{A} - \mathbf{r}\mathbf{B}\|$.

Proof: For a proof of the theorem consult [13, Section 4.1 and Appendix 1]. □

Analogous results for rotation and scaling in higher dimensions can be obtained, [13, Section 4]. The three-dimensional problem can be solved with the help of the "algebra" of quaternions, FAUGERAS and HEBERT [5], and the general n-dimensional problem with the help of the singular value decomposition, GREEN [8], and SHONEMANN [11]. The optimal rotation problem is known in literature as the *orthogonal Procrustes problem*, GOLUB and VAN LOAN [7], after the villainous son of Poseidon.*

The case that is not explicitly covered by the theorem, i.e. $(\mathbf{A}, \mathbf{B}) = 0$, may be treated by one of the standard lexicographical methods, ZIKAN and SILBERBERG [13].

In [13] one finds an approximation scheme, based on parametric linear programming, which solves (1-4) (and consequently (1-3)) within a specified error in $O(k^3)$ worst-case computational complexity. The availability of "hot start" at each step of the parametric "sweep" makes the approach practically attractive. In this paper we enhance the parametric algorithm. The new algorithm enables us to solve (1-3) and (1-4) exactly and (in fact) faster than by the original approximation method. In the process of developing the algorithm we also enrich the theory of the least-squares formulation of the image registration problem.

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* In Greek mythology, Procrustes forced travelers to fit into his bed by stretching their bodies or cutting off their legs.

3. Bilinear Functionals and the Parametric Linear Programming Problems

Define

$$X = \left\{ 0 \le x \in R^{k^2} : \sum_{j=1}^{k} x_{ij} = 1, \text{ and } \sum_{i=1}^{k} x_{ij} = 1, \text{for all } i,j = 1,2,\ldots,k \right\}. \qquad (3-1)$$

Recall the classical result due to Birkhoff (1946) that the vertices of X correspond to permutations of k elements. The vector x is a vertex of X if and only if there exists a permutation π such that $x_{ij} = 1$ whenever $i = \pi(j)$, and $x_{ij} = 0$ otherwise. The bases of X correspond to the spanning trees of the (complete) bipartite graph $\mathcal{B}_{kk}$.

Recall the definition (1-2) of $d_{ij}(\mathbf{r})$. Assume a lexicographic ordering of the $\{i,j\}$ pairs and let $d(\mathbf{r}) = (d_{ij}(\mathbf{r}))$ denote the resulting real k^2-dimensional vector. Note, that (1-3) and (1-4) can now be written as

$$\min_{\mathbf{r}} \min_{x \in X} \langle d(\mathbf{r}), x \rangle, \qquad (3-2)$$

and

$$\min_{\|\mathbf{r}\|=1} \min_{x \in X} \langle d(\mathbf{r}), x \rangle. \qquad (3-3)$$

Define the natural functionals,

$$F(\mathbf{r}, x) = \langle d(\mathbf{r}), x \rangle \qquad (3-4)$$

and

$$F(\mathbf{r}) = \min_{x \in X} F(\mathbf{r}, x), \qquad (3-5)$$

associated with (3-2) and (3-3). In addition to X, it is convenient to introduce analogues of $d(\mathbf{r})$, $F(\mathbf{r}, x)$, and $F(\mathbf{r})$. Thus, let us define the auxiliary functions

$$\delta_{ij}(\mathbf{r}) = -\langle \mathbf{a_i}, \mathbf{rb_j} \rangle, \qquad (3-6)$$

$$G(\mathbf{r}, x) = \langle \delta(\mathbf{r}), x \rangle, \qquad (3-7)$$

and

$$G(\mathbf{r}) = \min_{x \in X} G(\mathbf{r}, x). \qquad (3-8)$$

Note that $G(\cdot, \cdot)$ is a *bilinear functional*. Also note that

$$d_{ij}(\mathbf{r}) = \|\mathbf{a_i}\|^2 + \|\mathbf{r}\|^2 \|\mathbf{b_j}\|^2 + 2\delta_{ij}(\mathbf{r}), \qquad (3-9)$$

and

$$F(\mathbf{r}, x) = \|A\|^2 + \|\mathbf{r}\|^2 \|B\|^2 + 2G(\mathbf{r}, x). \qquad (3-10)$$

Lemma 1. *Functions $F(\mathbf{r})$ and $G(\mathbf{r})$ are related by*

$$F(\mathbf{r}) = \|A\|^2 + \|\mathbf{r}\|^2 \|B\|^2 + 2G(\mathbf{r}). \qquad (3-11)$$

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Proof: Equation (3-11) results from these identities:

$$F(\mathbf{r}) = \min_{x \in X} F(\mathbf{r}, x) = \min_{x \in X} \langle d(\mathbf{r}), x \rangle$$

$$= \min_{x \in X} \sum_{i,j=1}^{k} \left\{ ||\mathbf{a_i}||^2 + ||\mathbf{r}||^2 ||\mathbf{b_j}||^2 + 2\delta_{ij}(\mathbf{r}) \right\} x_{ij}$$

$$= \min_{x \in X} \sum_{i=1}^{k} \sum_{j=1}^{k} ||\mathbf{a_i}||^2 x_{ij} + \min_{x \in X} \sum_{j=1}^{k} \sum_{i=1}^{k} ||\mathbf{r}||^2 ||\mathbf{b_j}||^2 x_{ij} + 2 \min_{x \in X} \sum_{i,j=1}^{k} \delta_{ij}(\mathbf{r}) x_{ij} \qquad (3-12)$$

$$= \min_{x \in X} \sum_{i=1}^{k} ||\mathbf{a_i}||^2 \sum_{j=1}^{k} x_{ij} + \min_{x \in X} \sum_{j=1}^{k} ||\mathbf{r}||^2 ||\mathbf{b_j}||^2 \sum_{i=1}^{k} x_{ij} + 2 \min_{x \in X} \langle \delta(\mathbf{r}), x \rangle$$

$$= ||\mathbf{A}||^2 + ||\mathbf{r}||^2 ||\mathbf{B}||^2 + 2G(\mathbf{r}),$$

where the first three identities follow from (3-5), (3-4), and (3-9) respectively, and the last relation holds since $\sum_{i=1}^{k} x_{ij} = \sum_{j=1}^{k} x_{ij} = 1$, (3-1). Simple algebraic manipulations complete the proof. □

Theorem 2. *The complex number* $se^{i\theta^*}$ *solves* $\min_{||\mathbf{r}||=s} G(\mathbf{r})$ *if and only if it solves* $\min_{||\mathbf{r}||=s} F(\mathbf{r})$.

Proof: If the norm of $\mathbf{r}$ is fixed, then by Lemma 1, $F(\mathbf{r})$ and $2G(\mathbf{r})$ differ by a constant only. □

Above all, we are interested in the case $s = 1$. Theorem 2 implies that if the scaling factor is known, then the least-squares problem associated with the penalty function d is equivalent to the "maximal-projection" problem associated with the penalty function δ.

Lemma 3. *Assume that* $s \geq 0$ *is a nonnegative scalar. If* $x^* \in X$ *solves the linear program (3-8) associated with* $G(\mathbf{r})$, *then it solves the linear program associated with* $G(s\mathbf{r})$. *The optimal solutions are related by*

$$G(s\mathbf{r}) = sG(\mathbf{r}). \qquad (3-13)$$

Proof: Since $G(\mathbf{r}, x)$ is a bilinear functional,

$$\langle \delta(s\mathbf{r}), x \rangle = \langle s\delta(\mathbf{r}), x \rangle = s \langle \delta(\mathbf{r}), x \rangle.$$

If $s = 0$, then the result is immediate. If $s > 0$, then the lemma readily follows because positive scaling of the objective function of a linear program does not change its optimal solution set. □

Theorem 4. *Assume that $x^* \in X$ is a vertex solution to*

$$\min_{\|\mathbf{r}\|=s} \min_{x \in X} G(\mathbf{r}, x) \tag{3-14}$$

for some positive s. Assume that π^* is the permutation associated with x^*, and let

$$e^{i\theta^*} = \frac{\sum_{j=1}^k a_j \overline{b}_{\pi^*(j)}}{\|\sum_{j=1}^k a_j \overline{b}_{\pi^*(j)}\|}, \tag{3-15}$$

$$s^* = \frac{\|\sum_{j=1}^k a_j \overline{b}_{\pi^*(j)}\|}{\|\mathbf{B}\|^2}. \tag{3-16}$$

and

$$\mathbf{r}^* = s^* e^{i\theta^*} = \frac{\sum_{j=1}^k a_j \overline{b}_{\pi^*(j)}}{\|\mathbf{B}\|^2}. \tag{3-17}$$

Then $\{e^{i\theta^*}, x^*\}$ solves the problems

(a) $\quad \min_{\|\mathbf{r}\|=1} \min_{x \in X} G(\mathbf{r}, x),$

and

(b) $\quad \min_{\|\mathbf{r}\|=1} \min_{x \in X} F(\mathbf{r}, x), \tag{3-18}$

while $\{\mathbf{r}^*, x^*\}$ solves the problem

(c) $\quad \min_{\mathbf{r}} \min_{x \in X} F(\mathbf{r}, x).$

Proof: We assume that there exists $e^{i\theta'}$ such that

$$G(e^{i\theta'}, \pi^*) \leq G(e^{i\theta}, \pi) \tag{3-19}$$

for all $e^{i\theta}$ and π. Consequently, by Theorem 3.2,

$$F(e^{i\theta'}, \pi^*) \leq F(e^{i\theta}, \pi) \tag{3-20}$$

for all $e^{i\theta}$ and π. But we know from Theorem 2.2 that

$$F(e^{i\theta^*}, \pi^*) \leq F(e^{i\theta}, \pi^*), \tag{3-21}$$

for all $e^{i\theta}$. It follows from (3-20) and (3-21) that

$$F(e^{i\theta^*}, \pi^*) \leq F(e^{i\theta'}, \pi^*) \leq F(e^{i\theta}, \pi) \tag{3-22}$$

for all $e^{i\theta}$ and π. This establishes that $\{e^{i\theta^*}, x^*\}$ solves the problem (b). Invoke the Theorem 3.2 (again) and from (3-22) obtain

$$G(e^{i\theta^*}, \pi^*) \leq G(e^{i\theta}, \pi) \tag{3-23}$$

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for all $e^{i\theta}$ and π, which establishes that $\{e^{i\theta^*}, x^*\}$ solves the problem (a). Finally, for all nonnegative $s \geq 0$ we have

$$G(se^{i\theta^*}, \pi^*) = sG(e^{i\theta^*}, \pi^*) \leq sG(e^{i\theta}, \pi) = G(se^{i\theta}, \pi) \qquad (3-24)$$

by Lemma 3.3. Invoke Theorems 2.2 and 3.2 together and obtain

$$F(s^*e^{i\theta^*}, \pi^*) \leq F(se^{i\theta^*}, \pi^*) \leq F(se^{i\theta}, \pi). \qquad (3-25)$$

Since s is arbitrary, we have completed the proof that $\{r^*, x^*\}$ solves the problem (c). □

Let *least-squares registration problems* be a generic name for the three problems of (3-18). We have established a strong relation between the functionals $G(\cdot, \cdot)$ and $F(\cdot, \cdot)$. The bilinear structure of $G(\cdot, \cdot)$ provides us with another special property of the optimal solution sets. After we establish this last nontrivial theoretical property, we can give the promised algorithm. For a fixed G, define the corresponding complex plane equivalence relation by

$$r^{(1)} \sim_G r^{(2)} \text{ if and only if}$$

$$G(r^{(1)}, x^*) \leq G(r^{(1)}, x) \text{ for all } x \in X \iff G(r^{(2)}, x^*) \leq G(r^{(2)}, x) \text{ for all } x \in X. \qquad (3-26)$$

Theorem 5. *Relation "$\sim_G$" induces a convex conical subdivision of the complex plane.*

Proof: Assume, that (3-26) holds for some pair $r^{(1)}$ and $r^{(2)}$. Then by Lemma 3.3

$$G(sr^{(1)}, x^*) = sG(r^{(1)}, x^*) \leq sG(r^{(1)}, x) = G(sr^{(1)}, x) \qquad (3-27)$$

for all $s \geq 0$, and

$$\begin{aligned}
G(\lambda r^{(1)}, x^*) + G((1-\lambda)r^{(2)}, x^*) &= \\
\lambda G(r^{(1)}, x^*) + (1-\lambda)G(r^{(2)}, x^*) &\leq \lambda G(r^{(1)}, x) + (1-\lambda)G(r^{(2)}, x) \\
&= G(\lambda r^{(1)}, x) + G((1-\lambda)r^{(2)}, x),
\end{aligned} \qquad (3-28)$$

for all $0 \leq \lambda \leq 1$. □

The subdivision induced by F is, of course, identical to the one induced by G and consists of a finite number of cells. Thus the subdivision is the union of closures of its "two-dimensional" cells. (If we view C as a two-dimensional real plane.) For each two-dimensional cell, σ, we arbitrarily choose a *representative* $x(\sigma)$ from the vertices of the associated solution set of $\min_{x \in X} G(r_\sigma, x)$, where r_σ is any element of σ. Let $\pi(\sigma)$ be the permutation corresponding to $x(\sigma)$.

Proposition 6. *The set of representatives contains at least one optimal matching associated with the least-squares registration problems (3-18).*

Proof: The result follows from the fact that the relation $G(\mathbf{r}, x^*) \leq G(\mathbf{r}, x)$ for all $\mathbf{r} \in \sigma$ extends by continuity to the boundary of σ. □

Figure 3.1 exhibits a finite conical subdivision of the plane.
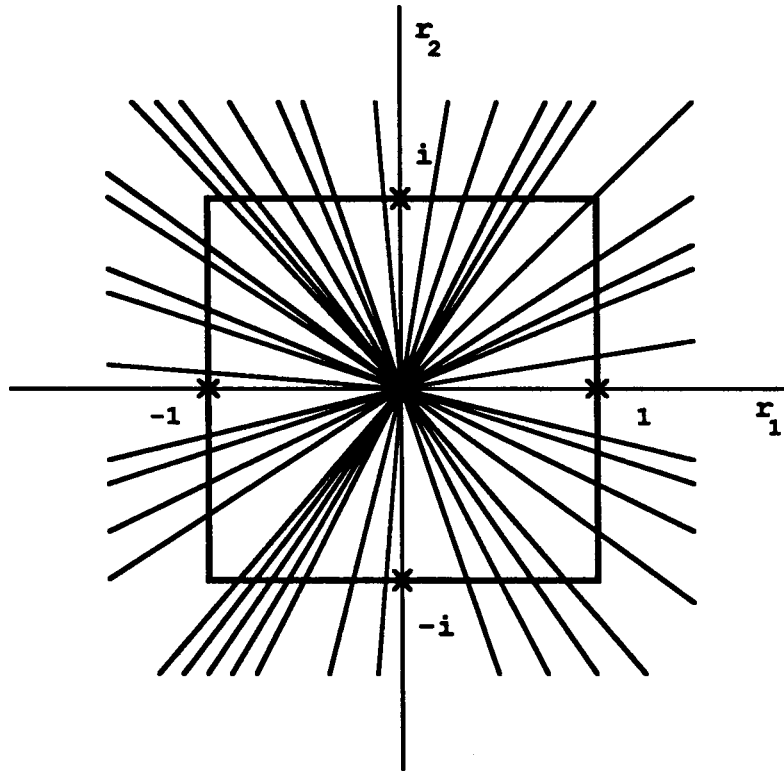

Figure 3.1 *A conical subdivision of C and a piecewise-linear path closed around origin*

Note, that any closed curve around the origin intersects all two-dimensional regions of the subdivision. In particular, this is true of all *piecewise-linear* curves around the origin.

4. The Algorithm

The algorithm to solve the least-squares registration problems (3-18) has four basic steps:

ALGORITHM

STEP 1: *Choose a closed piecewise-linear curve in the plane (γ), which contains origin in its interior.*

STEP 2: *Choose a starting point* $\mathbf{r}^0 \in \gamma$ *and solve the assignment problem associated with* $G(\mathbf{r}^0)$.

STEP 3: *Parametrically compute the optimal solutions of the assignment problems associated with* $G(\gamma(t))$ *and collect all distinct locally optimal permutations.*

STEP 4: *Use Theorem 2.2 to find the optimal rotation and scaling corresponding to each permutation obtained in Step 3. The best overall solution solves the registration problems (3-18).*

Let us briefly remark on each individual step.

Step 1. The unit square of Figure 3.1 is a convenient choice for γ. Note, that

$$\delta_{ij}(1) = \mathbf{a_i} \circ \overline{\mathbf{b}}_{\mathbf{j}} \qquad (4-1)$$

and that

$$\delta_{ij}(\mathbf{i}) = \mathbf{a_i} \times \overline{\mathbf{b}}_{\mathbf{j}}, \qquad (4-2)$$

the "dot" and "cross" products of $\mathbf{a_i}$ and $\overline{\mathbf{b}}_{\mathbf{j}}$ respectively. If $\mathbf{r} = r_1 + ir_2$, then

$$\delta_{ij}(\mathbf{r}) = r_1 \delta_{ij}(1) + r_2 \delta_{ij}(\mathbf{i}). \qquad (4-3)$$

Consequently, the costs on the unit square of Figure 3.1 are easy to construct.

Step 2. It is convenient to choose a cornerpoint of the unit square as a starting point, for instance, $\mathbf{r}^0 = 1 + \mathbf{i}$. The Hungarian method can be used to solve $G(1 + \mathbf{i})$ in $O(k^3)$ worst-case "time".

Step 3. The parametric version of the simplex algorithm specialized to the transportation problem can be used to perform this step. Since the vertices of X (3-1) are degenerate, few blocked pivots may be performed between successive permutations. It is believed that the overall number of pivots is quadratic in k, however, this question is still open.†

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†   Consider the conical subdivision of $R^{(k-1)^2}$ generated by the nonnegativity constraints in the circulations subspace, PAPADIMITRIOU and STEIGLITZ [10], KENNINGTON and HELGASON [9]. The complexity of our algorithm directly depends on the number of cones intersected by a line through this conical subdivision.

Step 4. In practice, Steps 3 and 4 are merged so that not all relevant matchings need be stored in memory. Note (Theorem 2.2) that for each permutation we mainly need to compute $(\mathbf{A}, \pi(\mathbf{B}))$, (1-1). This requires $O(k)$ arithmetic operations.

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5. References

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