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EVEN- AND ODD-PARITY FINITE-ELEMENT TRANSPORT SOLUTIONS IN THE THICK DIFFUSION LIMIT

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ABSTRACT

We analyze the behavior of odd-parity continuous finite-element methods (CFEMs) for problems that contain diffusive regions. We find that each of these methods produces a solution that, to leading order inside diffusive regions, satisfies a discretization of the diffusion equation. We find further that these leading-order solutions satisfy boundary conditions that can lead to large errors in the interior solution. We recognize, however, that we can combine an odd-parity CFEM solution and an even-parity CFEM solution and obtain a solution that satisfies very accurate boundary conditions. Our analysis holds in three-dimensional Cartesian geometry, with an arbitrary spatial grid. We give numerical results from slab-geometry; these invariably agree with the predictions of the analysis. Finally, we introduce a rapidly-convergent diffusion-synthetic acceleration scheme for the odd-parity CFEMs, which we believe is new.

INTRODUCTION

Thermal-radiation transport problems of practical interest often contain optically thick, diffusive regions. Practical considerations usually force the use of a spatial grid whose cells in such regions are thick relative to a mean-free path. We are therefore interested in the performance of numerical transport methods in diffusive regions with optically thick spatial cells. A recent paper[1] details the behavior of even-parity continuous finite-element methods (CFEMs) in such regions, pointing out a class of diffusive problems for which these methods produce large errors. Ackroyd has suggested that odd-parity CFEMs may produce equal and opposite errors in such problems.[2] (This is based on observations by Ackroyd and Nanneh of a similar phenomenon in certain neutron-transport calculations.[3]) If Ackroyd's conjecture is correct, then the average of an even-parity and an odd-parity CFEM solution will be much more accurate than either individual solution. In this paper we find that this is the case.

We begin by studying the behavior of odd-parity CFEMs for problems containing optically thick, diffusive regions with optically thick spatial cells. The first part of our study is an asymptotic analysis,[1,4-6] with which we address an entire family of odd-parity CFEMs in three dimensions assuming an arbitrary spatial grid. The second part is numerical testing, with which we examine the particular case of linear elements in slab geometry. We find that in thick, diffusive regions, every odd-parity CFEM produces a solution that (to leading order) satisfies a discrete diffusion equation. This is a highly desirable result, for we know that in the interior of such regions the exact transport solution (to leading order) satisfies a diffusion equation.[7-10] Despite this, we find that in general odd-parity CFEM solutions can be inaccurate in diffusive regions, because they satisfies boundary conditions that in general are inaccurate. We recall the similar result that *even*-parity CFEM solutions satisfy a discrete diffusion equation with boundary conditions that in general are inaccurate.[1] However, we find that the *average* of the odd-parity and even-parity boundary conditions is very accurate. Thus, Ackroyd's conjecture appears to be correct: the average of even- and odd-parity CFEM solutions is very accurate in diffusive regions even when neither individual solution is accurate. This is the main result of this paper.

We also present a diffusion-synthetic acceleration (DSA) method for the iterative solution of our odd-parity CFEM equations. DSA is a powerful iteration method that has been used extensively with the first-order form of the transport equation,[11-14] and more recently with the even-parity form.[1,15,16] To our knowledge, however, this is the first DSA method developed for the odd-parity form of the transport equation.

ASYMPTOTIC ANALYSIS

We begin with a brief review of the behavior of the exact transport solution in a diffusive region. This material is taken from references [7-10]. We write the transport equation for a single energy, assuming isotropic sources and scattering, as:

$$\mathbf{\Omega}\cdot\nabla\Psi + \sigma_t\Psi(r,\mathbf{\Omega}) = \frac{1}{4\pi}(\sigma_t - \sigma_a)\Phi(r) + \frac{1}{4\pi}Q(r) \ , \qquad r \in D \ , \tag{1a}$$

$$\Psi(r,\mathbf{\Omega}) = F(r,\mathbf{\Omega}) \ , \qquad r \in \partial D \ , \ \mathbf{n}\cdot\mathbf{\Omega} < 0 \ , \tag{1b}$$

where Ψ is the angular flux, Φ is the scalar flux, F is the incident angular flux, σ_t is the total cross section, σ_a is the absorption cross section, and Q is a given source. We consider the *scaled*[9] transport problem:

$$\mathbf{\Omega}\cdot\nabla\Psi + \frac{\sigma_t}{\varepsilon}\Psi(r,\mathbf{\Omega}) = \frac{1}{4\pi}\left(\frac{\sigma_t}{\varepsilon} - \varepsilon\sigma_a\right)\Phi(r) + \frac{\varepsilon}{4\pi}Q(r) \ , \qquad r \in D \ , \tag{2}$$

and ask how the solution behaves as the small parameter ε tends toward zero. We find that in the interior of the diffusive region D (i.e., more than a few mean-free paths from ∂D), the leading-order angular flux is isotropic and satisfies a diffusion equation:

$$\Psi(r,\mathbf{\Omega}) = \frac{1}{4\pi}\Phi(r) + O(\varepsilon) \ , \qquad r \textit{ away from } \partial D \ , \tag{3a}$$

$$-\nabla\cdot\frac{1}{3\sigma_t}\nabla\Phi + \sigma_a\Phi(r) = Q(r) \ , \tag{3b}$$

$$\Phi(r_s) = 2\int_{\mathbf{n}\cdot\mathbf{\Omega}<0} d\Omega \, W(|\mathbf{n}\cdot\mathbf{\Omega}|)F(r_s,\mathbf{\Omega}) \ , \qquad r_s \in \partial D \ . \tag{3c}$$

We note that the boundary condition (3c) is a Dirichlet condition equal to a weighted integral of the incoming intensity. The weight function W is defined in terms of Chandrasekhar's H-function for a purely scattering medium[17]:

$$W(\mu) = \frac{\sqrt{3}}{2}\mu H(\mu) \ . \tag{4}$$

When interpreting the results of the asymptotic analysis of *numerical* transport methods, we use the fact that W is approximately equal to a simple polynomial:

$$W(\mu) = 0.956\mu + 1.565\mu^2 \pm 0.0035 \qquad \approx \quad \mu + 1.5\mu^2 \ . \tag{5}$$

Derivation of Odd-Parity CFEMs

We turn now to the odd-parity CFEM equations. We begin with the odd-parity equation and its boundary condition:

$$-\mathbf{\Omega}\bullet\nabla\left(\frac{1}{\sigma_t}\mathbf{\Omega}\bullet\nabla\Psi_{odd}\right) + \sigma_t\Psi_{odd} = -\mathbf{\Omega}\bullet\nabla\left(\frac{[\sigma_t-\sigma_a]\Phi+Q}{4\pi\sigma_t}\right) , \tag{6a}$$

$$\nabla\bullet\mathbf{J} + \sigma_t\Phi = [\sigma_t-\sigma_a]\Phi+Q \ , \tag{6b}$$

$$\left[\Psi_{odd} - \frac{1}{\sigma_t}\mathbf{\Omega}\bullet\nabla\Psi_{odd} + \frac{[\sigma_t-\sigma_a]\Phi+Q}{4\pi\sigma_t}\right]_{(r_s,\mathbf{\Omega})} = F(r_s,\mathbf{\Omega}) \ , \qquad r_s \in \partial D \ , \ \mathbf{n}\bullet\mathbf{\Omega}<0 \ , \tag{6c}$$

where

$$\Psi_{odd}(r,\mathbf{\Omega}) \ \equiv \ \frac{1}{2}\left(\Psi(r,\mathbf{\Omega}) - \Psi(r,-\mathbf{\Omega})\right) \ ; \qquad \mathbf{J}(r) \ \equiv \ \int_{4\pi} d\Omega \, \mathbf{\Omega} \, \Psi_{odd}(r,\mathbf{\Omega}) \ .$$

We obtained these equations by writing the first-order transport equation for $+\mathbf{\Omega}$ and $-\mathbf{\Omega}$, adding and subtracting, and then manipulating the resulting equations. (This formulation of the odd-parity equations is used by Morel et. al.[16] Unlike many odd-parity formulations,[3] it does not involve explicit inversion of the scattering operator.) We note that the incoming angular flux, F, is defined only for $\mathbf{n}\bullet\mathbf{\Omega}<0$; we extend its definition to all $\mathbf{\Omega}$ in such a way that it is an odd function of $\mathbf{\Omega}$:

$$F(r_s,\mathbf{\Omega}) \ \equiv -F(r_s,-\mathbf{\Omega}) \text{ for } \mathbf{n}\bullet\mathbf{\Omega}>0 \ . \tag{7}$$

This allows us to manipulate the boundary condition (6c) into a more convenient form:

$$|\mathbf{n}\bullet\mathbf{\Omega}|\left[\frac{1}{\sigma_t}\mathbf{\Omega}\bullet\nabla\Psi_{odd} - \frac{[\sigma_t-\sigma_a]\Phi+Q}{4\pi\sigma_t}\right]_{(r_s,\mathbf{\Omega})} = |\mathbf{n}\bullet\mathbf{\Omega}|\left[\Psi_{odd} - F\right]_{(r_s,\mathbf{\Omega})} , \qquad r_s \in \partial D \ , \text{all } \mathbf{\Omega} \ . \tag{8}$$

We now define the family of CFEMs that we consider in this work. We define approximate functions as expansions in a chosen set of linearly independent basis functions $\{b_j\}$:

$$\begin{aligned} \Psi_{odd}(r,\mathbf{\Omega}) \ &\approx \ \psi_{odd}(r,\mathbf{\Omega}) \equiv \sum_{j=1}^{J} \psi_j(\mathbf{\Omega}) b_j(r) \ , \\ \Phi(r) \ &\approx \ \phi(r) \equiv \sum_{j=1}^{J} u_j |\nabla b_j(r)| \ . \end{aligned} \tag{9}$$

(The basis functions for the scalar flux are absolute values of gradients of the basis functions for the angular flux.) We then choose J linearly independent weight functions $\{w_i\}$, multiply the transport equation (6a) by each w_i, and integrate over the problem domain. We also multiply Eq. (6b) by each $|\nabla w_i|$ and integrate over the domain. We then insert the approximations (9), and incorporate the boundary condition (8) "naturally" (see reference [1]). The result is a set of equations for the unknowns $\{\psi_j\}$ and $\{u_j\}$:

$$\int_{\partial D} d^2r\, w_i |\mathbf{n}\cdot\boldsymbol{\Omega}|(\psi_{odd} - F) + \int_D d^3r(\boldsymbol{\Omega}\cdot\nabla w_i)\frac{1}{\sigma_t}\left(\boldsymbol{\Omega}\cdot\nabla\psi_{odd} - \frac{[\sigma_t - \sigma_a]\phi + Q}{4\pi}\right) + \int_D d^3r\, w_i\,\sigma_t\,\psi_{odd} = 0 \ , \quad (10a)$$

$$\int_D d^3r |\nabla w_i| [\nabla\cdot\mathbf{j} + \sigma_a\phi - q] = 0 \ , \quad (10b)$$

where

$$\mathbf{j}(r) \equiv \sum_{j=1}^{J} b_j(r) \int_{4\pi} d\Omega\, \boldsymbol{\Omega}\, \psi_j(\boldsymbol{\Omega}) \ . \quad (10c)$$

These equations define the family of odd-parity CFEMs that we consider in this paper.

Analysis of Odd-Parity CFEMs: Slab Geometry, Linear Elements

We now analyze the simplest method in the family of odd-parity CFEMs: piecewise linear basis functions in slab geometry, with "mass-matrix lumping[1]". When we apply this method to the *scaled* problem (2), we have:

$$-\mu^2\varepsilon\left[\frac{\psi_{j+3/2} - \psi_{j+1/2}}{\sigma_{t,j+1}\Delta x_{j+1}} - \frac{\psi_{j+1/2} - \psi_{j-1/2}}{\sigma_{tj}\Delta x_j}\right] + \frac{\sigma_{t,j+1}\Delta x_{j+1} + \sigma_{tj}\Delta x_j}{2\varepsilon}\psi_{j+1/2} = -\frac{\mu}{2}\left[\left(\frac{(\sigma_t - \varepsilon^2\sigma_a)\phi + Q}{\sigma_t}\right)_{j+1} - \left(\frac{(\sigma_t - \varepsilon^2\sigma_a)\phi + Q}{\sigma_t}\right)_j\right], \quad 1 \le j < J \ , \quad (11)$$

$$\mu\,\psi_{1/2}(\mu) - \mu^2\varepsilon\frac{\psi_{3/2} - \psi_{1/2}}{\sigma_{t,1}\Delta x_1} + \frac{\sigma_{t,1}\Delta x_1}{2\varepsilon}\psi_{1/2} = \mu F_{1/2}(\mu) - \frac{\mu}{2}\left(\frac{(\sigma_t - \varepsilon^2\sigma_a)\phi + Q}{\sigma_t}\right)_1 \ , \quad (12)$$

$$\mu\,\psi_{J+1/2}(\mu) + \mu^2\varepsilon\frac{\psi_{J+1/2} - \psi_{J-1/2}}{\sigma_{tJ}\Delta x_J} + \frac{\sigma_{tJ}\Delta x_J}{2\varepsilon}\psi_{J+1/2} = \mu F_{J+1/2}(\mu) + \frac{\mu}{2}\left(\frac{(\sigma_t - \varepsilon^2\sigma_a)\phi + Q}{\sigma_t}\right)_J \ ; \quad (13)$$

$$C_{j+1/2} = 2\int_0^1 d\mu\,\mu\,\psi_{j+1/2}(\mu) \qquad , \; 0 \le j \le J \; ; \tag{14}$$

$$\varepsilon \frac{C_{j+1/2} - C_{j-1/2}}{\sigma_{tj}\Delta x_j} + \phi_j = \left(\frac{\left(\sigma_t - \varepsilon^2 \sigma_a\right)\phi + Q}{\sigma_t} \right)_j \quad , \quad 1 \le j \le J \; . \tag{15}$$

where cell-center indices run from 1 to J and half-integers refer to cell edges. We assume an asymptotic expansion for our unknowns:

$$\psi_{j+1/2}(\mu) = \psi^{[0]}_{j+1/2}(\mu) + \varepsilon \psi^{[1]}_{j+1/2}(\mu) + \varepsilon^2 \psi^{[2]}_{j+1/2}(\mu) + \ldots \qquad , \tag{16a}$$

$$\phi_j = \phi^{[0]}_j(\mu) + \varepsilon \phi^{[1]}_j + \varepsilon^2 \phi^{[2]}_j + \ldots \qquad , \tag{16b}$$

We then require that Eqs. (11)-(15) hold for each power of ε. After some algebra, we find that:

$$\psi^{[0]}_{j+1/2} = 0 \; ,$$
$$-\left[\frac{\phi^{[0]}_{j+1} - \phi^{[0]}_j}{3(\sigma_t \Delta x)_{j+1/2}} - \frac{\phi^{[0]}_j - \phi^{[0]}_{j-1}}{3(\sigma_t \Delta x)_{j-1/2}} \right] + \sigma_{aj}\Delta x_j \phi_j = Q_j \Delta x_j \qquad , \tag{17}$$

where

$$(\sigma_t \Delta x)_{j+1/2} \equiv (\sigma_{tj}\Delta x_j + \sigma_{t,j+1}\Delta x_{j+1}) / 2 \qquad , \tag{18a}$$

$$\phi^{[0]}_0 \equiv 2\int_0^1 d\mu\, 3\mu^2 F(x_{1/2},\mu) \qquad , \tag{18b}$$

$$\phi^{[0]}_{J+1} \equiv 2\int_{-1}^0 d\mu\, 3\mu^2 F(x_{J+1/2},\mu) \qquad . \tag{18c}$$

Thus, the leading-order scalar flux satisfies a standard cell-centered difference equation, which is a reasonable approximation to the correct diffusion equation. However, the boundary condition satisfied by the leading-order scalar flux is not always accurate, because $3\mu^2$ is not an accurate approximation to $W(\mu)$. This can produce large errors in the interior solution, as we show below.

Analysis of Odd-Parity CFEMs: General

The preceeding discussion considered linear elements in slab geometry. Here we discuss the results of the analysis of the generic odd-parity CFEM (10), assuming three-dimensional Cartesian geometry and an arbitrary spatial grid. After some algebra, we find the following:

$$-\frac{1}{3}\mathbf{D}\cdot\mathbf{T}^{-1}\mathbf{G}\,\underline{u}^{[0]}+\mathbf{A}\underline{u}^{[0]}=\underline{q}-\frac{1}{3}\mathbf{D}\cdot\mathbf{T}^{-1}\underline{f} \quad , \tag{19}$$

where

$$\underline{u}^{[0]}=\left[u_1^{[0]},\ \dots\ ,u_J^{[0]}\right]^t \ ,$$

$$\underline{f}=[f_1,\ \dots\ ,f_J]^t \ , \qquad f_j=\int_{\partial D}\mathrm{d}^2r\,w_j(r)\int_{4\pi}\mathrm{d}\Omega\,3\mathbf{\Omega}|\mathbf{n}\cdot\mathbf{\Omega}|F(r,\mathbf{\Omega}) \ ,$$

$$\underline{q}=[q_1,\ \dots\ ,q_J]^t \ , \qquad q_j=\int_D \mathrm{d}^3r\,w_j(r)Q(r) \ ,$$

$$\mathbf{D}_{ij}=\int_D \mathrm{d}^3r\,\nabla w_i(r)\left|\nabla b_j(r)\right| \ ,$$

$$\mathbf{G}_{ij}=\int_D \mathrm{d}^3r\left|\nabla w_i(r)\right|\nabla b_j(r) \ .$$

These results define the leading-order coefficients $\{u_j\}$; the scalar flux is defined in terms of $\{u_j\}$ by Eq. (9b). We can show that if the incident angular flux F is azimuthally symmetric, then Eq. (19) is a discretization of the diffusion equation (3b), with the following boundary condition:

$$\phi^{[0]}(r_s)=2\int_{\mathbf{n}\cdot\mathbf{\Omega}<0}\mathrm{d}\Omega\,3|\mathbf{n}\cdot\mathbf{\Omega}|^2F(r_s,\mathbf{\Omega}) \ , \qquad r_s\in\partial D \ . \tag{20}$$

Thus, every member of our family of odd-parity CFEMs satisfies a discrete diffusion equation in the thick diffusion limit. The diffusion discretization may be unusual; this depends on the details of the basis and weight functions $\{b_j\}$ and $\{w_i\}$. Further, every odd-parity CFEM solution satisfies a boundary condition (20) that is not always an accurate approximation to the correct boundary condition (3c), for $3\mu^2$ is not an accurate approximation to $W(\mu)$.

However, we recall that *even*-parity CFEM solutions in the thick diffusion limit satisfy boundary conditions with a 2μ weighting. The arithmatic average of scalar fluxes from an even- and an odd-parity CFEM will therefore satisfy a boundary condition with weight function $(2\mu+3\mu^2)/2$, which is a reasonably accurate approximation to $W(\mu)$ [see Eq. (5)]. A more general average, given by

$$\Phi_{average}=\alpha\Phi_{from\ odd}+(1-\alpha)\,\Phi_{from\ even} \ , \tag{21}$$

will satisfy a boundary condition with the weight function $2\alpha\mu+3(1\text{-}\alpha)\mu^2$. (These comments strictly apply only if the incident flux F is azimuthally symmetric.) We are free to choose α so that this weight function "best" approximates $W(\mu)$. Our preliminary choice is $\alpha=0.455$; we have used this value to generate the numerical results shown below.

If the incident angular flux F is not azimuthally symmetric, the boundary condition imbedded in Eq. (19) is more difficult to describe. It can be less accurate than Eq. (20). Thus, in general the average flux given by Eq. (21) may be less accurate than we would like. This depends on the details of the weight functions, basis functions, and spatial grid. For the sake of brevity, we do not discuss these details here.

DIFFUSION-SYNTHETIC ACCELERATION

In this section we consider the iterative problem posed by the analytic (not discretized) odd-parity equations (6a,b). We propose a DSA scheme and analyze its performance on an idealized (slab-geometry, infinite medium, constant cross section) model problem. We find that in this case, the performance of our odd-parity DSA is identical to that of first-order DSA,[12-14] with a spectral radius less than 0.225 times the scattering ratio. We then describe our DSA scheme for the odd-parity CFEM equations.

Given an infinite homogeneous slab, the odd-parity equations are:

$$-\mu^2\frac{\partial^2\Psi_{odd}}{\partial x^2} + \Psi_{odd}(x,\mu) = -\frac{\mu^2}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[c\Phi + \frac{Q}{\sigma_t}\right] , \tag{22a}$$

$$J(x) = \int_{-1}^{1} \mathrm{d}\mu\,\mu\,\Psi_{odd}(x,\mu) , \tag{22b}$$

$$\frac{\mathrm{d}J}{\mathrm{d}x} + (1-c)\Phi(x) = Q(x) , \tag{22c}$$

where $c \equiv (\sigma_t - \sigma_a)/\sigma_t$. We propose the following DSA scheme, where (l) is the iteration index:

$$-\mu^2\frac{\partial^2\Psi_{odd}^{(l+1/2)}}{\partial x^2} + \Psi_{odd}^{(l+1/2)}(x,\mu) = -\frac{\mu^2}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[c\Phi^{(l)} + \frac{Q}{\sigma_t}\right] , \tag{23a}$$

$$J^{(l+1/2)}(x) = \int_{-1}^{1} \mathrm{d}\mu\,\mu\,\Psi_{odd}^{(l+1/2)}(x,\mu) , \tag{23b}$$

$$\frac{\mathrm{d}J^{(l+1/2)}}{\mathrm{d}x} + \Phi^{(l+1/2)}(x) = c\Phi^{(l)}(x) + Q(x) , \tag{23c}$$

$$-\frac{1}{3}\frac{d^2 f^{(l+1)}}{dx^2} + (1-c)f^{(l+1)}(x) = c\left(\Phi^{(l+1/2)} - \Phi^{(l)}\right) , \tag{24a}$$

$$\Phi^{(l+1)}(x) = \Phi^{(l+1/2)}(x) + f^{(l+1)}(x) . \tag{24b}$$

We apply a straightforward Fourier analysis[12] of this scheme to find the iteration eigenvalues ω:

$$\omega = \frac{\lambda^2}{\lambda^2 + 3(1-c)} \int_0^1 d\mu \frac{1-3\mu^2}{1+\lambda^2\mu^2} \quad . \tag{25}$$

This is exactly the expression obtained by previous researchers using the first-order transport equation.[12] The integral is bounded less than 0.2247 (reference [12]).

Previous research[11-16] has indicated that a successful DSA scheme for a given numerical transport method must employ a discretization of the diffusion equation (24a) that is "consistent" with that transport discretization. For odd-parity CFEMs, we propose the discretization that is satisfied by the odd-parity CFEM solution in the diffusion limit. This is given by Eqs. (19).

NUMERICAL RESULTS

In this section we consider three slab-geometry test problems, each of which contains a thick diffusive region. We solve each with an even-parity method and an odd-parity method. Each method uses linear continuous finite elements; all calculations employ the S_{16} Gauss-Legendre quadrature set. We note that the scalar flux from the even-parity method is piecewise linear and continuous, while that from the odd-parity method is piecewise constant and discontinuous:

$$\phi_{from\ even} = \int_{-1}^{1} d\mu\, \psi_{even} \ ;$$

$$\sigma_a \phi_{from\ odd} = Q - \frac{d}{dx} 2 \int_{-1}^{1} d\mu\, \mu\, \psi_{odd} \ .$$

Thus, the scalar flux from the even-parity method has the same spatial shape as the even-parity flux (i.e. linear and continuous), whereas the scalar flux from the odd-parity method has the shape of a *spatial derivative* of the odd-parity flux (constant and discontinuous). The average of the two scalar fluxes is therefore linear and discontinuous.

Our first test problem is a source-free purely-scattering slab 1000 mean-free paths thick, subject to an isotropic incoming flux at the left face and a vacuum at the right. We obtain a reference solution by using a linear-discontinuous discretization of the first-order transport equation[4], with a very fine spatial mesh. We obtain even- and odd-parity solutions with a coarse spatial mesh (10 zones). Results are given in Figure 1. (Wiggles in the reference solution are entirely an artifact of the graphics software.) We see that both methods perform very well, as expected, because their boundary conditions are correct when the incident flux is isotropic.

Our second test problem is identical to the first except that the incoming flux on the left is a delta-function in angle at $\mu \approx 0.1$. Results are given in Figure 2. The exact solution varies rapidly at the boundary, the even-parity solution is too high in the interior, and the odd-parity solution is too low. Our weighted average [see Eq. (21)] of the even- and odd-parity solutions is extremely accurate, exactly in agreement with our asymptotic analysis.

Our final test problem is just the second problem with a thin (.001 mean-free path) purely-absorbing region attached to the left face of the slab. (The incident flux must now penetrate the thin absorber before reaching the diffusive region.) Results are shown in Figures 3a and 3b. The solutions *in the diffusive region* behave as they did in our second test problem. In the thin region, which we emphasize is well-resolved by the spatial mesh, the even- and odd-parity solutions are very inaccurate. However, we see again that the average of the two is extremely accurate.

V. CONCLUSIONS

We have analyzed the behavior of continuous finite-element methods applied to the odd-parity form of the transport equation, for problems that contain thick diffusive regions. We have found that odd-parity CFEM solutions limit to a discretization of the correct diffusion equation, but that these solutions satisfy boundary conditions that can be inaccurate. Recalling a recent study of *even*-parity CFEM methods, we have recognized that the average of an even-parity and an odd-parity CFEM solution will satisfy accurate boundary conditions. We therefore have proposed a method whose solution is defined to be the average of an even-parity and an odd-parity CFEM solution. We have tested this method in slab geometry on diffusive test problems, and we have observed excellent results. These excellent results appear to hold in non diffusive regions that are adjacent to diffusive regions, where the individual even- and odd-parity solutions both exhibit large errors.

We have also introduced a diffusion-synthetic acceleration (DSA) method for the iterative solution of the odd-parity equations. We have analyzed this scheme for the continuous (not discretized) odd-parity equations. We have found that it converges rapidly, with spectral radius less than 0.2247. We have proposed that given a CFEM *discretization* of the odd-parity equations, the diffusion discretization used for acceleration should be that discretization satisfied by the odd-parity CFEM solution in the thick diffusion limit. We have not analyzed our DSA method for such a discrete problem, but we have observed excellent performance in our test problems.

An often-touted advantage of the even-parity or odd-parity method is that each requires a solution on only half of the angular domain. Our proposal eliminates this advantage by computing two such solutions. A natural question, then, is whether our proposed method has any advantages over a first-order method. Let us temporarily confine the discussion to slab geometry. We note that every first-order method that has been shown to behave well in the diffusion limit requires at least two unknowns per spatial cell per angular unknown. We note further that the even-parity linear CFEM requires approximately one unknown per spatial cell per angular unknown *for half of the angular domain*, as does the odd-parity linear CFEM. Thus, the total unknown count for our proposed method is approximately half that of a well-behaved first-order method. If we shift the discussion to two dimensions, the savings become greater; in three dimensions, the savings become even greater. These comments apply only to the unknown count; computational effort per unknown is an important issue that we do not address here. (See reference [16] for a discussion.)

Our analysis has taken place on transport equations with spatial discretization but no angular discretization. This keeps the analysis simple, and serves to isolate errors that are due to the spatial discretization scheme. It will hold for any reasonable angular discretization in the limit of fine angular resolution. We note that recent work by Jin and Levermore[18] shows that angular discretizations can also introduce errors in the boundary condition satisfied by the leading-order solution in diffusive regions. In the future we expect to address the behavior of transport methods in which both the angular and spatial variables are discretized.

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Figure 1. Results from Test Problem 1.

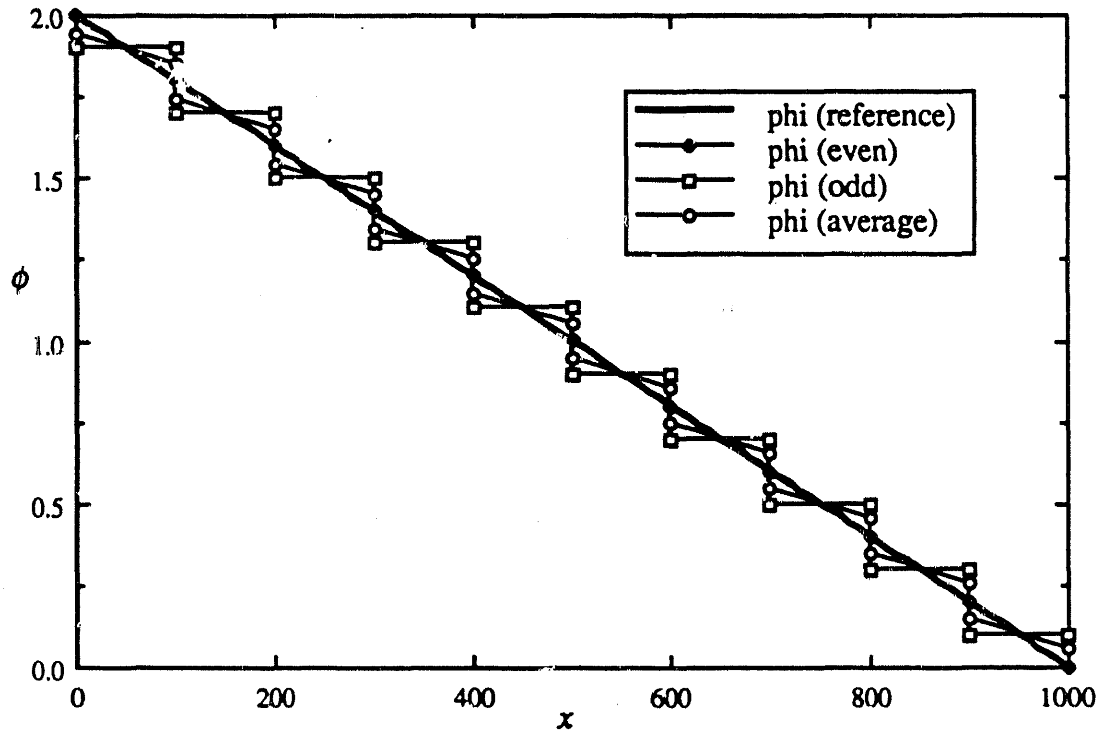


Figure 2. Results from Test Problem 2.

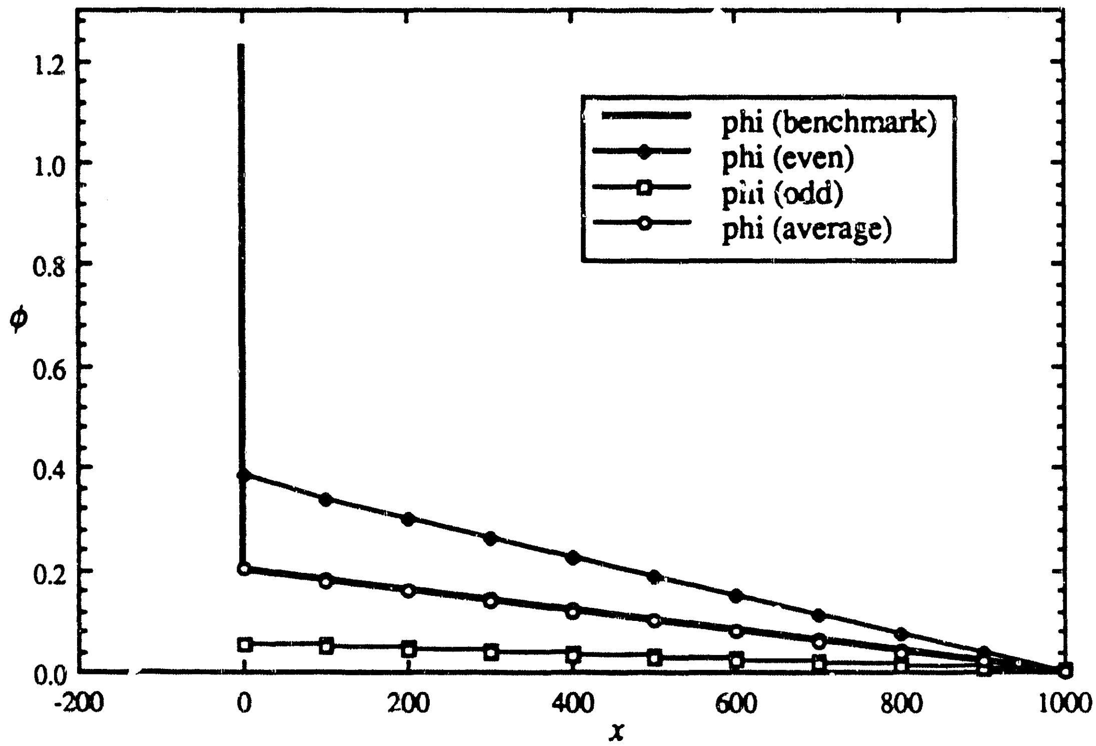


Figure 3a. Results from Test Problem 3.

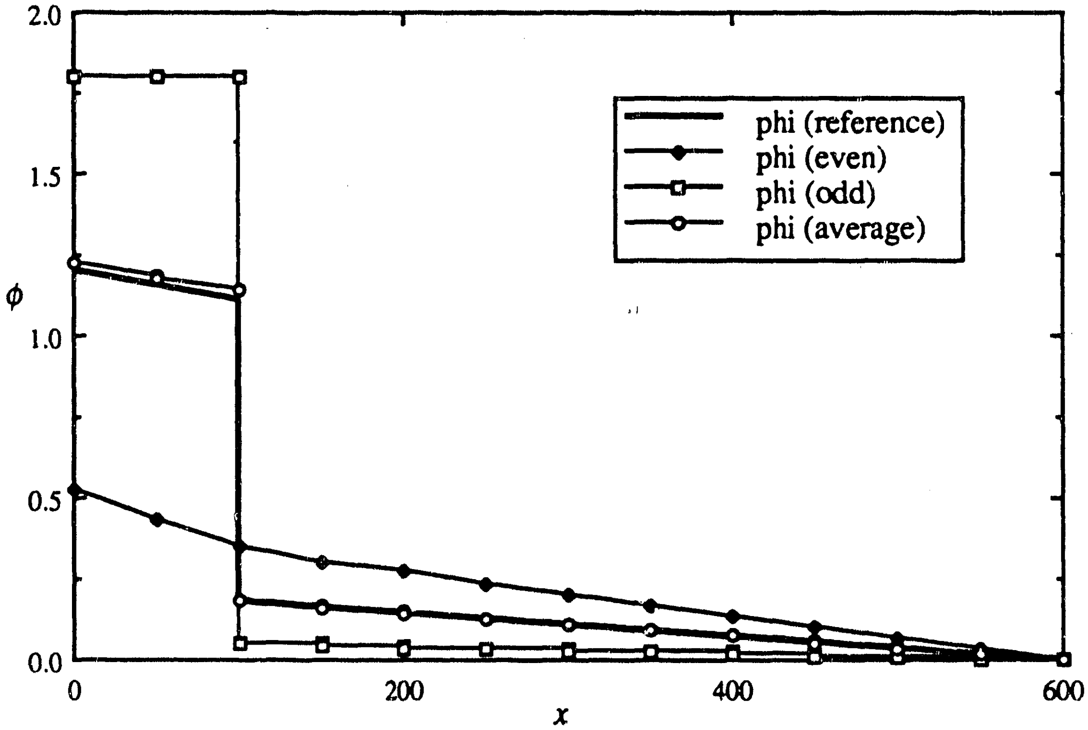
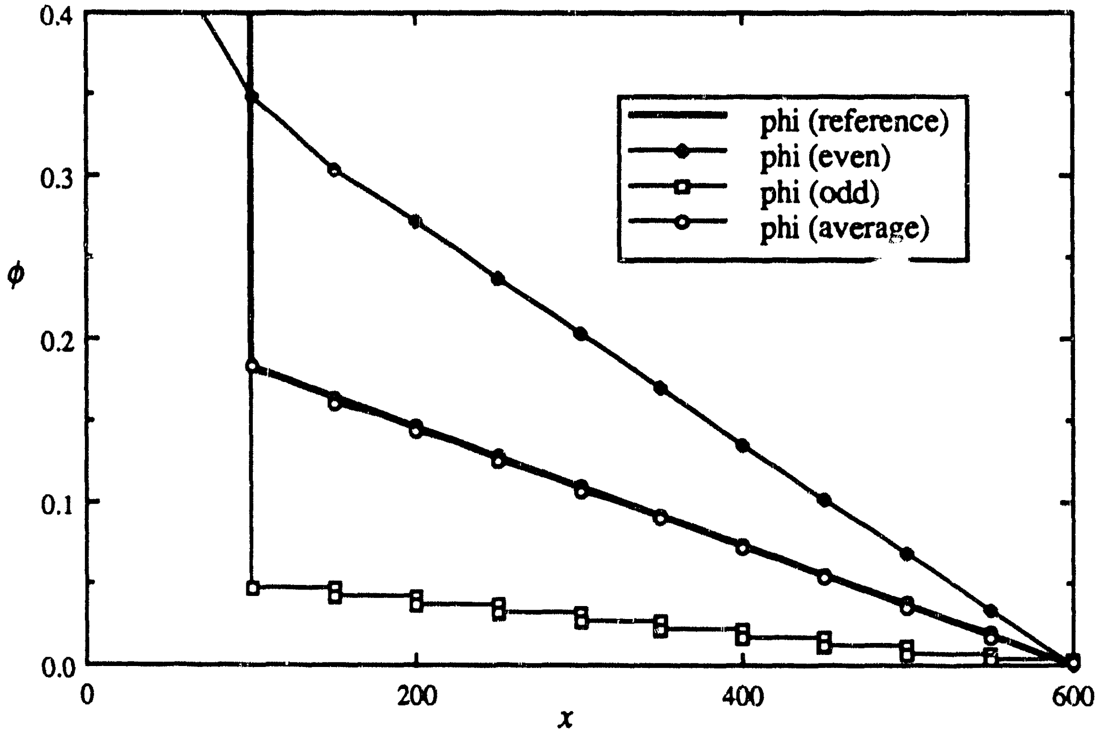


Figure 3b. Closeup of results from Test Problem 3.



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