LA-UR-83-623

LA-UR--83-623

DE83 010173

TITLE: EXACT INVARIANTS FOR TIME-DEPENDENT HAMILTONIAN SYSTEMS

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MASTER

SUBMITTED TO: Presented as an Invited Lecture at the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics Turin, Italy, June 7-11, 1982.

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FORM NO 836 R4
ST NO 2629 5/81

Exact Invariants for Time-Dependent Hamiltonian Systems

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I. Introduction

In this lecture I shall discuss a search for exact invariants for Hamiltonians of the form

$$H = \frac{1}{2} p^2 + V(q,t) . \tag{1.1}$$

By exact invariant I mean any function I(q,p,t) whose total time derivative vanishes,

$$\begin{aligned} 0 &= \frac{\partial I}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial I}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial I}{\partial p} \\ &= \frac{\partial I}{\partial t} + [I,H] . \end{aligned} \tag{1.2}$$

My principal collaborator in the work that I shall describe is Prof. P. G. L. Leach (La Trobe University; Department of Applied Mathematics; Bundoora, Victoria 3083; Australia). Dr. W. Sarlet (Rijksuniversiteit Gent; Instituut voor Theoretische Mechanica; Gebouw S9; Krijgslaan 271; B-9000 Gent; Belgium) has worked with us on closely related studies and continues to do so. Some persons whose work has contributed to the background of what I shall describe are L. Y. Bahar, C. J. Eliezer, M. Lutzky, G. E. Prince, J. R. Ray and J. L. Reid. References to the work of these persons are cited in Refs. 1 and 2.

The purpose of our search for exact invariants is two-fold: we wish to illuminate the underlying structure of dynamical systems and we wish to provide a basis for applications to fields like plasma physics and quantum theory. I am particularly interested in the possibility of application to plasma physics, where invariants enter in finding solutions to the highly nonlinear Vlasov-Maxwell system of integro-partial differential equations. As an illustrative example of the Vlasov-Maxwell equations, we can consider the special case in one spatial dimension of the Vlasov-Poisson equations for phase-space distribution functions $f_s(q,p,t)$ and an electric scalar potential $\phi(q,t)$,

$$\frac{\partial f_s}{\partial t} + \frac{\partial H_s}{\partial p}\frac{\partial f_s}{\partial q} - \frac{\partial H_s}{\partial q}\frac{\partial f_s}{\partial p} = 0 \ , \tag{1.3}$$

$$-\frac{\partial^2 \phi}{\partial q^2} = 4\pi \sum_s Q_s \int dp\ f_s(q,p,t) \ . \tag{1.4}$$

The characteristic equations of (1.3) are the equations of motion associated with the Hamiltonian

$$H_s(q,p,t) = \frac{p^2}{2M_s} + Q_s\phi(q,t) \ . \tag{1.5}$$

The subscript s denotes a particle species and the numbers M_s and Q_s are, respectively, the mass and charge of a particle of species s. The most general solution of (1.3) as a functional of $\phi(q,t)$ is an arbitrary differentiable function of two independent invariants of the particle motion associated with the Hamiltonian $H_s(q,p,t)$.

Methods for finding exact invariants that have been used recently include Noether's theorem, the Lie theory of extended groups, Ermakov's method, canonical transformations and direct methods. In this lecture I shall emphasize a new application of the direct method that is due to P. G. L. Leach and myself.[1]

II. The Direct Method

The direct method for finding exact invariants consists simply in assuming a functional form for the invariant $I(q,p,t)$ and substituting that form directly into the defining equation for an invariant, (1.2). For example, Lewis and Leach[2] assumed invariants with polynomial dependence on the momentum,

$$I(q,p,t) = \sum_{n=0}^{N} p^n f_n(q,t) \ . \tag{2.1}$$

With this assumption, directly from (1.2), they were able to derive the following result: For the Hamiltonian (1.1), an invariant quadratic in p exists if, and only if, the potential has the form

$$V(q,t) = \left(\frac{\ddot{\rho}\alpha}{\rho} - \ddot{\alpha}\right)q - \frac{1}{2}\frac{\ddot{\rho}}{\rho}q^2 + \frac{1}{\rho^2}G\left(\frac{q-\alpha}{\rho}\right) \ , \tag{2.2}$$

where $\rho(t)$, $\alpha(t)$, and $G\left(\frac{q-\alpha}{\rho}\right)$ are arbitrary functions. For these potentials, the invariants quadratic in p are

$$I(q,p,t) = \frac{1}{2}\left[\rho(p-\dot{\alpha}) - \dot{\rho}(q-\alpha)\right]^2 + G\left(\frac{q-\alpha}{\rho}\right) \ . \tag{2.3}$$

This result has now been found by other methods as well. These quadratic invariants have been applied to the single-species Vlasov-Poisson equations by Lewis and Symon.[3] Interesting new solutions were found, although those solutions are still rather restrictive from a physical standpoint. The invariants of degree higher than two have not been found.

The physical limitations of the solutions of the Vlasov-Poisson equations that are associated with the quadratic invariants and the fact that invariants of higher degree have not been found have motivated Leach and myself to study a new ansatz for the direct method.[1] This ansatz, which is a representation of the invariant in terms of simple poles in the complex momentum plane (called resonances), is the subject of the remainder of my lecture.

III. Resonance Formulation

The ansatz that we make for the dependence of the invariant on the momentum is that the invariant be a rational function of the momentum with only simple poles. Singularities of the rational function may occur at complex values of the momentum, but of course the only physically realizable values of the momentum are real. The motivation for choosing rational functions, with or without the restriction to simple poles, is that rational functions are good approximating functions that have singularities. The Padé approximants are good examples of the use of rational functions for approximation. Having singularities in the assumed form for the invariants is desirable because they can correspond to the singularities that occur in the general solution of the linearized Vlasov-Maxwell equations. (Recall that the solutions of the Vlasov equation are invariants of the underlying single-particle motion.) Also, singularities may correspond to the well-known existence of adiabatic invariants that are valid in localized regions of the phase space. The restriction to simple poles in the rational function is not serious. Consider p as a complex variable,

$$p = \xi + i\eta \tag{3.1}$$

and write a particular rational function with simple poles as

$$R(p) = u(\xi,\eta) + iv(\xi,\eta) \ . \tag{3.2}$$

Now suppose that the singularities of R(p) coincide with those of some function f(p), but allow the nature of the singularities of the two functions to differ. Then, almost everywhere, f(p) can be written as

$$f(p) = g[R(p)] \ , \tag{3.3}$$

where g is a nonsingular function. The reason is that the Cauchy-Riemann condition allows the Jacobian of the transformation from (ξ,η) to (u,v) to be written as

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} = \left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \eta}\right)^2 . \tag{3.4}$$

The Jacobian is nonzero except where each of the first derivatives of u and v with respect to ξ and η vanish. Because any function of an invariant is an invariant, we can study all invariants of the form $f(p)$ by studying functions $R(p)$.

It is convenient to represent our rational function with only simple poles in a canonical form in terms of resonance denominators,

$$I(q,p,t) = c(q,t) + \sum_{n=0}^{N} \frac{v_n(q,t)}{p - u_n(q,t)} . \tag{3.5}$$

This choice of representation has the advantage that the equations that $c(q,t)$ and the functions $u_n(q,t)$ and $v_n(q,t)$ must satisfy only couple in a single condition the functions associated with different subscripts n. The equations that must be satisfied can be determined by substituting (3.5) into (1.2). The result is of the form

$$w_0(q,t) + pw_1(q,t) + \sum_n \left[\frac{y_n(q,t)}{x_n} + \frac{z_n(q,t)}{x_n^2}\right] = 0 , \tag{3.6}$$

where x_n is defined by

$$x_n \equiv p - u_n(q,t) . \tag{3.7}$$

Because (3.6) must hold for all values of p, the conditions to be satisfied are that the functions w_0 and w_1 must vanish and that the functions y_n and z_n must vanish for each subscript n separately. These conditions are necessary and sufficient for I(q,p,t) given by (3.5) to be an invariant. A result of the conditions is that c(q,t) must not be a function of q: $c(q,t) \rightarrow c(t)$. Taking that into account, we can conveniently write the conditions explicitly as

$$c'(t) + \sum_n \frac{\partial v_n}{\partial q} = 0 \ , \tag{3.8}$$

$$\frac{\partial v_n}{\partial t} + \frac{\partial}{\partial q}(u_n v_n) = 0 \ , \tag{3.9}$$

$$\frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial q} = - \frac{\partial V}{\partial q} \ . \tag{3.10}$$

Equations (3.9) and (3.10) must be satisfied for each subscript n separately. If one function $u_n(q,t)$ is specified then $\partial V/\partial q$ can be calculated from (3.10). If there are two or more functions $u_n(q,t)$, then they must yield the same $\partial V/\partial q$ according to (3.10).

Simple Harmonic Oscillator

In order to obtain a feeling for invariants written in this resonance form, let us put some well-known invariants for the simple harmonic oscillator in resonance form. Consider the Hamiltonian

$$H = \frac{1}{2} \left(p^2 + q^2\right) \ . \tag{3.11}$$

The solution of the equations of motion is

$$q = p_0 \sin(t) + q_0 \cos(t) \ , \qquad p = p_0 \cos(t) - q_0 \sin(t) \tag{3.12}$$

or, in another form,

$$q = A\sin(t-\phi)\ , \qquad p = A\cos(t-\phi)\ . \tag{3.13}$$

The constants of integration are q_0 and p_0 or A and ϕ. We can write $1/H$, $1/q_0$, $1/p_0$ and $\tan(\phi)$ in resonance form:

$$\frac{1}{H} = \frac{-\frac{i}{q}}{p - iq} + \frac{\frac{i}{q}}{p + iq}\ , \tag{3.14}$$

$$\frac{1}{q_0} = \frac{-\frac{1}{\sin(t)}}{p - q\,\frac{\cos(t)}{\sin(t)}}\ , \qquad \frac{1}{p_0} = \frac{\frac{1}{\cos(t)}}{p + q\,\frac{\sin(t)}{\cos(t)}}\ , \tag{3.15}$$

$$\tan(\phi) = -\frac{q_0}{p_0} = \frac{\sin(t)}{\cos(t)} - \frac{\frac{q}{\cos^2(t)}}{p + q\,\frac{\sin(t)}{\cos(t)}}\ . \tag{3.16}$$

Thus, H, q_0, p_0 and ϕ can each be written as a function of a rational function of p with simple poles.

Lagrangian Variables

Equations (3.9) and (3.10) are of the form of particle conservation and momentum balance equations in fluid dynamics. The treatment of those equations is facilitated by a transformation to a set of Lagrangian variables. This is discussed in Ref. 1 by Lewis and Leach. For each function $u_n(q,t)$ we introduce a Lagrangian mapping function $F_n(q,t)$ that is related to $u_n(q,t)$ by

$$\frac{\partial F_n}{\partial t} + u_n\,\frac{\partial F_n}{\partial q} = 0\ . \tag{3.17}$$

Equation (3.9) can be solved in terms of $F_n(q,t)$ and we view (3.10) as a formula for <u>calculating</u> $\partial V/\partial q$. The problem then is to determine a suitable set of mapping functions, each of which corresponds to the same $\partial V/\partial q$, such that the consistency condition (3.8) is satisfied. In the analysis it is convenient to introduce Lagrangian variables x_n and inverse mappings $J_n(x_n,t)$ defined by

$$x_n = F_n(q,t) \quad \leftrightarrow \quad q = J_n(x_n,t) \ . \tag{3.18}$$

The functions $F_n(q,t)$ and $J_n(x_n,t)$ satisfy the identity

$$q \equiv J_n[F_n(q,t),t] \ . \tag{3.19}$$

Generally speaking, the mapping given by $F_n(q,t)$ will be invertible only locally. For each value of t, it will be single-valued and thus invertible between each pair of adjacent extrema. The domains in which the mapping is single-valued can change with time. These domains may be associated with the subdivision of the phase plane into more or less isolated regions that is often observed in numerical solutions of equations of motion. Also, they may be associated with the existence of adiabatic invariants that are only valid in localized regions of the phase plane.

In terms of the set of Lagrangian mapping functions, the consistency condition (3.8) can be written as

$$c'(t) + \sum_n \frac{\partial^2 F_n}{\partial q^2} = 0 \ , \tag{3.20}$$

which can be integrated readily. The expression for $\partial V/\partial q$ obtained from (3.10) is

$$\frac{\partial V}{\partial q} = \left\{\frac{\partial^2 F_n}{\partial q^2}\left(\frac{\partial F_n}{\partial t}\right)^2 - 2\,\frac{\partial^2 F_n}{\partial q \partial t}\,\frac{\partial F_n}{\partial t}\,\frac{\partial F_n}{\partial q} + \frac{\partial^2 F_n}{\partial t^2}\left(\frac{\partial F_n}{\partial q}\right)^2\right\} / \left(\frac{\partial F_n}{\partial q}\right)^3 \ . \tag{3.21}$$

Single Resonance

If there is only one resonance in the invariant (only one Lagrangian mapping function), then the consistency condition (3.20) requires that $F_1(q,t)$ be quadratic in q,

$$F_1(q,t) = [\alpha(t)q + \beta(t)]^2 + \gamma(t) \ . \qquad (3.22)$$

The potential generated from (3.21) by this mapping function is

$$V(q,t) = \left(\frac{\alpha''}{\alpha^2} - \frac{2\alpha'^2}{\alpha^3}\right)\frac{1}{2\alpha}(\alpha q + \beta)^2 + \left(\frac{\gamma''}{2\alpha^2} - \frac{\gamma'\alpha'}{\alpha^3}\right)\log(\alpha q + \beta)$$

$$- \frac{\gamma'^2}{8\alpha^2}\frac{1}{(\alpha q + \beta)^2} + \left(\frac{\beta''}{\alpha^2} - \frac{2\beta'\alpha'}{\alpha^3} - \frac{\beta\alpha''}{\alpha^3} + \frac{2\beta\alpha'^2}{\alpha^4}\right)(\alpha q + \beta) + g(t) \ , \qquad (3.23)$$

where g(t) is an arbitrary function. An invariant for this potential has been derived by Sarlet[4] and, recently, by a canonical transformation technique.[5] Sarlet used a generalization to time-dependent transformations of the techniques in Ref. 6. Note that the potential (3.23) contains three arbitrary functions of time. The invariant associated with (3.22) is

$$I_1(q,p,t) = -\int^t \alpha^2(t')dt'$$

$$+ \frac{\alpha\,(\alpha q + \beta)}{p - \left(\frac{\beta}{\alpha}\right)' + \frac{\alpha'}{\alpha^2}(\alpha q + \beta) + \frac{\gamma'}{2\alpha\,(\alpha q + \beta)}} \ . \qquad (3.24)$$

There exists a two-parameter family of mappings different than (3.22), each quadratic in q but with different functions of time, that give the same V(q,t) as (3.23) but different functions $u_1(q,t)$. This means that the invariants are different and we can construct two independent invariants for the potential

(3.23). Thus, the dynamics has been solved completely for the potentials corresponding to invariants with a single resonance.

Multiple Resonances

Suppose that we are given one mapping function for a particular $\partial V/\partial q$ and that we have defined the Lagrangian variable x_1,

$$x_1 = F_1(q,t) \quad \Rightarrow \quad q = J_1(x_1,t) \; . \tag{3.25}$$

Suppose also that $F_1(q,t)$ is not quadratic in q, so that at least one other mapping function is required in order to satisfy the consistency condition (3.20). In such a case, how many mapping functions are required in order to be able to satisfy the consistency condition and how can they be found?

We seek a different mapping function, $F_2(q,t)$, that corresponds to the same $\partial V/\partial q$. Of course, the function $u_2(q,t)$ produced by $F_2(q,t)$ must be different than the function $u_1(q,t)$ produced by $F_1(q,t)$. If the consistency condition can be satisfied by using just these two mapping functions, then we shall have a two-resonance example. Otherwise, one or more additional mapping functions must be found. Let us define a Lagrangian variable x_2 associated with $F_2(q,t)$,

$$x_2 = F_2(q,t) \quad \Rightarrow \quad q = J_2(x_2,t) \; . \tag{3.26}$$

We want to find a constructive relation between the two mappings. For that purpose it turns out to be convenient to express x_2 in terms of x_1,

$$x_2 = F_2(q,t) = F_2[J_1(x_1,t),t] = P(x_1,t) \; , \tag{3.27}$$

and to seek the relation between $P(x_1,t)$ and $J_1(x_1,t)$.

In order to simplify the notation, I shall drop the subscript 1 during the remainder of this discussion. Requiring that the two mapping functions give the same $\partial V/\partial q$ implies the following equation that must be satisfied by $P(x,t)$ and $J(x,t)$:

$$\frac{\partial P}{\partial x}\left(\frac{\partial P}{\partial t}\right)^2 \frac{\partial^2 J}{\partial x^2} - 2\left(\frac{\partial P}{\partial x}\right)^2 \frac{\partial P}{\partial t}\frac{\partial^2 J}{\partial x \partial t} = \left\{\frac{\partial^2 P}{\partial x^2}\left(\frac{\partial P}{\partial t}\right)^2 - 2\frac{\partial^2 P}{\partial x \partial t}\frac{\partial P}{\partial t}\frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial t^2}\left(\frac{\partial P}{\partial x}\right)^2\right\}\frac{\partial J}{\partial x} . \quad (3.28)$$

It can be shown that the solution of this equation is

$$\frac{\partial J}{\partial x} = \frac{\partial \eta/\partial x}{(\partial \eta/\partial t)^{1/2}} , \quad (3.29)$$

where η is any solution of

$$\frac{\partial P}{\partial t}\frac{\partial \eta}{\partial x} - 2\frac{\partial P}{\partial x}\frac{\partial \eta}{\partial t} = 0 . \quad (3.30)$$

A result of this form of the solution of (3.28) is that there exists only one $J(x,t)$ for a given $P(x,t)$.

A Two-Resonance Example

This formalism can be applied to give the potentials for which there exist invariants quadratic in the momentum and to give the quadratic invariants for those potentials. The result can be obtained by taking $F(q,t)$ in the form

$$F(q,t) = N\left(\frac{q - \alpha}{\rho}\right) - \int^t \rho^{-2}(t')dt' , \quad (3.31)$$

where $N\left(\frac{q-\alpha}{\rho}\right)$, $\alpha(t)$ and $\rho(t)$ are arbitrary functions. Then $F_2(q,t)$ is given by

$$F_2(q,t) = -N\left(\frac{q-\alpha}{\rho}\right) - \int^t \rho^{-2}(t')dt' \tag{3.32}$$

and the Hamiltonian is

$$H = \frac{1}{2}p^2 - \frac{q}{\rho}\left(\frac{1}{2}q\ddot{\rho} + \ddot{\alpha}\rho - \alpha\ddot{\rho}\right) - \frac{1}{\rho^2}\,\frac{1}{2N'^2\left(\frac{q-\alpha}{\rho}\right)} \,. \tag{3.33}$$

The potential in this Hamiltonian is of the same form as (2.2), which is the most general potential for which there exists an invariant quadratic in p.

General Two-Resonance Case

The general case in which there exists an invariant that can be written in the form (3.5) with $N = 2$ can be formulated as follows. Let $M(x,t)$ be a function such that $\eta(x,t) = M(x,t)$ is a particular solution of (3.30). Define a new independent variable ζ by

$$\zeta = M(x,t) \quad \Rightarrow \quad x = N(\zeta,t) \,, \tag{3.34}$$

where $N(\zeta,t)$ is the inverse of $M(x,t)$. Because (3.30) is a homogeneous, first-order partial differential equation for $\eta(x,t)$, its general solution can be written in terms of the single variable ζ as

$$\eta(x,t) = \nu(\zeta) \,, \tag{3.35}$$

where $\nu(\zeta)$ is some function of a single argument. By virtue of (3.34), $P(x,t)$ can be expressed as some function $Q(\zeta,t)$,

$$P(x,t) = Q(\zeta,t) \ . \tag{3.36}$$

In principle we can express q as some function $g(\zeta,t)$,

$$q = g(\zeta,t) \ , \tag{3.37}$$

by inverting

$$\zeta = M(x,t) = M[F(q,t),t] = f(q,t) \ . \tag{3.38}$$

By substituting (3.38) into (3.34), we see that specifying $N(\zeta,t)$ is equivalent to specifying $F(q,t)$ in the form

$$x = F(q,t) = N[f(q,t),t] \ . \tag{3.39}$$

This form is a generalization of the ansatz (3.31). In (3.31), the function $f(q,t)$ is $(q - \alpha)/\rho$.

A way of proceeding to investigate two-resonance cases in detail is to specify various forms for the function $N(\zeta,t)$ and to determine the allowable functions $Q(\zeta,t)$ and $g(\zeta,t)$. The conditions that these three functions must satisfy are

$$\frac{\partial N}{\partial \zeta}\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial \zeta}\frac{\partial N}{\partial t} = 0 \ , \tag{3.40}$$

which is (3.30) rewritten,

$$\frac{\partial g}{\partial \zeta} = \nu'(\zeta)\left[\frac{-\partial N/\partial \zeta}{\nu'(\zeta)\partial N/\partial t}\right]^{1/2} \ , \tag{3.41}$$

which is (3.29) rewritten, and

$$\frac{1}{2}\, c'(t)g^2(\zeta,t) - b(t)g(\zeta,t) - a(t) + N(\zeta,t) + Q(\zeta,t) = 0 \;, \qquad (3.42)$$

which is the consistency condition (3.20) and where a(t) and b(t) are arbitrary.

IV. Conclusion

In the main part of this lecture, I have presented a resonance formulation for exact invariants of Hamiltonian systems that describe the motion of a particle in a one-dimensional potential. The formulation is due to Prof. P. G. L. Leach and myself. We have obtained earlier results as simple applications of the new formulation. I have presented a possible formulation for studying the general two-resonance case. There is a possibility that the resonance formulation for exact invariants will have applications in plasma physics and quantum theory.

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