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Acoustic Damping for Explicit Calculations of Fluid Flow at Low Mach Number

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ACOUSTIC DAMPING FOR EXPLICIT CALCULATIONS OF FLUID FLOW AT LOW MACH NUMBER

by

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ABSTRACT

A method is proposed for damping the sound waves in explicit calculations of fluid flow at low Mach number, where sound waves are usually not of interest but may distract attention from other flow features. The method is based on the introduction of an artificial pressure q of the form $q = - q_0 \rho c^2 \Delta t(\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{u}_0)$, where q_0 is a coefficient of order unity, ρ is the density, c is the sound speed, Δt is the time step, and $\mathbf{u}_0$ is the velocity field that would obtain at zero Mach number. When $\nabla \cdot \mathbf{u}_0$ is zero, the method becomes equivalent to the use of an artificial bulk viscosity $q_0 \rho c^2 \Delta t$. However, $\nabla \cdot \mathbf{u}_0$ can be substantially different from zero in problems with heat or mass sources (e.g., combustion), and its inclusion is then essential to obtain the correct pressure field. The method is well suited for use in conjunction with explicit numerical schemes that employ acoustic subcycling or artificial reduction of the sound speed for improved efficiency at low Mach number. The beneficial effects of the method are illustrated by means of calculations with an acoustic subcycling computer program.

I. INTRODUCTION

As is well known, explicit numerical calculations of fluid flow at low Mach number M (M << 1) tend to be inefficient because of the wide disparity between the convection and sound-speed stability limits, the latter of which is more restrictive than the former by roughly a factor of M. For this reason, numerical calculations of low Mach number flow have usually been performed with implicit or partially implicit methods such as the ICE method [1,2]. However, the inefficiency can also be alleviated within the framework of an explicit numerical scheme. Two recent methods for doing so are acoustic subcycling [3-5] and artificial reduction of the sound speed [6,7].

A notable difference between such explicit techniques and the more traditional implicit ones is that the latter usually have a much greater tendency to damp sound waves. This tendency is primarily due to a temporal truncation error that arises when the pressure gradient in the momentum equation is evaluated at the advanced time level. In the absence of heat and mass sources, this truncation error has the character of an artificial bulk viscosity. In contrast to the familiar artificial diffusivities associated with upwind differencing of convection terms, an artificial bulk viscosity is actually a desirable feature at low Mach number because it damps sound waves. At low Mach number sound waves are usually not of interest, but their presence in the calculation may tend to distract attention from the flow features that are of interest. This tendency is increased in explicit calculations with the pressure gradient scaling method [7], because this method artificially increases the amplitudes of the sound waves in the process of artificially reducing their speeds.

A further motivation for damping sound waves at low Mach number is that the calculated sound waves are often unphysical in any case. Artificial sound waves with artificially large amplitudes may be generated by various types of truncation errors. Moreover, sound waves with wavelengths less than a few cell widths are rapidly rendered unphysical by dispersion errors. These effects frequently conspire to produce transient short-wavelength irregularities, acoustic in character but of no physical significance, in the computed velocity field.

The above considerations suggest that in explicit calculations of fluid flow at low Mach number, it may be desirable to explicitly introduce an acoustic damping mechanism similar in its effect to that provided by the truncation errors in implicit schemes. Our purpose here is to propose and discuss such a mechanism, and to present calculational results that illustrate the beneficial effects of its use.

II. ARTIFICIAL PRESSURE FOR ACOUSTIC DAMPING

The simplest way to implement an acoustic damping mechanism would be to introduce an artificial bulk viscosity μ_b^*. This is equivalent to adding an artificial pressure $-\mu_b^* \nabla \cdot \underset{\sim}{u}$ to the thermodynamic pressure p. The optimal value of μ_b^* would be expected to be problem-dependent, but its order of magnitude may be estimated by requiring that the characteristic damping time for wavelengths of order Δx be of order Δt. This requirement makes μ^* of order $\rho \Delta x^2 / \Delta t$. In an explicit calculation Δx is of order $c \Delta t$, so an equivalent statement is that μ_b^* is of order $\rho c^2 \Delta t$.

However, a simple bulk viscosity damping mechanism would be insufficiently general for many purposes, because a damping of this form would tend to oppose <u>any</u> nonuniform $\nabla \cdot \underset{\sim}{u}$, regardless of its physical origin. Even at very low Mach number, $\nabla \cdot \underset{\sim}{u}$ can be substantially different from zero for reasons that have nothing to do with sound waves. This is true, in particular, in problems with heat or mass sources [6,8], such as combustion. Since the objective is to damp only the sound waves, one should adopt a damping mechanism that tends to drive $\nabla \cdot \underset{\sim}{u}$ not necessarily to zero, but rather to the value $\nabla \cdot \underset{\sim}{u}_0$ that would obtain in the limit of zero Mach number. This requirement is met by an artificial pressure of the form

$$q = - q_0 \rho c^2 \Delta t (\nabla \cdot \underset{\sim}{u} - \nabla \cdot \underset{\sim}{u}_0) \quad , \tag{1}$$

where q_0 is a dimensionless coefficient of order unity. It is noteworthy that the acoustic damping provided by the principal temporal truncation error of the common implicit schemes is of precisely this form, including the $\nabla \cdot \underset{\sim}{u}_0$ term (see Appendix). In situations where $\nabla \cdot \underset{\sim}{u}_0$ is in fact zero, Eq. (1) becomes equivalent to the use of an artificial bulk viscosity $q_0 \rho c^2 \Delta t$.

Equation (1) is the form of acoustic damping mechanism that we propose. It is implemented simply by replacing the thermodynamic pressure p by p + q. In the momentum equation, this replacement gives rise to a term $-\underset{\sim}{\nabla} q$ which clearly produces a damping force with the desired tendency. Strictly speaking, q should also be added to p in the internal energy equation to properly account for the energy dissipated by the damping. However, this is not really necessary because viscous dissipation is negligible at low Mach number [6], where the kinetic energy of the flow is negligible compared to the internal energy. We therefore replace p by p + q in the momentum equation only.

Use of Eq. (1) requires that $\nabla \cdot \mathbf{u}_0$ be evaluated. This may be done by manipulating the elliptic system of equations that results when sound waves are strictly suppressed [7]. In this way one readily derives the relation

$$\rho c^2 \nabla \cdot \mathbf{u}_0 = \left(\frac{\partial p}{\partial e}\right)_{\rho_i} \dot{Q} + \sum_i \left(\frac{\partial p}{\partial \rho_i}\right)_e \dot{R}_i - \frac{d\bar{p}}{dt} \quad , \tag{2}$$

where e is the thermal internal energy per unit mass, ρ_i is the partial mass density of chemical species i, $\bar{p}$ is the spatial average uniform pressure level in the system, and $\dot{Q}$ and $\dot{R}_i$ represent all contributions to $\partial e/\partial t$ and $\partial \rho_i/\partial t$, respectively, other than convection and compression effects. Thus $\dot{Q}$ contains the effects of viscous dissipation, thermal diffusion, and heat sources (e.g., chemical heat release), but does not include the convection term $-\mathbf{u}\cdot\nabla e$ or the pdV work term $-(\bar{p}/\rho)\nabla\cdot\mathbf{u}_0$. Similarly, $\dot{R}_i$ contains the effects of mass diffusion, chemical reactions, and mass sources, but does not include the convective term $-\mathbf{u}\cdot\nabla\rho_i$ or the compression term $-\rho_i\nabla\cdot\mathbf{u}_0$.

Owing to the definitions of $\dot{Q}$ and $\dot{R}_i$, the first two terms in the right member of Eq. (2) can be interpreted as the contributions of all terms other than convection and compression terms to the local value of $\partial p/\partial t$. It is convenient to represent this partial or incomplete value of $\partial p/\partial t$ by $(\partial p/\partial t)^*$, so that Eq. (2) becomes simply

$$\rho c^2 \nabla \cdot \mathbf{u}_0 = \left(\frac{\partial p}{\partial t}\right)^* - \frac{d\bar{p}}{dt} \quad . \tag{3}$$

The direct evaluation of $(\partial p/\partial t)^*$ from $\dot{Q}$, $\dot{R}_i$, and the state function $p(\rho_i,e)$ can often be circumvented in computer codes based on time-splitting schemes. In such codes the very terms that contribute to $(\partial p/\partial t)^*$ are often split off and used to generate intermediate values of ρ_i and e defined by

$$\tilde{\rho}_i = \rho_i^n + \Delta t\ \dot{R}_i \quad ,$$

$$\tilde{e} = e^n + \Delta t\ \dot{Q} \quad , \tag{4}$$

where Δt is the time step and superscript n denotes the old time level. It is then natural to approximate $(\partial p/\partial t)^*$ by

$$\left(\frac{\partial p}{\partial t}\right)^* = \frac{1}{\Delta t}\left[p(\tilde{\rho}_i, \tilde{e}) - p^n\right] \quad . \tag{5}$$

This is the procedure now used in KIVA [5].

Since q is introduced into the momentum equation only, it enters into the equation system only through its gradient ∇q. The spatially uniform term $-d\bar{p}/dt$ in Eqs. (2) and (3) may therefore be omitted, as it does not contribute to ∇q. The evaluation of $d\bar{p}/dt$ is then unnecessary, although it would have been trivial in any case.

One might be concerned that use of the artificial pressure q would overwhelm any legitimate physical effects due to the true molecular bulk viscosity. The main such effect, of course, is to damp sound waves, and we have no desire to represent this effect faithfully; we wish to exaggerate it. But the molecular bulk viscosity will also have a tendency to oppose the expansion (or compression) represented by $\nabla \cdot u_0$, and this is a physical effect with which we would not wish to tamper. The use of q will not indeed alter this effect, because q acts by construction only on the deviation of $\nabla \cdot u$ from $\nabla \cdot u_0$, and not on $\nabla \cdot u$ itself. In any case, the effect in question is ordinarily negligible, as may be shown by a simple order-of-magnitude estimate.

The role played by the $\nabla \cdot u_0$ term in q is actually somewhat more subtle than has so far been indicated, and this warrants some further discussion. The role of this term may be clarified by considering the nature of the errors that would result from its omission. One might at first expect the main error to be an incorrect velocity field, since q would then tend to drive $\nabla \cdot u$ to zero rather than $\nabla \cdot u_0$. Curiously, however, no such error occurs! The reason lies in the fact that at low Mach number, the pressure gradients are effectively determined by the velocity divergence and not vice versa [7]. Thus, in the absence of sound waves, the velocity divergence is constrained by Eq. (3) regardless of the presence or absence of the $\nabla \cdot u_0$ term in q. The pressure gradients simply adjust themselves to whatever values are necessary for the velocity divergence of Eq. (3) to result. The situation is analogous to that in the pressure gradient scaling method [7], where falsification of the ∇p term in the momentum equation merely forces a readjustment of p to preserve the same $\nabla \cdot u$.

Thus the $\nabla \cdot u_0$ term in q is not needed to obtain the correct velocity field, but it is needed to obtain the correct pressure field in the absence of sound waves. Once the sound waves have been essentially eliminated by the acoustic damping, the combined pressure gradient $\nabla(p + q)$ will have whatever values are needed to make $\nabla \cdot u = \nabla \cdot u_0$. These values are the same regardless of how q is defined; different definitions of q merely apportion the total gradient differently between ∇p and ∇q. The correct physical pressure gradient ∇p_{true} is that which obtains in the absence of q, and it too must take on the same values. Therefore $\nabla p_{true} = \nabla p + \nabla q$. Thus, in order for the pressure gradients calculated in the presence of q to be correct, ∇q must vanish in the absence of sound waves. This is true by construction for the q of Eq. (1), which itself vanishes when $\nabla \cdot u = \nabla \cdot u_0$. However, if the $\nabla \cdot u_0$ term were omitted from Eq. (1), q would become $-q_0 \rho c^2 \Delta t \nabla \cdot u_0$ in the absence of sound waves, and ∇p would then differ from ∇p_{true} by a term $q_0 \Delta t \nabla(\rho c^2 \nabla \cdot u_0)$. Simple estimates show that this term is comparable to ∇p_{true} at low Mach number, so it represents a substantial error.

Inclusion of the $\nabla \cdot u_0$ term in q is therefore necessary to obtain the correct pressure variations in the acoustically damped flow field. Of course, in many low Mach number calculations one is not interested in the pressure variations, and the $\nabla \cdot u_0$ term could then be omitted without significant effect. However, in calculations where pressure variations are important (such as flows driven by a pressure drop, or problems involving pressure drag) the $\nabla \cdot u_0$ term in q is essential, and we therefore recommend its inclusion as a general procedure.

III. STABILITY CONSIDERATIONS

The use of the artificial pressure q imposes an additional stability restriction on the calculation, and this restriction provides an upper bound to the value of q_0 that may be used. This stability restriction may be inferred by taking the divergence of the momentum equation to obtain an evolution equation for $\nabla \cdot u$. Assuming uniform density for simplicity, one finds that the q term therein involves the Laplacian of $\nabla \cdot u$ with a coefficient of $q_0 c^2 \Delta t$. Thus the evolution equation for $\nabla \cdot \underset{\sim}{u}$ has a diffusional character with a diffusivity of $q_0 c^2 \Delta t$, and this diffusivity must satisfy the usual explicit diffusional stability limit. For example, in a two-dimensional mesh of square cells the stability condition becomes $q_0 c^2 \Delta t \leq \Delta x^2 / 4 \Delta t$, or

$$q_0 \leq \frac{\Delta x^2}{4 c^2 \Delta t^2} \quad , \tag{6}$$

and if $c \Delta t / \Delta x \sim 1/2$ this implies $q_0 \lesssim 1$. It should be noted that Δt here represents an explicit time step which must satisfy the Courant stability condition $c \Delta t / \Delta x < 1$. In contrast, implicit schemes are not subject to this condition and will ordinarily be run with a considerably larger value of Δt. It follows that the acoustic damping effect inherent in an implicit scheme (see Appendix) will ordinarily be considerably larger than can be achieved by the use of the artificial pressure q in an explicit scheme. In practice, however, the degree of damping permitted in an explicit calculation appears to be quite sufficient to remove the unwanted sound waves, as illustrated in Sec. V.

IV. ASPECT RATIO CONSIDERATIONS

In multidimensional calculations that utilize finite-difference grids with large cell aspect ratios, use of Eq. (1) may not provide sufficient damping of sound waves propagating in the direction of the larger spatial increments. This difficulty may be alleviated by the use of any artificial pressure <u>tensor</u> $\underset{\approx}{Q}$ instead of the scalar q. For simplicity we restrict attention to the case of two-dimensional Cartesian coordinates (x,y), with spatial increments of Δx and Δy respectively. A suitable form for $\underset{\approx}{Q}$ is then

$$\underset{\approx}{Q} = - \rho c^2 \Delta t (\nabla \cdot \underset{\sim}{u} - \nabla \cdot \underset{\sim}{u}_0)(q_{0x} \underset{\sim}{i}\underset{\sim}{i} + q_{0y} \underset{\sim}{j}\underset{\sim}{j}) \quad , \tag{7}$$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the x- and y-directions respectively. The corresponding force term in the momentum equation is

$$-\nabla \cdot \underset{\approx}{Q} = q_{0x} \Delta t \underset{\sim}{i} \frac{\partial}{\partial x} \left[\rho c^2 (\nabla \cdot \underset{\sim}{u} - \nabla \cdot \underset{\sim}{u}_0) \right] + q_{0y} \Delta t \underset{\sim}{j} \frac{\partial}{\partial y} \left[\rho c^2 (\nabla \cdot \underset{\sim}{u} - \nabla \cdot \underset{\sim}{u}_0) \right] \quad . \tag{8}$$

Thus the effect of replacing q by $\underset{\sim}{Q}$ is to introduce different damping coefficients q_{0x} and q_{0y} into the x and y momentum equations.

Of course, $\underset{\sim}{Q}$ is not a true tensor with a coordinate-free existence, as it has arbitrarily been defined to be diagonal in the particular coordinate system in use. However, this presents no problems because $\underset{\sim}{Q}$ acts by construction only on the sound waves, which are not of interest, and not on the remainder of the solution. All that we ask of $\underset{\sim}{Q}$ is that it damp the sound waves, and if it does so properly its tensor character is unimportant.

The stability restriction on q_{0x} and q_{0y} for this case is readily found to be

$$(c\Delta t)^2(q_{0x}/\Delta x^2 + q_{0y}/\Delta y^2) < 1/2 \quad . \tag{9}$$

Within this restriction, the relative magnitudes of q_{0x} and q_{0y} can now be selected so that sound waves propagating in the x-direction with wavelengths of order Δx are damped at the same rate as sound waves propagating in the y-direction with wavelengths of order Δy. The former rate is proportional to $q_{0x}/\Delta x^2$, while the latter is proportional to $q_{0y}/\Delta y^2$. In order for these rates to be the same for arbitrary Δx and Δy, we must set $q_{0x} = \alpha\Delta x^2$ and $q_{0y} = \alpha\Delta y^2$, where α is a constant. Substitution into Eq. (9) then gives $\alpha < (2c\Delta t)^{-2}$. Let $\Delta = \min(\Delta x, \Delta y)$ be the smaller of Δx and Δy. For $c\Delta t/\Delta \sim 1/2$, the restriction on α becomes $\alpha < 1/\Delta^2$ or $\alpha = \alpha_0/\Delta^2$, where $0 < \alpha_0 < 1$. We thereby obtain

$$\begin{aligned} q_{0x} &= \alpha_0 \Delta x^2/\Delta^2 \quad , \\ q_{0y} &= \alpha_0 \Delta y^2/\Delta^2 \quad , \end{aligned} \tag{10}$$

with $\alpha_0 \lesssim 1$, as our recommended values for the damping coefficients in the present situation. Similar arguments may readily be given for three-dimensional grids or cylindrical coordinates.

Suppose for purposes of discussion that $\Delta x << \Delta y$, so that $\Delta = \Delta x$. We then obtain $q_{0x} = \alpha_0$ and $q_{0y} = \alpha_0(\Delta y/\Delta x)^2$. Thus the damping coefficient in the x-direction is of order unity, while that in the y-direction is of the order of the square of the aspect ratio. This larger value of q_{0y} is just what is needed to damp sound waves in the y-direction at the same rate as those in the x-direction. Use of the scalar q of Sec. II in such a situation would be tantamount to using a much smaller q_{0y}, of order unity, which of course would be inadequate to damp the sound waves in the y-direction. This is why the use of $\underset{\approx}{Q}$ instead of q is necessary in problems with large cell aspect ratios.

V. COMPUTATIONAL RESULTS

Here we present numerical results for a simple test problem, which was solved both with and without acoustic damping to show some of the benefits of the method. The calculations were performed with KIVA [4,5], a compressible-flow fluid dynamics code that uses the acoustic subcycling method for efficiency in low Mach number problems. In the absence of acoustic damping, the acoustic subcycling algorithm in KIVA is neutrally stable [9]. That is, acoustic modes of all wavelengths are undamped, although shorter wavelengths are subject to distortion by dispersion errors.

The problem was to calculate the one-dimensional motion of a gas in a tube driven by a piston at one end. Piston motion was started impulsively at time t = 0.0, and a constant piston velocity UP was maintained thereafter. The Mach number based on UP was approximately 0.01. Without acoustic modes, the solution for the gas velocity is the linear profile

$$u(x,t) = \frac{x}{XP(t)} \, UP \quad ,$$

where x = 0 is the fixed end of the tube and x = XP(t) is the location of the piston. Twenty computational cells were used, and the mesh was continuously remapped into a new mesh that had cells of uniform size $\Delta x = XP(t)/20$.

Figure 1 gives computed velocity profiles from the calculation with (top) and without (bottom) the acoustic damping method at early (left) and late (right) times. At the early time $t = 5.0 \times 10^{-5}$, the pressure wave produced by piston motion has traveled about one third the length of the tube. Dispersion errors cause large oscillations behind the wave at $t = 5.0 \times 10^{-5}$ in the calculation without acoustic damping. In the calculation with acoustic damping the oscillations are eliminated, and the wave front is slightly broader. At the late time $t = 5.0 \times 10^{-3}$, the linear velocity profile is recovered in the calculation with acoustic damping, but without acoustic damping long-wavelength, large-amplitude acoustic waves persist in the tube.

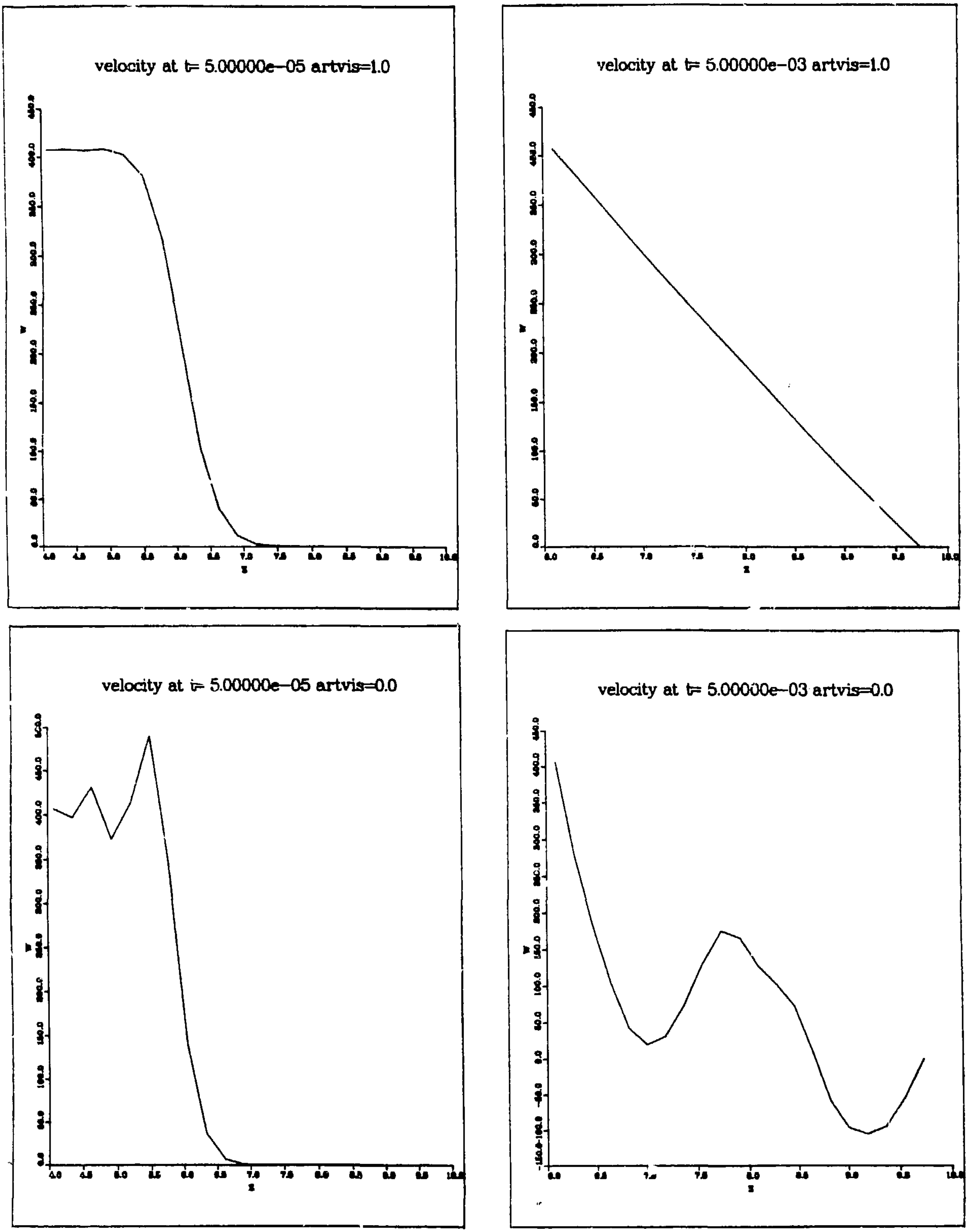

Fig. 1. Velocity profiles from calculations with (top) and without (bottom) acoustic damping.

The benefits of acoustic damping in this problem are typical, and similar improvements have been realized in a wide variety of other calculations. It therefore seems likely that the acoustic damping method will be of general utility in explicit calculations of fluid flow at low Mach number.

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APPENDIX

TRUNCATION ERROR DUE TO ADVANCED TIME PRESSURE GRADIENT

In partially implicit numerical schemes for low-speed fluid flow, the pressure gradient in the momentum equation is typically evaluated at the advanced time level. Here we wish to identify the primary temporal truncation error that thereby results. For this purpose the spatial differencing is immaterial and will therefore be suppressed. The temporal differencing to be examined, then, is

$$\frac{(\rho \underset{\sim}{u})^{n+1} - (\rho \underset{\sim}{u})^{n}}{\Delta t} = - \underset{\sim}{\nabla} p^{n+1} + \underset{\sim}{S}^{n} \quad , \qquad \text{(A1)}$$

where n is the time level and $\underset{\sim}{S}^n$ represents explicit difference approximations to the convection and viscous terms. Now to first order in Δt, we have

$$p^{n+1} = p^{n+1/2} + \frac{1}{2} \Delta t \left(\frac{\partial p}{\partial t}\right)^{n+1/2} \quad , \qquad \text{(A2)}$$

and in the presence of heat (or mass) sources the pressure equation is just [6,8]

$$\frac{\partial}{\partial t} (p - \bar{p}) + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} p = - \rho c^2 (\underset{\sim}{\nabla} \cdot \underset{\sim}{u} - \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_0) \quad , \qquad \text{(A3)}$$

where $\bar{p}$ is the spatially averaged uniform pressure level [6,7]. Equation (A1) thus becomes

$$\frac{(\rho \underset{\sim}{u})^{n+1} - (\rho \underset{\sim}{u})^{n}}{\Delta t} = - \underset{\sim}{\nabla} p^{n+1/2} + \underset{\sim}{S}^{n} + \underset{\sim}{T} \quad , \qquad \text{(A4)}$$

where the temporal truncation error $\underset{\sim}{T}$ is given by

$$\underset{\sim}{T} = \frac{1}{2} \Delta t \ \underset{\sim}{\nabla} \left[\underset{\sim}{u} \cdot \underset{\sim}{\nabla} p + \rho c^2 (\underset{\sim}{\nabla} \cdot \underset{\sim}{u} - \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_0) \right] \quad . \qquad \text{(A5)}$$

The convective term $\underset{\sim}{u} \cdot \nabla \underset{\sim}{p}$ is immaterial to sound wave propagation, so the acoustically significant part of $\underset{\sim}{T}$ is just $\underset{\sim}{\nabla}[(1/2)\rho c^2 \Delta t(\underset{\sim}{\nabla} \cdot \underset{\sim}{u} - \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_0)]$. This corresponds to an artificial pressure of $-(1/2)\rho c^2 \Delta t(\underset{\sim}{\nabla} \cdot \underset{\sim}{u} - \underset{\sim}{\nabla} \cdot \underset{\sim}{u}_0)$, which is of exactly the same form as Eq. (1) of the main text, with $q_0 = 1/2$.

It should be noted that in this analysis Δt represents a typical time step for the implicit scheme. This scheme is not subject to the Courant sound-speed stability restriction, so this time step will ordinarily be substantially larger than a typical explicit time step.

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