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BIAXIAL BIANCHI TYPE IX QUANTUM COSMOLOGY

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Abstract

We investigate the quantum cosmology of spatially homogeneous models with compact spatial sections admitting a $u(2)$ isometry algebra. The metric ansatz in these models is that of Bianchi type IX with two scale factors set to be equal. We apply the Hartle-Hawking no-boundary path integral prescription and find the semi-classical contributions to the wave function. Exact formulae are obtainable for certain contributions and otherwise the limits of large and small anisotropy (for the pure vacuum case) and large spatial volume or small anisotropy (for the case with a positive cosmological constant) are considered. For the pure vacuum case we find no semiclassical components which would correspond to Lorentzian universes. For the case with a cosmological constant the Hartle-Hawking boundary conditions formally constrain one of the parameters in the Lorentzian solutions to be purely imaginary. Possible interpretations of this imaginary parameter are discussed.

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1. INTRODUCTION

Most of the current understanding of the no-boundary proposal by Hartle and Hawking for the quantum state of the Universe [1–3] has been obtained from applying this proposal in spatially homogeneous cosmological models. This proposal defines the wave function Ψ_{HH} by the Euclidean path integral

$$\Psi_{HH}(h_{ij}) = \int d[g_{\mu\nu}]\, e^{-I} \tag{1.1}$$

where I is the Euclidean action of the gravitational field $g_{\mu\nu}$. The integral is formally taken to extend over Euclidean metrics on compact four-manifolds with a boundary, such that the induced metric on the boundary is given by the three-metric appearing as the argument of the wave function. The inclusion of matter fields is straightforward. In spatially homogeneous models whose dynamics admits a Hamiltonian formulation, the wave function reduces to a function of finitely many parameters characterising the homogeneous three-metrics, and the path integral (1.1) reduces to a path integral in a constrained quantum mechanical system. Furthermore, as the Hamiltonian dynamics of these models retains the essentials of the Hamiltonian dynamics of the full theory, one can analyse the interpretation of the wave function in these models by methods which may be extendible also to the full theory.

The most notorious problem in defining the Hartle-Hawking integral (1.1), both in the general formalism and in spatially homogeneous models, is that the Euclidean Einstein action on real positive definite metrics is not bounded from below. If one stays within pure Einstein gravity, the remedy usually suggested is that some of the degrees of freedom in the metric should be integrated over a complex contour. One such suggestion is the conformal rotation prescription of Gibbons, Hawking and Perry [4], which has been shown to lead into sensible integrals in the linearised theory on a flat background [5], in a perturbation expansion on an asymptotically flat background [6], and also in some cosmological models [7]. Another suggestion is the contour prescription of Mazur and Mottola for the linearised theory on more general backgrounds [8]. For integrals with the Hartle-Hawking

boundary conditions, however, the status of these contour prescriptions is less clear. It has been suggested that a way towards a proper definition of (1.1) would be to look at general contours in the space of complex metrics, possibly using steepest descent contours as a guideline [9,10]. Examples in simple models [11] indicate that different contours can in general give drastically different wave functions, and specification of the contour must therefore be regarded as an essential part of the definition of the integral.

The contour of integration remains an issue even when one only wishes to approximate (1.1) by the semiclassical contributions coming from the saddle point configurations. These saddle point configurations are four-metrics solving the Euclidean Einstein equations and satisfying the boundary data of the integral. For given boundary data there may be many solutions, and choosing the contour of integration translates into choosing which of the saddle points actually contribute to the integral. Note that although the boundary data enforces the induced metric on the boundary to be positive definite, the saddle point solutions in the interior of the four-manifold can be complex-valued metrics (while still being defined on a real manifold).

In this paper we shall investigate the Hartle-Hawking proposal in spatially homogeneous models where the four-metric admits an isometry group with Lie algebra $u(2)$ acting transitively on the spatial three-surfaces. We consider pure Einstein gravity with a non-negative cosmological constant Λ. As will be explained in the Appendix, these models are obtained from the general (diagonal) Bianchi type IX model by setting two of the three scale factors to be equal. Our purpose is to find the saddle point metrics in the Hartle-Hawking integral and analyse the possible semiclassical contributions to the wave function. It would be in principle straightforward to set up the machinery of Ref. [11] for a steepest descent analysis of the integration contours, but the practical calculations appear to become complicated. In this paper we shall just look separately at all the different saddle point contributions.

The general classical solutions in our models are known in closed form, both in their Lorentzian and Euclidean versions. They are usually referred to as Taub-NUT space for $\Lambda = 0$ and as Taub-NUT-de Sitter space for $\Lambda > 0$ [12–14]. We can therefore find the purely Euclidean saddle points by matching the known form of the general real Euclidean solution to the boundary data of the integral. It turns out that we find with this method also some complex valued saddle point metrics. These complex metrics are formally obtained from the real Euclidean solution by making the parameters and the Euclidean time coordinate complex-valued, but it can be verified that they indeed represent complex metrics on a real manifold and satisfy the boundary data of the integral. We do not know, however, whether all complex-valued saddle point metrics can be found in this way.

In practice, finding the actions of the saddle point metrics as a function of the boundary data reduces to solving an algebraic equation. This equation can in most cases not be solved in terms of simple expressions, but it is possible to analyse the behaviour of the classical actions by taking suitable limits in the boundary data. When $\Lambda = 0$, we do not find any saddle point metrics whose Euclidean action would be approximately imaginary. If we relate the wave function to classical space-times using a semiclassical interpretation [15–17], this would mean that the contributions to the wave function are not rapidly oscillating and do therefore not correspond to Lorentzian universes. When $\Lambda > 0$, on the other hand, we find two saddle point metrics whose Euclidean action is approximately imaginary in certain regions in the configuration space. One would therefore like to say that these semiclassical contributions to the wave function are rapidly oscillating and correspond to certain families of Lorentzian universes. Even in this case, however, the real part of the Euclidean action does not vanish exactly. Such a non vanishing (and non-constant) part of the Euclidean saddle point action may raise subtle questions about a semiclassical interpretation of the wave function.

The cases $\Lambda = 0$ and $\Lambda > 0$ will be analysed respectively in Sections 2 and 3. The results are summarised and discussed in Section 4. In the Appendix we collect some elementary results on 3 and 4 geometries admitting a $u(2)$ isometry algebra acting transitively on the 3 surfaces.

2. VACUUM THEORY

We consider metrics which are spatially homogeneous and with symmetry group having the Lie algebra $u(2)$, the group generically having 3 dimensional orbits, 3-spheres or Lens spaces. The form of the Lorentzian four-metric is assumed to be

$$ds^2 = -N^2dt^2 + a^2(\sigma_1^2 + \sigma_2^2) + c^2\sigma_3^2 \tag{2.1}$$

where

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k \tag{2.2}$$

and σ_i are left-invariant one forms on the generic 3 surfaces regarded as identified three spheres.

The action is the usual Einstein action with a cosmological constant Λ [18]:

$$S_L = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K \tag{2.3}$$

Here K_{ij} is the extrinsic curvature on the boundary $\partial\mathcal{M}$, K its trace, and h_{ij} the induced metric on $\partial\mathcal{M}$. A convenient parametrisation of the spatial metric is

$$h_{ij} = V^{\frac{2}{3}}\exp(2\beta_{ij}) \tag{2.4}$$

where $V = a^2c$ scales like the volume of three-space. The matrix β_{ij} is traceless and diagonal with entries $\beta_{11} = \beta_{22} = -\beta_{33}/2 \equiv \beta$, *i.e.* $\exp(3\beta) = a/c$, and represents the deviation of the three-surface from isotropy. Substituting this into the action (2.3) we find the following,

$$S_L = \frac{6\pi}{G}\int dt\ \frac{V}{N}\left(-\frac{1}{9}\left(\frac{\dot V}{V}\right)^2 + \dot\beta^2 + \frac{N^2}{6}({}^3R - 2\Lambda)\right) \tag{2.5}$$

where the spatial curvature is

$${}^3R = 2V^{-\frac{2}{3}}(e^{-2\beta} - \frac{1}{4}e^{-8\beta}) \tag{2.6}$$

The equations of motion may now be derived from (2.5). In this section we shall consider the pure vacuum case, $\Lambda = 0$. The solutions are, in the gauge $Nc = 2l$,

$$ds^2 = -\frac{(t^2+l^2)}{(t^2-2mt-l^2)}dt^2 + 4l^2\frac{(t^2-2mt-l^2)}{(t^2+l^2)}{\sigma_3}^2 + (t^2+l^2)(\sigma_1^2+\sigma_2^2) \qquad (2.7)$$

known as the Taub-NUT solutions [12]. The singularities of these spaces were discussed in [19]. We need to consider the Euclidean vacuum solutions. These take the form:

$$ds^2 = \frac{(l^2-t^2)}{(l^2-2mt+t^2)}dt^2 + 4l^2\frac{(l^2-2mt+t^2)}{(l^2-t^2)}\sigma_3^2 + (l^2-t^2)(\sigma_1^2+\sigma_2^2) \qquad (2.8)$$

or precisely -1 times this metric. Metric (2.8) generally has curvature singularities at $t = \pm l$ as can be seen from the fact that the space is left and right Petrov type D with Weyl tensor

$$\Psi_2 = -\frac{(l+m)}{(t+l)^3} \qquad \tilde{\Psi}_2 = \frac{(l-m)}{(t-l)^3} \qquad (2.9)$$

In order to obtain the semi-classical contributions to the wave function with Hartle-Hawking boundary conditions we need to find vacuum solutions (both real and complex) on a real compact manifold that matches on to a given three geometry on its only boundary (the geometry being the argument of the wave function). To implement this no-boundary proposal we must examine ways in which metric (2.8) can close. By symmetry considerations this amounts to discussions of ways in which the range of t can be limited to leave a manifold without boundary. Since the metric is non singular except at points where $(l^2-t^2) = 0$ or where $(l^2-2mt+t^2) = 0$ we need only consider these possible initial t values for else t is a good coordinate. In this case the analysis of Refs. [13,14] shows that these t values must correspond to either points (NUTs) or 2-surfaces (BOLTs). We consider these closings in turn. We will assume first that we have three surfaces with topology of S^3 and deal at the end of this section with the case of RP^3 surfaces and the generalisations to higher Lens spaces $L(s,1)$, $s > 2$.

NUT case

This case corresponds to closing at $t = \pm l$. Removal of the curvature singularity thus requires that $m = \pm l$ and we obtain the self-dual Taub NUT space. The classical action for the solution matching on to final (a,c) values is found by solving for l and t_{final} and substituting into the expression for the action. We find

$$I(a,c) = -\frac{\pi}{G}(4ac - c^2) \qquad (2.10)$$

There is a real classical solution matching on to **any** positive (a,c) values and the action is real. Thus in no region of the space of 3-geometries we consider is the wave function rapidly oscillating, and we cannot interpret the wave function as corresponding to Lorentzian universes.

BOLT case (S^3 surfaces)

Here we have to consider the case of closing at the roots of $(l^2 - 2mt + t^2) = 0$ which we take to be $t_+ \geq t_-$. Since ψ is an angular variable (with period 4π for the S^3 topology) there is the possibility of a conical singularity at one of these roots. Elimination of the singularity at $t_\pm$ requires $m = \mp\frac{5}{4}l$ respectively. Taking the lower sign implies that $t_- = l/2$, $t_+ = 2l$ and yields the metric

$$ds^2 = \frac{(l^2-t^2)}{(t-\frac{l}{2})(t-2l)}dt^2 + 4l^2\frac{(t-\frac{l}{2})(t-2l)}{(l^2-t^2)}\sigma_3^2 + (l^2-t^2)(\sigma_1^2+\sigma_2^2) \qquad (2.11)$$

This metric is positive definite for $\frac{l}{2} > t > -l$, indefinite for $2l > t > \frac{l}{2}$ and negative definite otherwise. For metrics ending at $t = 2l$ we first consider the range $t \geq 2l$. This is the (negative definite) Taub BOLT metric discovered by Page [20]. First we note that the ratio

$$\frac{c^2}{a^2} = \frac{4l^2(t-\frac{l}{2})(t-2l)}{(l^2-t^2)^2} = f(t,l) \qquad (2.12)$$

takes a range of values that is independent of l in the range $t \geq 2l$ and is bounded above by some value $f_c = f(l\xi^*, l)$ where $\xi^* \approx 3$. Thus for $(c/a)^2$ less than this critical value we find two Euclidean BOLT solutions matching on to the required geometry but for $(c/a)^2$ greater than this value we find *no* real BOLT solutions. For these values we have to resort to considering complex metrics defined on real manifolds, but first we deal with the case $(c/a)^2 < f_c$.

Let $2 < \xi_1 < \xi^* < \xi_2$ be the unique such solutions to the equation:

$$f(l\xi, l) = \frac{c^2}{a^2} \tag{2.13}$$

Then we find that we have two classical solutions with actions

$$-\frac{2\pi a^2}{G}\frac{4\xi_i^3 - \frac{15}{2}\xi_i^2 + \frac{5}{2}}{(\xi_i^2 - 1)^2} \tag{2.14}$$

We cannot find a closed expression for all ξ_i, however for $\frac{c^2}{a^2} \ll 1$ we find the two actions

$$I_1(c,a) = -\frac{4\pi}{G}(ac - \frac{5}{8}c^2 + a^2 O(\frac{c^3}{a^3})) \tag{2.15}$$

$$I_2(c,a) = -\frac{\pi}{G}(a^2 + 2c^2 + a^2 O(\frac{c^3}{a^3})) \tag{2.16}$$

As both I_1 and I_2 are real in this part of superspace, we cannot interpret the semi-classical wave functions as corresponding to Lorentzian universes. For $f > f_c$ we cannot find any real Euclidean solutions with a BOLT closing that match onto the prescribed boundary and obey the Hartle Hawking condition. Thus we resort to considering complex metrics. This is the smallest complex vector space containing the cone of Euclidean metrics on a given four manifold and seems the closest to what one does in analytic continuation theory, but where in that case one needs to be careful in specifying the branch taken in the analytic continuation. Complex metrics on real manifolds are bizarre objects; the geodesic equation has generally no solutions (even locally). In addition the relation between an analytic continuation of the path integral and gauge fixing needs to be carefully addressed. We have assumed that such issues are not relevant for computing the possible semiclassical contributions to the wave function.

Now clearly complex metrics on a real manifold solving the vacuum field equations may be obtained by performing the transformation $t \to t(\tau)$ where $t(\tau)$ is a complex function of τ. We will assume in what follows that relevant solutions for the 'BOLT case' are obtained from the Taub BOLT metric merely by performing such a transformation. It is clear that any such transformed metric has the desired properties of non-singularity and regular closing. However it is not clear that all such complex solutions are obtained in this way.

For all values of $\frac{c^2}{a^2}$ we can find two real solutions to equation (2.13), ξ_3, ξ_4 obeying $\frac{1}{2} > \xi_3 > -1 > \xi_4$. These final values cannot be joined in the real ξ line to $\xi = 2$,the BOLT, without passing through a curvature singularity at $\xi = 1$, or more crucially undergoing a change of signature. However there is a contour in the complex $\xi = t/l$ plane which yields a complex metric when that contour is given a real parametrisation. Thus we obtain complex solutions with the actions given by equation (2.13). We can again evaluate these contributions for large and small values of $(c/a)^2$ For $\frac{c^2}{a^2} \ll 1$ we obtain:

$$I_3(c,a) = -\frac{4\pi}{G}(a^2 + \frac{c^2}{8} + a^2 O(\frac{c}{a})^3) \tag{2.17}$$

$$I_4(c,a) = -\frac{4\pi}{G}(ac - 5c^2 + a^2 O(\frac{c}{a})^3) \tag{2.18}$$

whilst for $\frac{c^2}{a^2} \gg 1$

$$I_3(c,a) = \frac{\pi}{G}(2c^2 - 3ca + c^2 O(\frac{a}{c})^2) \tag{2.19}$$

$$I_4(c,a) = \frac{\pi}{G}(2c^2 + 3ca + c^2 O(\frac{a}{c})^2) \tag{2.20}$$

Although these semi-classical contributions have been calculated from complex metrics, because the final values of l and ξ are real the actions are real to all orders. This is unlike the case of the contributions from the other two roots of (2.13) for $\frac{c^2}{a^2} \gg 1$ which give

$$I_{1,2}(c,a) = -\frac{\pi}{G}(2c^2 \mp 3\sqrt{2}ica + c^2 O(\frac{a}{c})^2) \tag{2.21}$$

These yield complex solutions, which are not however of the Lorentzian WKB type in the region of (c, a) space where they are valid. We thus see that if $\frac{c^2}{a^2}$ is much less than or greater than 1, we do not have a wave function that corresponds to Lorentzian universes. Although we do not have a good approximation to $\exp(-I_1)$ and $\exp(-I_2)$ outside these two limiting regions, it appears unlikely that these components would become rapidly oscillating anywhere in the configuration space. However, even leaving open what happens outside the above two limits, these limits allow us to conclude that the wave function cannot correspond to Lorentzian universes that reach a singularity. To see this, suppose that in some region one of these wave function components takes the Lorentzian WKB form

$$\Psi \approx A(a, c) e^{iS(a,c)} \tag{2.22}$$

where the exponential varies rapidly relative to A. Then this component corresponds to the one-parameter family of space-times for which S is taken as Hamilton's principal function. At the singularity of every Lorentzian solution the ratio $\frac{c}{a}$ tends to zero, and at this limit we have shown that the wave function is not of the form (2.22). Thus the wave function does not correspond to Lorentzian universes reaching a singularity.

BOLT case (Lens Space surfaces)

For completeness we now give the case of a 'BOLT' closing where the surfaces of homogeneity have the topology of RP^3 or more generally the Lens spaces $L(s, 1)$, $s > 2$. For the case of RP^3 the relevant Euclidean solution is the Eguchi Hanson metric [14,21]

$$ds^2 = (1 - \frac{m^4}{t^4})^{-1} dt^2 + \frac{t^2}{4}(1 - \frac{m^4}{t^4})\sigma_3^2 + \frac{t^2}{4}(\sigma_1^2 + \sigma_2^2) \tag{2.23}$$

which has a BOLT at $t = m$.

For $c < a$ we can match on to a real solution with action,

$$I = -\frac{\pi}{2G}(2a^2 + c^2). \tag{2.24}$$

There is also a complex solution with the same action which in the form (2.23) has a bolt at $t = \pm im$. It is easy to show that for $c \geq a$ we can match a complex metric closing at a bolt which is obtained by setting $t = a + \tau(m - a)$ where $m^4 = (16a^4)(1 - \frac{c^2}{a^2})$ m is complex and τ is a variable running between 0 and 1. The form of the action is the same as (2.24) and does not correspond to Lorentzian universes.

The higher Lens spaces $L(s, 1)$ are defined by regarding the 3 sphere as a subset of C^2 defined by $|Z_1|^2 + |Z_2|^2 = 1$. Then the Z_s action $(Z_1, Z_2) \rightarrow \exp(\frac{2\pi i}{s})(Z_1, Z_2)$ has no fixed points and we can thus form the quotient manifold $L(s, 1)$. Since this action is isometric for $U(2)$ invariant metrics the quotient space will have a well defined metric. The classical solutions are then given by (2.8) with $m = (1 + \frac{c^2}{4})\frac{1}{s}$ for $s > 2$. It can be shown that for any (c, a) values, we have three complex geometries and one real geometry with a BOLT closing. They all have real action and thus the wave function is not Lorentzian. It is interesting to note that the real classical solution cannot be extended arbitrarily far into the future beyond the final surface without encountering a singularity.

3. COSMOLOGICAL THEORY

In this section we let the cosmological constant have a positive value. This considerably complicates the classical solutions and a complete treatment is not possible. The Lorentzian solutions are

$$ds^2 = -\frac{(t^2 + l^2)}{\dot{\Delta}} dt^2 + 4l^2 \frac{\dot{\Delta}}{(t^2 + l^2)}\sigma_3^2 + (t^2 + l^2)(\sigma_1^2 + \sigma_2^2) \tag{3.1}$$

where

$$\dot{\Delta} = \Lambda(l^4 - 2l^2t^2 - \frac{t^4}{3}) \tag{3.2}$$

The Euclidean solutions, which are generally known (at least by the authors) as the Taub-NUT de Sitter family are:

$$ds^2 = \frac{(l^2 - t^2)}{\Delta} dt^2 + 4l^2 \frac{\Delta}{(l^2 - t^2)}\sigma_3^2 + (l^2 - t^2)(\sigma_1^2 + \sigma_2^2) \tag{3.3}$$

where

$$\Delta = l^2 - 2mt + t^2 - \Lambda(l^4 + 2l^2t^2 - \frac{t^4}{3}) \qquad (3.4)$$

This space is left and right Petrov type D (as is evident from the $u(2)$ symmetry acting on 3 dimensional orbits) with Weyl tensor

$$\Psi_2 = -\frac{l + m - \frac{4}{3}\Lambda l^3}{(t + l)^3} \qquad \tilde{\Psi}_2 = \frac{l - m - \frac{4}{3}\Lambda l^3}{(t - l)^3} \qquad (3.5)$$

The Lorentzian solution (3.4) with $l^2 = 3\Lambda/4$ and $m = 0$ is de Sitter space. Similarly, the Euclidean solution (3.5) with $l^2 = 3\Lambda/4$ and $m = 0$ is the Euclidean version of de Sitter space, or the round metric on S^4.

The closings in the Euclidean solution are again of NUT and BOLT type, BOLT being possible for all spatial topologies but NUT only for S^3. We shall now consider these closings in turn.

NUT case

We consider first the case of S^3 spatial topology with a NUT type closing. Without loss of generality we take the NUT to be at $t = -l$, then (3.5) tells us that to avoid a curvature singularity we must set $m = -l(1 - \frac{4}{3}\Lambda l^2)$. This space is thus half conformally flat and has been discussed by Gibbons and Pope [13] and by Pedersen [22]. In general we will still encounter conical singularities at the zeros of Δ. However these are relevant only for certain real solutions and we can find *complex* solutions ending at $-l$ which avoid these conical singularities. For the NUT closing Δ simplifies to give:

$$\Delta = (l + t)^2(1 - \frac{\Lambda}{3}(l + t)(3l - t)) \qquad (3.6)$$

It can be shown that matching the solution to the specified boundary values of a and c reduces to solving a single algebraic equation cubic in l^2. Unfortunately, the coefficients in this equation depend on (a, c) in a complicated way and the exact solution is not

practical. Also, deriving this equation requires taking squares of certain expressions, and without an explicit solution it is not easy to see whether this might produce roots which do not correspond to actual saddle points. We therefore resort to two limits at which we can find the asymptotic forms of two solutions. These limits are those of large volume V and small anisotropy β, where V and β are defined through (2.4).

For fixed β and large V we may obtain the action of one complex solution matching onto the boundary three geometry as an asymptotic series in V. The parameters l, t_{final} resulting in the desired metrics can be shown to have the following asymptotic expansions:

$$t_{final} = iV^{\frac{1}{3}}(e^{\beta} - \frac{3}{8}e^{-7\beta}\Lambda^{-1}V^{-\frac{2}{3}} + O(V^{-\frac{4}{3}})) \qquad (3.7)$$

$$\begin{aligned} l = \frac{\sqrt{3}}{2}\Lambda^{-\frac{1}{2}}\Big(&e^{-3\beta} + \frac{3}{2}\Lambda^{-1}V^{-\frac{2}{3}}(e^{-5\beta} - e^{-11\beta}) \\ &+ i\frac{3\sqrt{3}}{2}\Lambda^{-\frac{3}{2}}V^{-1}(e^{-9\beta} - e^{-15\beta}) + O(V^{-\frac{4}{3}})\Big) \end{aligned} \qquad (3.8)$$

When substituted into the action we obtain

$$\begin{aligned} I = \frac{\pi}{G}\Big(&\frac{4i}{\sqrt{3}}\Lambda^{\frac{1}{2}}V - 2i\sqrt{3}\Lambda^{-\frac{1}{2}}V^{\frac{1}{3}}(e^{-2\beta} - \frac{1}{4}e^{-8\beta}) \\ &- 3\Lambda^{-1}(e^{-6\beta} - \frac{1}{2}e^{-12\beta}) + O(V^{-\frac{1}{3}})\Big) \end{aligned} \qquad (3.9)$$

This expansion is valid for $V \gg \Lambda^{-3/2}\exp(-12\beta)$. A second complex solution is obtained by taking the complex conjugate in (3.7)–(3.9).

The dominant β-dependence of (3.9) is in the second term. We remark that this is exactly the same dependence as that appearing in the spatial curvature in the minisuperspace action for metrics of our form, see eq. (2.6). (A similar result may hold even for the general Bianchi IX model.) This is a necessary consequence of the particular exponents of V occurring in (3.9). Because (3.9) is not exactly imaginary, it is difficult to be clear about the predictions for Lorentzian space-times. The expansion in volume yields a

Hamilton-Jacobi function which implies that the 'Lorentzian' solutions are given by the family (3.1) with

$$m = \pm il(1 - \frac{4\Lambda l^2}{3}) \qquad (3.10)$$

Knowing that the Lorentzian action S_L satisfies the Hamilton-Jacobi equation, this result should not come as a surprise. The continuation from the Lorentzian Taub-NUT line element (3.1) to the Euclidean (3.3) requires in addition to the usual $t \to it$ also that we let $m \to im$. So the solutions to the Hamilton-Jacobi equation derived from the Euclidean action must necessarily correspond to *imaginary* values of the Lorentzian m. The particular form (3.10) for m comes from the constraint that was imposed to eliminate the curvature singularity at the NUT of the Euclidean solution. There are regions where this complex metric is approximately real (and Lorentzian), but it is not clear what in a situation like this should be understood as the predictions of the wave function. We shall return to this question in Section 4.

The other limit we consider is that of fixed V and small absolute value of β. Here we are essentially considering slightly distorted three spheres as the final surface. To find solutions, we make the assumption that the anisotropy is small not only on the final surface but everywhere in the four-geometries. We can thus first solve the equations with β set strictly equal to zero (see for example Ref. [2]), and then to solve the equations for β by treating it as a perturbation on the isotropic background. Let $r^2 = \frac{1}{3}\Lambda V^{\frac{2}{3}}$. When r^2 is less than 1, we find

$$I_\pm = \frac{3\pi}{2G\Lambda}(-1 \pm (1 - r^2)^{\frac{3}{2}} + 12r^2(3 \pm (1 - r^2)^{\frac{1}{2}})^{-1}\beta^2 + O(\beta^3)) \qquad (3.11)$$

The upper and lower signs correspond to four-geometries which are close to respectively the smaller and larger parts of the round four-sphere bounded by a round three-sphere [2]. For r^2 greater than 1 we obtain

$$I_\pm = \frac{3\pi}{2G\Lambda}(-1 \mp i(r^2 - 1)^{\frac{3}{2}} + 12r^2(3 \pm i(r^2 - 1)^{\frac{1}{2}})^{-1}\beta^2 + O(\beta^3)) \qquad (3.12)$$

The region of parameter space in which (3.11) and (3.12) remain valid corresponds to $|\beta| \ll \frac{1}{3}$. The two expansions (3.9) and (3.12) agree in the limit of large volume V, and we find that also (3.12) leads to the relation (3.10) for the parameters of the 'Lorentzian' solutions. This indicates that the actions obtained by the two different expansions come from the same pair of saddle points.

We remark that our expressions (3.11) and (3.12) differ from those obtained by Amsterdamski [23] for the general Bianchi type IX model at the limit of small anisotropy. The reason is that the anisotropy equation of motion derived and used in Ref. [23] contains an incorrect numerical factor in its zeroth derivative term.

The BOLT case

We now turn to the BOLT type closing with the spatial topology S^3, RP^3 and the higher Lens spaces. It is very difficult to deal with this case in detail, but a qualitative treatment at large volume is possible. A BOLT closing occurs where we close the manifold at a zero of Δ not equal to $\pm l$. Thus the orbits of the isometry group go from being Lens spaces $L(s,1)$, $s \geq 1$ $(0 \leq \psi \leq \frac{4\pi}{s})$, to being two spheres. To obtain a compact manifold which is also complete would fix both m and l and lead to the Page solution [24] $(m = 0)$. However, for our case of seeking solutions (possibly complex) inducing given three geometries, we need merely eliminate the possible conical singularity at one zero of Δ, t_*,say. The condition is then:

$$(t_*^2 - l^2) = \pm 2nl\Delta'(t_*) \qquad (3.13)$$

The consistency of these conditions leads to a (multi-valued) relation $m = f(l)$ which is straightforward, but tedious, to find. The crucial points for the large volume expansion however are that the same expansions of t_{final}, l as in (3.7), (3.8) are obtained and that given this, the term in m does not contribute to the first two terms in the action (3.9). The argument is as follows. One may work with dimensionless quantities $\tau_{final} = l^{-1}t_{final}$,

$\alpha = (\Lambda l^2)^{-1}$. Then on dimensional grounds the condition for the removal of conical singularities takes the form

$$m = log(\alpha) \tag{3.14}$$

By examining equation (3.13) and the condition $\Delta(t_*) = 0$ one concludes that $g(\alpha) = O(1)$ for $\alpha = O(1)$ and α small. For α large we find that $g = O(\alpha^3)$, but this is not a consistent condition for matching on to large V at fixed β. Thus we may exclude this latter possibility. The m term contribution did not appear in (3.9) and thus it still does not for the BOLT case, whatever the detailed form of $g(\alpha)$. Thus we can deduce that we have the same large volume form. A direct corollary is then that for the Lens space case we have:

$$\hat{I} = \frac{1}{n} I(V, \beta) + O(1) \tag{3.15}$$

where $I(V, \beta)$ is the expression appearing in (3.9).

4. DISCUSSION

In this paper we have investigated the quantum cosmology of spatially homogeneous cosmological models with $u(2)$-invariant spatial hypersurfaces. The metric ansatz in these models is given by the diagonal Bianchi type IX metric with two of the scale factors set to be equal, and the possible spatial topologies are S^3, RP^3 or higher Lens spaces $L(s, 1)$ with $s > 2$. The solutions to the Wheeler-DeWitt equation were chosen by the Hartle-Hawking no-boundary prescription. We found the saddle points of the path integral which formally defines the no-boundary wave function, and we analysed separately the semiclassical contributions arising from these saddle points. Which of the semiclassical components actually are present in the wave function would depend on how the integral is defined, in particular how the the contour of integration is chosen. This question was left subject to future work.

The saddle points were found by imposing conditions which make the real Euclidean solutions to the Einstein equations match with the boundary conditions of the the Hartle-Hawking integral. This method clearly finds all the real Euclidean saddle points. In most cases we found that the matching conditions had complex-valued solutions, and we verified that these complex-valued solutions can be written as complex valued metrics on a real manifold, satisfying the Einstein equations and matching with the boundary data of the integral. These complex solutions are thus saddle points of the integral, and (with a suitable choice of the contour) they can be expected to give semiclassical contributions to the wave function. We do not know, however, whether all complex saddle points can be found in this manner.

In the case of a vanishing cosmological constant we found saddle points for all the spatial topologies. Some of these saddle points were real and some complex. However, the only topology for which there were saddle points with complex-valued actions was S^3. In the regions in the configuration space where we were able to obtain approximate expressions for these complex actions, we found that the corresponding semiclassical components are not rapidly oscillating and therefore cannot be interpreted in terms of Lorentzian space-times. From this and the known form of the general Lorentzian solutions it was possible to conclude that the wave function cannot correspond to Lorentzian universes that reach a singularity. It appears unlikely that these components should become rapidly oscillating anywhere in the remaining regions of the configuration space. If this is true, it would mean that the no-boundary prescription rules out all $u(2)$-invariant Lorentzian universes with pure vacuum.

In the case of a positive cosmological constant we again found saddle points for all the spatial topologies. The expressions for the actions of these saddle points are more difficult to analyse, and we had to resort to expansions in the volume and the anisotropy. The situation analysed in most detail was that of S^3 spatial topology with a NUT-type closing of the four-geometry. Here we found a complex conjugate pair of saddle points whose Euclidean actions were approximately imaginary in a certain region of the configuration space. These saddle points therefore gave rise to rapidly oscillating semiclassical wave function components, and one would expect such components to correspond to Lorentzian space-times. There is a subtlety, however, arising from the fact that the Euclidean action is only approximately imaginary. The quantity $\pm iI$ is, by construction, a solution to the Lorentzian Hamilton-Jacobi equation, but it is now only approximately real. When solving the Hamilton-Jacobi equations of motion with $\pm iI$ as Hamilton's principal function, we therefore obtained a set of genuinely complex valued metrics, given by the general 'Lorentzian' solution (3.1) with the relation

$$m = \pm il(1 - \frac{\Lambda l^2}{3}) \tag{4.1}$$

between the parameters m and l. Although we obtained (4.1) as an approximate relation from our expansions in the volume and anisotropy, it is obvious from comparing the Lorentzian and Euclidean solutions (3.1) and (3.3), and from the NUT-regularity condition for the saddle point solutions, that (4.1) is in fact exact and could have been foreseen even without doing the expansions. Note that (4.1) is consistent with a real Lorentzian metric only when $l^2 = 3/4\Lambda$ and $m = 0$, which gives the metric of de Sitter space.

What one should understand as the predictions from these wave-function components which are rapidly oscillating, yet not quite of a Lorentzian semiclassical form, is not clear to us. The simple model discussed in this paper is based on a minisuperspace approximation reducing the number of degrees of freedom to just two. Naturally such a model cannot provide an accurate description of the complex quantum phenomena in the early universe. However models of this type may help us gain some insight as to what to expect from more complicated models. Furthermore, in a semi-classical calculation such as the one presented here, one hopes with the aid of the Hamilton-Jacobi equations to relate the wave function to classical solutions to Einstein's equations. With this in mind, we would here like to offer two possible interpretations of the results presented in this paper.

The first, and perhaps most straightforward, point of view would be to say that the wave function indeed corresponds to the family of genuinely complex metrics (on a real manifold) given by the 'Lorentzian' solution (3.2) with the relation (4.1). As the large scale

geometry of our own universe is for all apparent purposes described by a real Lorentzian metric, one would therefore say that a wave function like this is inappropriate for describing our universe, except perhaps if the universe were exactly de Sitter. Although such a conclusion in a single model may not seem alarming, experience from simple solvable models [25-27] suggests that the semiclassical components of the no-boundary wave function generically do not take a purely Lorentzian semiclassical form even when they are rapidly oscillating. (Some exceptions are shown in Refs. [2,24,26].) Hence, the no-boundary wave function in 'realistic' models would not be expected to take a purely Lorentzian semiclassical form. One is thus led to conclude that the no-boundary wave function is not likely to provide a good description of our own universe. This is perhaps slightly discouraging, since a priori, one would have expected the Hartle-Hawking prediction to result in a *real* relation between m and l instead of (4.1).

The second point of view would be to regard a wave function as predicting correlations only insofar as those correlations concern purely Lorentzian quantities. One way of justifying this idea might be to regard the Wigner function $W(p,q)$ as a measure of correlations in the phase space [17]: in certain quantum mechanical examples it can be verified that the Wigner function of a wave function of the Lorentzian semiclassical form $\exp(iS)$ (with real S) is indeed peaked around the expected correlation $p_\alpha = \partial S/\partial q^\alpha$, whereas the Wigner function of a wave function of the Euclidean semiclassical form $\exp(-I)$ (with real I) is not peaked around any correlation between p_α and q^α. To apply this interpretation to our wave function, we observe that $\mathrm{Re}(I)$ can be made arbitrarily slowly varying compared with $\mathrm{Im}(I)$ by going sufficiently deep into the region where the expansion (3.9) is valid. Deep in this region, $\exp[-\mathrm{Re}(I)]$ is therefore slowly varying compared with not only the oscillating factor $\exp[-i\mathrm{Im}(I)]$ but also with the pre-exponential factor. In this region one would thus interpret the wave function as corresponding to those Lorentzian space-times which satisfy

$$p_\alpha = \pm \frac{\partial(\mathrm{Im}(I))}{\partial q^\alpha} \quad . \tag{4.2}$$

In the region where $\mathrm{Re}(I)$ becomes comparable to $\mathrm{Im}(I)$, one would conversely interpret the wave function as simply not predicting Lorentzian spacetimes. This interpretation remains admittedly vague without more concrete criteria as to what can be regarded as "rapidly varying" and "slowly varying." Also, since $\ln(I)$ is only an approximate solution to the Lorentzian Hamilton-Jacobi equation, the solutions of (4.2) are only approximate solutions to the Einstein equations. Deep in the region where the expansion (3.9) is valid, the solutions of (4.2) are close to exact Lorentzian solutions, but it is difficult to give a more quantitative characterisation of this closeness. It may be, however, that a vagueness of this kind is inevitable when an intrinsically semiclassical interpretation is applied in a simple model like ours. The ambiguities may be resolved only when a model is sufficiently complicated to allow a realistic description of cosmological observations including the observers themselves.

APPENDIX

We summarise here some elementary results on 3 and 4 geometries admitting a $u(2)$ isometry algebra acting on 3-surfaces. First consider the case where the 3-surface has the topology S^3. Then we can identify the 3-surface with the group manifold $SU(2)$ and write the metric on the surface in the form

$$ds^2 = A_{ij}\sigma^i\sigma^j \tag{A.1}$$

where σ^i are left invariant one-forms under $SU(2)$ and can be chosen to satisfy (2.2). By a further choice of basis (A.1) may be diagonalised, and the extra generator for $u(2)$ forces two of the eigenvalues to be equal. Thus any such metric may be brought to the form

$$ds^2 = a^2(\sigma_1^2 + \sigma_2^2) + c^2(\sigma_3^2) \tag{A.2}$$

The case where the 3-surface is RP^3 or a higher Lens space can be obtained from (A.2) by the $u(2)$ invariant identifications presented at the end of Section 2.

We now discuss the restricted form that the solution 4 geometries must have. We have restricted our class to be foliated by 3 geometries with $u(2)$ acting on 3-surfaces. Pick one

of these as an initial geometry and we can thus put it in the form (A.2). Now the field equations imply that the second fundamental form is diagonal with equal eigenvalues in the *same* basis. Thus at least locally the metric takes the form (2.1) and hence can be brought to the form of the solutions that we use in the main text.

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