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By

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I. INTRODUCTION

Line successive over relaxation (LSOR) is an iterative method for solving elliptic differential equations.[1] LSOR takes advantage of the CRAY vector capabilities as compared to the point successive over relaxation (SOR) method, which does not vectorize. The substantial advantages of LSOR on a vectorizing machine are not well-known, except in the field of aerodynamics.[2,3] By minor modification of the traditional SOR elliptic equation solver, we find that in certain coordinates an increase of a factor of two or greater in convergence time can be realized.

As a model problem for comparison of SOR and LSOR, the numerical solution of Poisson's equation will be reviewed in Sec. II. In Sec. III, we discuss the decreased computation time on the National Magnetic Fusion Energy Computer Center (NMFECC) CRAY computers found with LSOR applied to the iterative solution of plasma equilibria. In Sec. IV, the conditions for which LSOR is most useful are summarized.

II. SOLUTION OF POISSON'S EQUATION BY SOR AND LSOR

Centered finite differencing of Poisson's equation

$$\nabla^2 P = S$$

for a 10x10 cylindrical $[r(i), \theta(j)]$ grid leads to

$$\frac{r}{36(r^2+1)}\left[P_{i+1,j} - P_{i-1,j}\right] + \frac{r^2}{2(r^2+1)}\left[P_{i+1,j} + P_{i-1,j}\right] + \frac{1}{2(r^2+1)}\left[P_{i,j+1} + P_{i,j-1}\right] - P_{ij} = \frac{S_{ij}\, r^2}{162(r^2+1)} .$$

In the SOR technique, the nth iteration of P is calculated from

$$P^n_{ij} = P^{n-1}_{ij} + \omega R_{ij} .$$

Here the residual R(i,j) is

$$R_{ij} = \frac{r}{36(r^2+1)}\left[P^{n-1}_{i+1,j} - P^n_{i-1,j}\right] + \frac{r^2}{2(r^2+1)}\left[P^{n-1}_{i+1,j} + P^n_{i-1,j}\right] + \frac{1}{2(r^2+1)}\left[P^{n-1}_{i,j+1} + P^n_{i,j-1}\right] - P^{n-1}_{ij} - \frac{S^{n-1}_{ij}\, r^2}{162(r^2+1)} .$$

Varying the SOR parameter ω optimizes convergence time.

In the LSOR[4] method, we write the set of finite differenced equations for all points or the (i,j) grid in matrix form:

$$\begin{bmatrix} D_1 & U_2 & & \\ L_1 & D_2 & U_3 & \\ & L_2 & D_3 & U_4 \\ & & & \ddots \end{bmatrix} P = S$$

Here $P = (P_1, P_2 \ldots\ldots)$ and each vector P_j represents a whole row of grid points. Each row P_j can be found from

$$P_j^T = D_j^{-1}\left[S_j - L_j P_{j-1}^n - U_{j+1} P_{j+1}^{n-1}\right],$$

via a tridiagonal matrix inversion. The vector P_j^T is used to increment the n-1st vector P_j^{n-1}.

$$P_j^n = (1 - \omega)\, P_j^{n-1} + \omega\, P_j^T .$$

The point SOR method uses the advanced (k+1) values at two neighboring points (i-1,j) and (i,j-1). Line SOR uses the advanced (k+1) values at three neighboring points and so slightly improves the convergence rate on a scalar computer.[4]

To compare the speed of SOR and LSOR on a vectorizing computer, solutions of Poisson's equation with a nonlinear source

$$\nabla^2 P = P^2 \exp\left(P^2\right)$$

were obtained in a cylindrical geometry with cyclic boundary conditions. The equation was solved with tridiagonal solution for rows at constant radius and also for rows at constant θ (Fig. 1). It was found that LSOR with tridiagonal solution along θ (at constant radius) was fastest. This converged ten times faster than SOR when sweeping radial grid points at constant θ. These conclusions also hold for a stationary linear source.

In cylindrical coordinates LSOR was significantly faster than SOR for computations requiring many iterations. With an error criterion requiring at least 400 SOR iterations for convergence, LSOR was much faster than SOR (Fig. 1). LSOR and SOR were found to be equally fast when only ten SOR iterations were needed. Imposed updown symmetry would further speed LSOR convergence.

When Poisson's equation was solved in a rectangular geometry, optimized SOR and LSOR were found to have equal running times on the CRAY-1 for both linear and nonlinear source terms. Increasing the number of iterations required did not increase the relative convergence time of LSOR to SOR in rectangular coordinates.

III. SOR AND LSOR SOLUTIONS OF PLASMA EQUILIBRIA

Replacing SOR and LSOR in the Princeton equilibrium code EQ accelerates convergence for a typical plasma fixed boundary equilibrium. EQ solves the Grad-Shafranov equation[5,6] for the poloidal flux function, X, which is derived from the equilibrium force balance between the magnetic and kinetic pressures in tokamak plasmas. This equation has a nonlinear source since the pressure and toroidal field functions are dependent on X.

The equation is solved in magnetic flux coordinates by the method of DeLucia, Jardin, and Todd.[7] The equation is solved by iteration for the poloidal flux function. This is used to compute a new magnetic coordinate system via the Jacobean constraint as well as by matching the total measured current and central plasma pressure. Then the Grad-Shafranov equation is solved again in the new coordinate system. The coordinate readjustment and the iterative Grad-Shafranov solution continue until the error criterion is no longer exceeded.

In EQ, the equilibrium solution is obtained in a nonorthogonal cylindrical geometry (θ,r) in the poloidal plane. The convergence time required for EQ with SOR was the same whether θ or r is the direction of successive sweeps. However, the convergence time for EQ with LSOR is about three times faster for tridiagonal solutions along θ rather than along r for a typical Poloidal Divertor Experiment (PDX) plasma simulation.

Figure 2 presents the total cpu time required for convergence of EQ as a function of the relaxation parameter. The results of SOR and LSOR calculations are shown for a Joint European Torus (JET) simulation. We find that optimized LSOR reduces cpu time required for convergence. The LSOR JET simulation convergence is faster than SOR by a factor of four, while a LSOR PDX simulation (not shown) converged faster by a factor of two. The JET simulation represents a noncircular, low aspect ratio, high current plasma. This is an equilibrium with steeper gradients, a more nonlinear grid, and requires a more time-consuming calculation. For this case LSOR is faster than SOR at all values of ω.

Another approach to obtaining a fast accurate two-dimensional plasma MHD equilibrium is that of Lao, Hirschman, and Wieland.[8] They developed a variational moments method which takes about 0.2 seconds of CRAY time to compute an equilibrium with a relative error of 10^{-3} for three amplitude functions. For a Princeton PDX simulation, EQ with SOR iteration and error criterion

$$\frac{\max_{ij} |\psi^n - \psi^{n-1}|}{\langle|\psi^n|\rangle_{ij}} < 10^{-5} ,$$

converges in 9 seconds of CRAY time for 80 poloidal points and imposed up-down symmetry. ψ is the toroidal flux function. EQ with LSOR obtains the same solution in 4.5 seconds for this nearly circular fixed boundary case.

The LSOR-modified EQ code runs more quickly than EQ with SOR. It is potentially more accurate than the Lao-Hirschman-Wieland moments code for high beta equilibria having steep gradients, since it computes with all Fourier amplitude functions. It should more dramatically surpass the SOR EQ code's convergence speed for noncircular, free boundary equilibria.

IV. CONCLUSION

On scalar computing machines, LSOR is only slightly faster than SOR.[9] LSOR converges in fewer iterations than point SOR but each iteration may take longer because of the implicit tridiagonal solution required. SOR is an adequate iterative method for solution of elliptic equations when the problem (a) is cast in rectangular coordinates, (b) does not require a great many iterations, or (c) must be solved on a scalar computer. On a vectorizing machine problems requiring many iterations of a nonlinear elliptic equation in cylindrical geometry can be solved faster by LSOR.

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FIGURE CAPTIONS

FIG. 1. Cpu convergence time to solve Poisson's equation by LSOR and SOR methods as a function of over relaxation parameter ω. For LSOR-R and SOR-R the tridiagonal solution is obtained at constant θ. For LSOR-θ and SOR-θ the tridiagonal solution is obtained at constant radius.

FIG. 2. Cpu convergence time for EQ for a JET simulation for SOR and LSOR as a function of relaxation parameter ω.

Symbol Identification:

ω	lc omega
ψ	lc psi
χ	lc chi
θ	lc theta

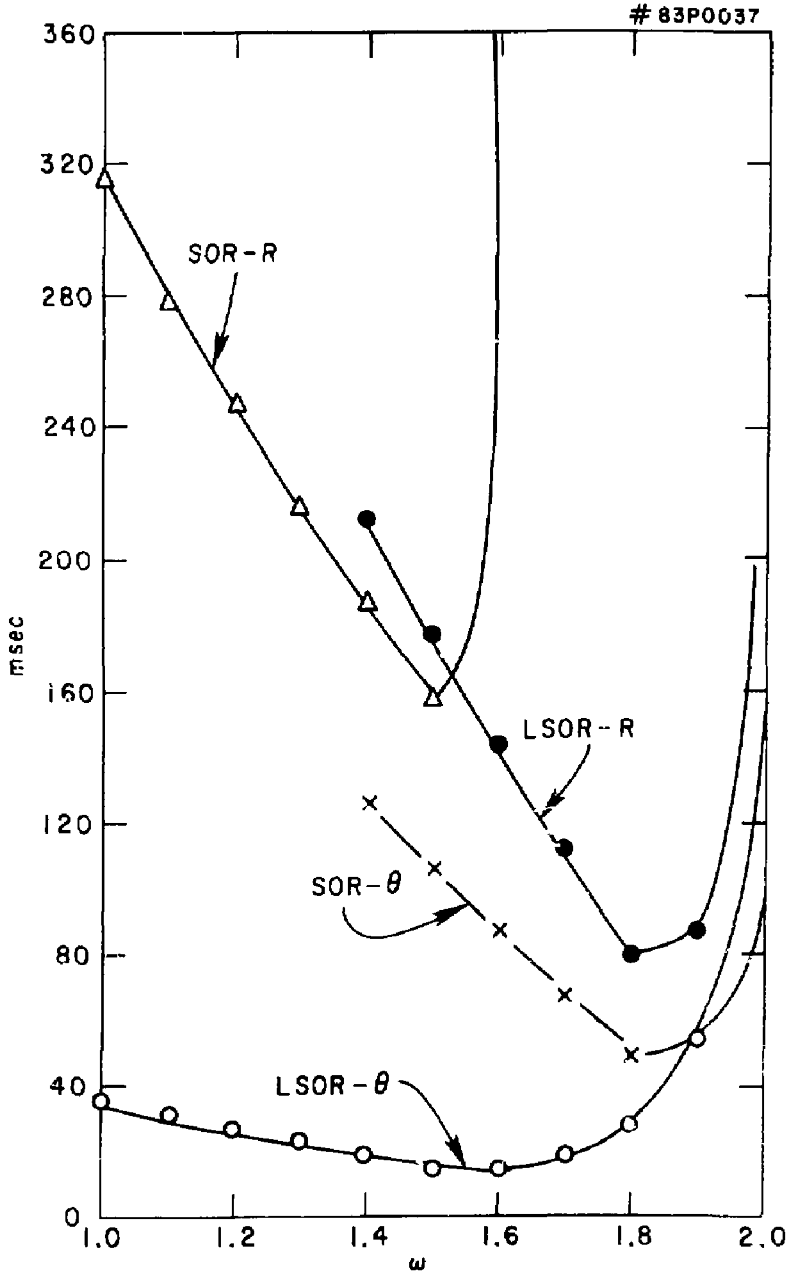
#83P0037
360
320
280
240
200
160
120
80
40
0
msec
1.0
1.2
1.4
1.6
1.8
2.0
ω
SOR-R
LSOR-R
SOR-θ
LSOR-θ

Fig. 1

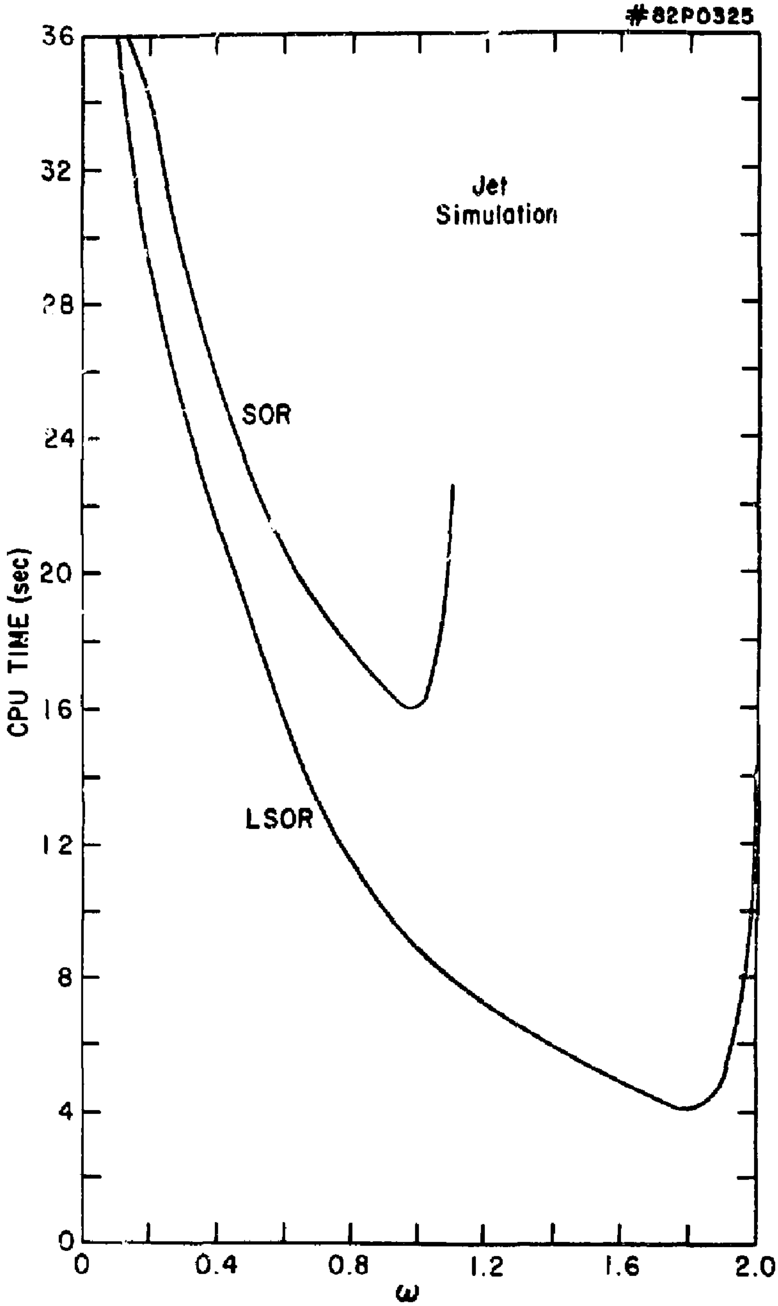
#82P0325
Jet Simulation
SOR
LSOR
CPU TIME (sec)
ω
0
4
8
12
16
20
24
28
32
36
0.4
0.8
1.2
1.6
2.0

Fig. 2

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