DOE/ER/03443-80

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

University of Maryland
College Park

Technical Note BN-906

ON UNIQUENESS OF NEUMANN-TRICOMI PROBLEM IN $\mathbb{R}^2$

MASTER

by

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and

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April 1979

DISCLAIMER

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*Research supported in part by the Department of Energy under contract DE AS05-76ERO 3443.

1. Introduction

Consider the equation

$$L[u] = K(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y) \quad , \tag{1.1}$$

in a bounded simply connected region G, where $K(y) \gtrless 0$ whenever $y \gtrless 0$ and the region G is bounded by the curves: A piecewise smooth curve Γ_o lying in the half-plane $y > 0$ which intersects the line $y = 0$ at the points $A(-1,0)$ and $B(0,0)$. For $y < 0$ by a piecewise smooth curve Γ_1 through A which meets the characteristic of (1.1) issued from B at the point P and the curve Γ_2 which consists of the portion PB of the characteristic through B.

In this paper using a variation of the energy-integral method (a,b,c-method) similar to the one used in [1], we obtain sufficient conditions for the uniqueness of the solution of the boundary value problem

$$L[u] = f \quad \text{in} \quad G \tag{1.2}$$

$$d_n u := K(y)u_x dy - u_y dx|_{\Gamma_o} = \psi(s)ds \tag{1.3}$$

$$u|_{\Gamma_1} = \phi(s) \quad . \tag{1.4}$$

The question of uniqueness and existence of solutions of this Neumann-Tricomi boundary value problem has been dealt with for instance by L. S. Čubenko [4] in the case $K(y) = \text{sign } y|y|^m$, $m > 0$ when Γ_1 coincides with the characteristic through A and instead of (1.3) the values $d_n u + \alpha(s)u\, ds$ on Γ_o are known. For $r(x,y) \leq 0$, $\alpha(s) \geq 0$ the uniqueness is proved with the maximum-minimum principle and the existence of a solution in a special function class under further conditions on Γ_o by the integral-equation method. V. F. Egorov proved in [5] the uniqueness of the described boundary value problem in the case $K(y) = y$, $r(x,y) \equiv 0$ if Γ_1 has the special form $x = -1 + C(-y)^{3/2}$, $C \geq \frac{2}{3}$ and Γ_o satisfies the boundary condition $3xdy - 2ydx|_{\Gamma_o} \geq 0$.

In the papers [6] and [7] he generalized this uniqueness theorem to the case $K(y) = \text{sign}\, y|y|^m$, $m \geq 0$, $r(x,y) \equiv 0$,

$\Gamma_1 : x = -1 + C(-y)^{\frac{m}{2}+1}$, $C \geq \frac{2}{m+2}$ and $|C - \frac{2}{m+2}|$ is "sufficient small". -

D. V. Koračev and K. I. Mihaĭlov [8] treated the Neumann-Tricomi problem in case $K(y) = y^m$ for $y > 0$, $K(y) = -(-y)^n$ for $y < 0$, $K(0) = 0$, $r(x,y) \equiv 0$, when Γ_1 is a characteristic through A . By use of the maximum-minimum principle the uniqueness and, with the integral-equation method, the existence of a solution are shown. For a representation of the integral-equation method for the Neumann-Tricomi problem when Γ_1 is a characteristic we refer to [11] and for the existence of weak solution to [3].

To our knowledge in all the papers connected with uniqueness results for the problem (1.2), (1.3), (1.4) when Γ_1 is not a characteristic, the assumptions $K(y) = \text{sign}|y|y^m$, $m > 0$, $r(x,y) = 0$ and that Γ_1 is of special form play an essential role.

In the present paper we give an uniqueness theorem for a "general" function $K(y)$, when $r(x,y)$ is not necessarily zero and Γ_1 is of a more general form.

2. Preliminary Lemmas

We consider the differential operator

(2.1) $$L[u] = K(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y) \quad ,$$

where

(2.2)
$$K(y) \gtrless 0 \quad \text{for} \quad y \gtrless 0, \quad K(y) \in C^o(\overline{G}) \cap C^3(\overline{G}_+) \cap C^3(\overline{G}_-) \quad ,$$

$$r(x,y) \in C^1(\overline{G}_+) \cap C^1(\overline{G}_-), \quad f(x,y) \in C^o(\overline{G})$$

and

$$G_+ = G \cap \{y > 0\}, \quad G_- = G \cap \{y < 0\} \quad .$$

Remark 1

The assumptions (2.2) can be weakened; see theorem 3.1. We introduce the Pfaffian form of first degree

(2.3)
$$\begin{aligned}\Omega &= 2(\alpha^o u+\alpha^1 u_x+\alpha^2 u_y)d_n u+(ru^2-K(y)u_x^2-u_y^2)(\alpha^1 dy-\alpha^2 dx)-u^2 d_n\alpha^o \\ &= (K(y)u_x^2-u_y^2)(\alpha^1 dy+\alpha^2 dx)+2u_x u_y(K\alpha^2 dy-\alpha^1 dx) \\ &\quad + 2\alpha^o u(K(y)u_x dy-u_y dx) \\ &\quad + u^2\left\{(r\alpha^1-K(y)\alpha^o_x)dy-(r\alpha^2-\alpha^o_y)dx\right\} \quad ,\end{aligned}$$

where $d_n u = K(y)u_x dy - u_y dx$, and obtain

(2.4)
$$\begin{aligned}[d,\Omega] &= 2(\alpha^o u+\alpha^1 u_x+\alpha^2 u_y)L[u][dx,dy] \\ &\quad + (Au_x^2+2Bu_x u_y+Cu_y^2+Du^2)[dx,dy]\end{aligned}$$

where

$$(2.5)\quad \begin{aligned} A &= K(y)\left(\alpha^1_x - \alpha^2_y\right) - \alpha^2 K'(y) + 2K(y)\alpha^o \quad , \\ B &= K(y)\alpha^2_x + \alpha^1_y \quad , \\ C &= -\left(\alpha^1_x - \alpha^2_y\right) + 2\alpha^o \quad , \\ D &= -K(y)\alpha^o_{xx} - \alpha^o_{yy} - 2\alpha^o r + (\alpha^1 r)_x + (\alpha^2 r)_y \quad . \end{aligned}$$

By use of Green's theorem we have

$$(2.6)\qquad \int_{\partial G_+ \cup \partial G_-} \Omega = \iint_{G_+ \cup G_-} [d,\Omega] \quad .$$

We shall show that under suitable assumptions on Γ_o , Γ_1 and the coefficients of (2.1), the functions α^i, $i = 0,1,2$ can be determined so that

$$(2.7)\qquad 0 \leq \int_{\partial G_+ \cup \partial G_-} \Omega = \iint_{G_+ \cup G_-} [d,\Omega] \leq 0 \quad ,$$

thus it will follow from (2.7) that $u \equiv 0$.

Lemma 2.1

If

$$\alpha^1 = -(x+1)$$

$$\alpha^2 = +|K(y)|^{-1/2} \int_{t=y}^{0} |K(t)|^{1/2} dt$$

$$\alpha^0 = \text{sign } y \ \frac{1}{4} K'(y)|K(y)|^{-3/2} \int_{t=y}^{0} |K(t)|^{1/2} dt$$

then

$$A = B = C \equiv 0 \quad .$$

Proof

The conclusion of the lemma follows at once by substitution of α^i, $i = 0,1,2$ (as given in the lemma) in (2.5).

Remark 2

For the special case

$$K(y) = \text{sign } y|y|^m, \quad m > 0 \tag{2.8}$$

we conclude from Lemma 2.1

$$\alpha^1 = -(x+1), \quad \alpha^2 = -\frac{2}{m+2}\, y, \quad \alpha^0 = -\frac{1}{2}\,\frac{m}{m+2} \quad .$$

Lemma 2.2

If the Pfaffian form Ω is as in (2.3) and the function u satisfies the boundary conditions (1.3) and (1.4) then

(a) $$\int_{\Gamma_o} \Omega = \int_{\Gamma_o} \{(ru^2-K(y)u_x^2-u_y^2)(\alpha^1 dy-\alpha^2 dx)-u^2 d_n\alpha^0\} \quad ,$$

(b) $$\int_{\Gamma_1} \Omega = \int_{\Gamma_1} (K(y)u_x^2+u_y^2)(\alpha^1 dy-\alpha^2 dx)$$

$$= \int_{\Gamma_1} u_y^2 \left\{K(y)\left(\frac{dy}{dx}\right)^2+1\right\} (\alpha^1 dy-\alpha^2 dx) \quad ,$$

(c)
$$\int_{\Gamma_2} \Omega = \int_P^B \Omega = -\alpha^0(-K)^{1/2}\, u^2 \Big|_P^B$$

$$+ \int_P^B \Big\{ -[(-K)^{1/2} u_x + u_y]^2 (\alpha^1 dy + \alpha^2 dx) +$$

$$+ u^2 \Big[d[\alpha^0(-K)^{1/2}] + (-K)^{1/2} d\alpha^0 + r(\alpha^1 dy - \alpha^2 dx) \Big] \Big\} .$$

Proof

The statement (a) follows directly from (2.3) and (1.3). $u|_{\Gamma_1} = 0$ implies $u_x dx + u_y dy|_{\Gamma_1} = 0$ on the smooth parts of Γ_1. Thus from (2.3) we have

$$\Omega|_{\Gamma_1} = K(y) u_x^2 \alpha^1 dy + K(y) u_x^2 \alpha^2 dx - u_y^2 \alpha^1 dy - u_y^2 \alpha^2 dx +$$

$$+ 2u_x u_y K(y) \alpha^2 dy - 2u_x u_y \alpha^1 dx .$$

Observing that on Γ_1

$$2u_x u_y K(y) \alpha^2 dy = -2u_x^2 K(y) \alpha^2 dx ,$$

$$-2u_x u_y \alpha^1 dx = 2u_y^2 \alpha^1 dy ,$$

we have

$$\Omega|_{\Gamma_1} = \left(K(y) u_x^2 + u_y^2\right)\left(\alpha^1 dy - \alpha^2 dx\right) .$$

Now since $u_x dx + u_y dy|_{\Gamma_1} = 0$, (b) follows. Since on $\Gamma_2 : x + \int_{t=y}^{0} (-K(t))^{\frac{1}{2}} dt = 0$ we have $dx - (K(y))^{\frac{1}{2}} dy = 0$, $d_n u|_{\Gamma_2} = K(y)u_x dy - u_y dx = -(-K(y))^{\frac{1}{2}} du$. Thus

$$\Omega|_{\Gamma_2} = -2\alpha^o(-K(y))^{\frac{1}{2}} u du + 2(\alpha^1 u_x + \alpha^2 u_y)(K(y)u_x dy - u_y dx) \tag{2.9}$$
$$- (K(y)u_x^2 + u_y^2)(\alpha^1 dy - \alpha^2 dx) + u^2\left[r(\alpha^1 dy - \alpha^2 dx) + (-K)^{\frac{1}{2}} d\alpha^o\right] .$$

An integration by parts yields

$$\int_P^B -2\alpha^o(-K)^{\frac{1}{2}} u du = -\int_P^B \alpha^o(-K)^{\frac{1}{2}} du^2$$

$$= -\alpha^o(-K)^{\frac{1}{2}} u^2 \Big|_P^B + \int_P^B u^2 d[\alpha^o(-K)^{\frac{1}{2}}]$$

so we obtain by a simple calculation from (2.9) the statement (c).

<u>Remark 3</u>

If Γ_1 has the characteristic direction, then $\int_{\Gamma_1} \Omega = 0$. This follows at once from the observations that

$$(-K(y))^{\frac{1}{2}} dy - dx|_{\Gamma_1} = 0 , \quad u_x dx + u_y dy|_{\Gamma_1} = 0 ,$$

which implies $K(y)u_x^2 + u_y^2|_{\Gamma_1} = 0$. In this case see [2] .

3. Neumann-Tricomi Problem

We call a function $u(x,y) \in C^{o}(G)$ a quasi-regular solution of (2.1) if the following hold ([9], p. 234):

i) $u(x,y)$ satisfies (2.1);

ii) The integral $\int_{G \cap \{y=0\}} \Omega$ exists;

iii) If $G_{\pm}(\varepsilon)$ are regions with boundary $\partial G_{\pm}(\varepsilon)$ lying entirely in G_+ and G_-, then the line integrals along $\partial G_{\pm}(\varepsilon)$ which result from the application of Green's theorem to the integrals:

$$\iint_{G_{\pm}(\varepsilon)} u\, L[u]dxdy \ , \quad \iint_{G_{\pm}(\varepsilon)} u_x\, L[u]dxdy \ , \quad \iint_{G_{\pm}(\varepsilon)} u_y\, L[u]dxdy$$

have a limit when $\partial G_{\pm}(\varepsilon)$ approaches the boundary of G_+ and G_-.

First we prove the uniqueness theorem for a special case:

Theorem 3.1

The equation

$$L[u] = K(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y)$$

where

(2.8) $K(y) = (\text{sign } y)|y|^m, \quad m > 0; \quad r(x,y) \in C^1(\overline{G}_+) \cap C^1_-(G), \quad f(x,y) \in C^o(G)$

has at most one quasi-regular solution in G satisfying the boundary conditions

$$d_n u|_{\Gamma_o} = K(y) u_x dy - u_y dx = \psi(s) ds \quad ,$$

$$u|_{\Gamma_1} = \phi(s)$$

if the following conditions are satisfied:

i) $\alpha = (\alpha^1, \alpha^2) = (-(x+1), -\frac{2}{m+2} y)$

$-D = \frac{4}{m+2} r - \alpha \cdot \text{grad } r \geq 0$ in G, and $-D \not\equiv 0$ in G_+ in case $r(x,y) \not\equiv 0$ in G_+ .

ii) Γ_1 lies inside the characteristic triangle determined by the line $\overline{AB}$ and the two characteristics of (2.1) through A and B which intersect in C and satisfies

(3.2) $$\left(K(y)\left(\frac{dy}{dx}\right)^2 + 1\right)\left(-(m+2)(x+1)dy + 2ydx\right)\Big|_{\Gamma_1} \geq 0 \quad .$$

iii) $r(x,y)|_{\Gamma_2} \leq 0$, $r(x,y)|_{\Gamma_o} \leq 0$

(3.3) $$(m+2)(x+1)dy - 2ydx|_{\Gamma_o} > 0 \quad .$$

Remark 4

See the remarks at the end of the proof concerning the possible forms of the curves Γ_o and Γ_1 .

Proof

Suppose there exist two solutions u_1 and u_2 . Let $u = u_1 - u_2$, by (2.4)

$$[d,\Omega] = (Au_x^2 + 2Bu_xu_y + Cu_y^2 + Du^2)[dx,dy]$$

and with the choice (see Lemma 2.1)

$$(3.4) \qquad \alpha^1 = -(x+1), \quad \alpha^2 = -\frac{2}{m+2}\, y, \quad \alpha^o = -\frac{1}{2}\,\frac{m}{m+2}$$

we obtain

$$Q(u_x,u_y) = Au_x^2 + 2Bu_xu_y + Cu_y^2 \equiv 0 \quad \text{in} \quad G,$$

$$D = -\frac{4}{m+2}\, r + \alpha\cdot \text{grad } r \quad .$$

From Lemma 2.2 and (3.4), under the assumptions (3.2) and iii) we obtain

$$(3.5) \qquad \int_{\Gamma_o} \Omega \geq 0, \qquad \int_{\Gamma_1} \Omega \geq 0 \quad .$$

On $\Gamma_2 : x + \frac{2}{m+2}(-y)^{\frac{m}{2}+1} = 0$, $dx - (-y)^{\frac{m}{2}} dy = 0$, and

$$-(\alpha^1 dy + \alpha^2 dx) = \left((x+1)dy + \frac{2}{m+2}\, y\, dx\right)$$

$$= \left\{(x+1) - \frac{2}{m+2}(-y)^{\frac{m}{2}+1}\right\} dy \geq 0$$

because Γ_2 consists of points above the characteristic AC given by $x+1 - \frac{2}{m+2}(-y)^{\frac{m}{2}+1} = 0$. From $\alpha^o = -\frac{1}{2}\,\frac{m}{m+2}$ and $r|_{\Gamma_2} \leq 0$ if follows that

$$d[\alpha^o(-K)^{\frac{1}{2}}] + r(\alpha^1 dy - \alpha^2 dx)|_{\Gamma_2} \geq 0 \quad .$$

Thus we obtain under the hypothesis of theorem 3.1

$$0 \leq \int_{\partial G_+ \cup \partial G_-} \Omega = \iint_{G_+ \cup G_-} [d,\Omega] \leq 0$$

and $u|_{\Gamma_2} = 0$. By hyperbolic theory we have $u \equiv 0$ in G_-, but this implies $u \equiv 0$ in case $r(x,y) \equiv 0$ in G_+ by the maximum principle, otherwise we get $u \equiv 0$ in G_+ by the hypotheses i).

- As it can be seem from the proof, the condition $r(x,y)|_{\Gamma_2} \leq 0$ can be replaced by the "weaker" condition

$$d[\alpha^o(-K)^{\frac{1}{2}}] + r(\alpha^1 dy - \alpha^2 dx)|_{\Gamma_2} \geq 0 \quad . \quad -$$

<u>Remark 5</u> (Boundary curve Γ_1)

By theorem 3.1 Γ_1 is a piecewise smooth curve inside the characteristic triangle and must satisfy

$$\left(K(y)\left(\frac{dy}{dx}\right)^2 + 1\right)\left(-(m+2)(x+1)dy + 2ydx\right)\Big|_{\Gamma_1} \geq 0 \quad .$$

The solutions of

$$\omega_o = -(m+2)(x+1) + 2ydx = 0$$

through $A(-1,0)$ lying inside the characteristic triangle are given by

(3.6) $$x = -1 + C(-y)^{\frac{m}{2}+1} \quad \text{with} \quad C \geq \frac{2}{m+2} \quad .$$

(3.6) gives the possible forms of the boundary curves Γ_1 which satisfy (3.2) and where we have $\int_{\Gamma_1} \Omega = 0$. For $m = 1$ we obtain the result of V. P. Egorov in [5].

We observe that the characteristic directions $Q(dx,dy) = 0$ through an arbitrary point $(x_o,y_o) \in G_-$ divide a neighborhood of (x_o,y_o) in four parts where $Q(dx,dy) = K(y)\left(\frac{dy}{dx}\right)^2 + 1 \gtrless 0$. Similarly the solution of $\omega_o \equiv -(m+2)(x+1)dy + 2y\,dx = 0$ through (x_o,y_o) divides a neighborhood of (x_o,y_o) in two parts where $\omega_o > 0$ and $\omega_o < 0$ respectively. From the point (x_o,y_o) the curve Γ_1 may proceed in a direction (dx,dy) where $Q\cdot\omega_o \geq 0$.

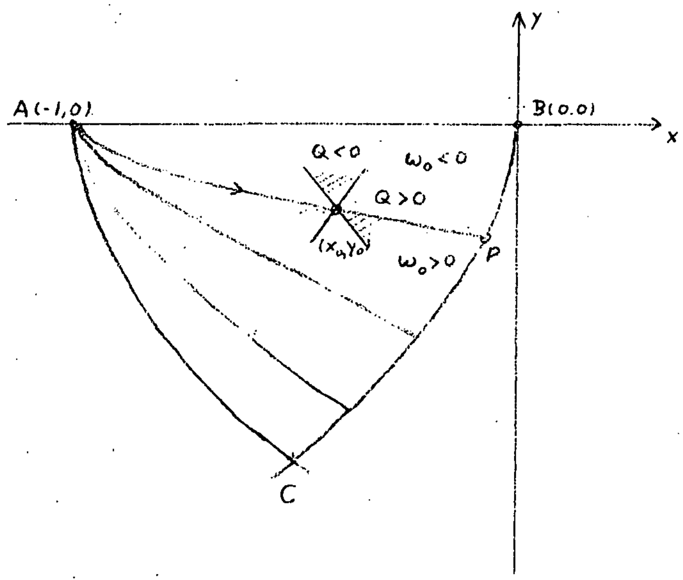

Fig. 1

The possible directions are indicated in figure 1. It is clear from theorem 3.1 (condition 3.2) that all possible boundary curves Γ_1 have a vertical tangent in A(-1,0).

<u>Remark 6</u> (Boundary curve Γ_0)

By theorem 3.1 Γ_0 is a piecewise smooth curve which must satisfy

(3.3) $$\omega_1 \equiv (m+2)(x+1)dy - 2ydx|_{\Gamma_0} \geq 0 \ .$$

We note that the boundary of the rectangle in Figure 2 below satisfies the condition (3.3). We are interested in other possible piecewise smooth curves which intersect the line $y = 0$ in $A(-1,0)$ and $B(0,0)$.

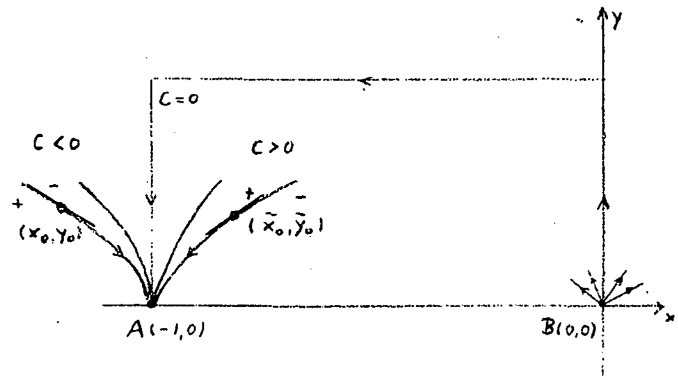

Fig. 2

A(-1,0): The condition (3.3) is satisfied for the solutions of $\omega_1 = 0$, which are given by

(3.7) $$x = -1 + Cy^{\frac{m}{2}+1} \ , \quad -\infty < C < +\infty \ .$$

Let (x_o,y_o) be an arbitrary point such that $x_o < -1, y_o > 0$, then there exists a solution of $\omega_1 = 0$ through (x_o,y_o) which is given by

(3.8) $$x = -1 + \frac{1+x_o}{y_o^{\frac{m}{2}+1}} \ y^{\frac{m}{2}+1} \ .$$

The solution (3.8) through (x_o,y_o) divide a neighborhood of (x_o,y_o) in two parts where $\omega_1 > 0$ and $\omega_1 < 0$ respectively. Starting from (x_o,y_o) to reach $A(-1,0)$ the curve Γ_o may proceed in a direction (dx,dy) where $\omega_1 \geq 0$. As it is easily seen, the line which connects (x_o,y_o) is a possible part of Γ_o .

Starting from a point $(\tilde{x}_o, \tilde{y}_o)$ such that $-1 < \tilde{x}_o$, $\tilde{y}_o > 0$, the same consideration shows that by condition (3.3) a possible curve Γ_o which intersects $y = 0$ in $A(-1,0)$ must have either a vertical tangent in $A(-1,0)$ or a negative derivative.

B(0,0): Writing the condition (3.3) $\omega_1 \geq 0$ in the form

$$\omega_1 = \Big((m+2)(x+1),\ 2y\Big) \cdot n \geq 0 \ , \ n = (dy, -dx)$$

we see, that for a curve starting at $B(0,0)$ all direction are possible.

For other types of conditions on Γ_o for the Neumann-problem see C.S. Morawetz [10].

Before returning to the Neumann-Tricomi problem for a general function $K(y)$, we make some remarks concerning a problem which is similar to the problem considered by D. V. Koračev and K. I. Mihaĭlov in [8].

Remark 7

Instead of the special case (2.8) we consider the problem

$$K(y) = \begin{cases} y^n & \text{for } y > 0 \ , \ n > 0 \\ -(-y)^m & \text{for } y < 0 \ , \ m > 0 \ . \end{cases} \tag{3.9}$$

With the choice

$$G_+ : \alpha^1 = -(x+1),\ \alpha^2 = -\frac{2}{n+2}\, y,\ \alpha^0 = -\frac{1}{2}\frac{n}{n+2} \quad ;$$

$$G_- : \alpha^1 = -(x+1),\ \alpha^2 = -\frac{2}{m+2}\, y,\ \alpha^0 = -\frac{1}{2}\frac{m}{m+2}$$

the considerations in theorem 3.1 remain the same. On Γ_o we get instead of (3.3)

$$(n+2)(x+1)dy - 2ydx\Big|_{\Gamma_o} \geq 0 \tag{3.10}$$

and theorem 3.1 is valid if the line integral (see (2.7))

$$\int_A^B (-\Omega_- + \Omega_+) = \int_A^B \Big\{(\alpha_-^2 - \alpha_+^2)u_y^2 + 2(\alpha_-^1 - \alpha_+^1)u_x u_y \tag{3.11}$$

$$+ 2(\alpha_-^0 - \alpha_+^0)uu_y + u^2\Big[r(\alpha_-^2 - \alpha_+^2) - (\alpha^0_{y_+} - \alpha^0_{y_-})\Big]\Big\} dx$$

$$= 2\int_A^B (\alpha_-^0 - \alpha_+^0)uu_y dx \geq 0 \quad .$$

As it can be seen from (2.3), (2.4) with $\alpha^1 = \alpha^2 = 0$, $\alpha^0 = -1$, considered in G^+

$$\int_A^B uu_y dx \leq 0 \quad \text{for} \quad r \leq 0 \quad \text{in} \quad G_+ ,$$

thus (3.11) is valid if $\alpha_-^0 - \alpha_+^0 \leq 0$, which means $n \leq m$. - The theorem 3.1 remains true in the case $K(y)$ is given by (3.9) if the condition (3.3) is replaced by (3.10), $r(x,y) \leq 0$ in G_+ and $0 < n \leq m$. -

The proof of theorem 3.1 indicates the approach for proving an analogous uniqueness theorem for the Neumann-Tricomi problem in the general case when the function $K(y) \gtrless 0$ for $y \gtrless 0$. The central idea is to choose the functions α^i , $i = 0,1,2$ so that the form $Q(u_x,u_y) = Au_x^2 + 2Bu_x u_y + Cu_y^2 \equiv 0$ in G .

Theorem 3.2

The equation

$$L[u] = K(y)u_{xx} + u_{yy} + r(x,y)u = f(x,y)$$

where

$$K(y) \gtrless 0 \quad \text{for} \quad y \gtrless 0 \ , \quad K(y) \in C^0(\overline{G}) \cap C^3(\overline{G}_+) \cap C^3(\overline{G}_-) \ ,$$

$$r(x,y) \in C^1(\overline{G}_+) \cap C^1(\overline{G}_-) \ , \quad f(x,y) \in C^0(\overline{G})$$

has at most one quasi-regular solution in G satisfying the boundary conditions $d_n u|_{\Gamma_o} = K(y)u_x d_y - u_y dx|_{\Gamma_o} = \psi(s)ds$, $u|_{\Gamma_1} = \phi(s)$ if the following conditions hold:

With

$$(3.12) \qquad \begin{aligned} \alpha^1 &= -(x+1) \ , \quad \alpha^2 = |K(y)|^{-1/2} \int_y^o |K(t)|^{1/2} dt \ , \\ \alpha^0 &= \text{sign } y \ \frac{1}{4} K'(y) \ |K(y)|^{-3/2} \int_y^o |K(t)|^{1/2} dt \ , \end{aligned}$$

i) $-D = K(y)\alpha^o_{xx} + \alpha^o_{yy} + 2r\alpha^o - (\alpha^1 r)_x - (\alpha^1 r)_y \geq 0$ in G with $-D \not\equiv 0$ in G_+ in case $r(x,y)$ is not identical zero in G_+ .

ii) Γ_1 lies inside the characteristic triangle determined by the line $\overline{AB}$ and the characteristics of (2.1) through A and B which intersect in C and satisfies

$$(3.13) \qquad \left(K(y) \left(\frac{dy}{dx}\right)^2 + 1\right)\left(\alpha^1 dy - \alpha^2 dx\right)\Big|_{\Gamma_1} \geq 0 \ ,$$

iii) $r(x,y)|_{\Gamma_2} \leq 0$, $2(-K)^{1/2}d\alpha^o + \alpha^o(-K)^{1/2}|_{\Gamma_2} \geq 0$,

$-(\alpha^1 dy - \alpha^2 dx)|_{\Gamma_o} \geq 0$; $r(\alpha^1 dy - \alpha^2 dx) - d_n\alpha^o|_{\Gamma_o} \geq 0$,

$\lim_{y\to 0\pm} \alpha^2 = 0$.

The proof of theorem 3.2 is similar to that of theorem 3.1 by the choice (3.12) of the functions $\alpha^0, \alpha^1, \alpha^2$ instead of (3.4). As it can be seen, all conditions on the functions $\alpha^0, \alpha^1, \alpha^2$ in theorem 3.2 can be written as conditions on the function $K(y)$. The corresponding remarks 4 and 5 are valid, where the curves (3.6) have to be replaced by the curves

$$x = -1 + C \int_{t=y}^{o} (-K(t)^{1/2}dt \ , \ C \geq 1 \ ,$$

for which we have

$$\alpha^1 dy - \alpha^2 dx = 0 \quad .$$

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