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Introduction

In the dynamics of single, charged particles in a storage ring, the effect of nonlinear magnets on the particle motion can reduce the stable region of phase space (the dynamic aperture) and lead to short beam lifetimes. The nonlinearity can also increase the effective size of the particle distribution in a beam and decrease the luminosity of a colliding beam machine. For these practical reasons, as well as for theoretical ones, the approximate solution of the Hamilton-Jacobi equation, and the resulting torus, prove useful for understanding strong nonlinear perturbations on linear motion.

The existence of a continuous family of accurate, approximate invariant tori in a region of phase space is related to the stability of orbits in that region. Using the canonical transformation associated with that family, one can set long-term bounds on nonlinear motion, following an argument¹ in the spirit of the Nekhoroshev theorem.² Also, the shift of the tune from its linear value as a function of the amplitude can be found from the solutions of the Hamilton-Jacobi equation. This gives a picture of nearby resonances that can affect the particle motion.

The linear motion of a charged particle in a transverse magnetic field can be described by the Hamiltonian,

$$H_0(x_1, p_1, x_2, p_2, s) = \frac{p_1^2}{2} + K_1(s) \frac{x_1^2}{2} + \frac{p_2^2}{2} + K_2(s) \frac{x_2^2}{2}, \quad (1)$$

where x_i are the transverse displacements, p_i are the components of the transverse momentum normalized by the longitudinal momentum, and s is the arclength around the storage ring. The focusing functions, $K_i(s)$, describe the magnetic field and are periodic with period C , the ring circumference: $K_i(s+C) = K_i(s)$. Eqn. (1) describes the linear motion of the particle. In storage rings there are also nonlinear magnets, usually sextupoles, that give a nonlinear perturbation to H_0 . The perturbation from sextupoles is $V(x_1, x_2, s) = S(s)(x_1^3 - 3x_1x_2^2)/3!$, where $S(s)$ gives the strength and distribution of the sextupoles. The perturbation is also periodic with period C .

It is convenient to use the action-angle variables, $\mathbf{I} = (I_1, I_2)$, $\Phi = (\Phi_1, \Phi_2)$ of the linear Hamiltonian. The transformation equations are $x_i = \sqrt{2\beta_i(s)} I_i \cos \Phi_i$ and $p_i = -\sqrt{2I_i/\beta_i(s)} (\sin \Phi_i + \alpha_i(s) \cos \Phi_i)$, with $i = 1, 2$. Courant and Snyder³ and Ruth⁴ describe the single particle motion, the action-angle transformation, and the lattice functions (β_i and α_i) in greater detail.

After the change of variables, the new Hamiltonian is

$$H(\Phi, \mathbf{I}, s) = \Omega(s) \cdot \mathbf{I} + V(\Phi, \mathbf{I}, s), \quad (2)$$

with $\Omega(s) = (1/\beta_1, 1/\beta_2)$. We see from Eqn. (2) that if there were no nonlinear-

Computation of Invariant Tori in 2 1/2 Degrees of Freedom[†]

W. E. Gabella

Department of Physics, University of Colorado, Boulder, CO 80309[‡]

R. D. Ruth and R. L. Warnock

Stanford Linear Accelerator Center, Stanford, CA 94309

Abstract

Approximate invariant tori in phase space are found using a non-perturbative, numerical solution of the Hamilton-Jacobi equation for a nonlinear, time-periodic Hamiltonian. The Hamiltonian is written in the action-angle variables of its solvable part. The solution of the Hamilton-Jacobi equation is represented as a Fourier series in the angle variables but not in the 'time' variable. The Fourier coefficients of the solution are regarded as the fixed point of a nonlinear map. The fixed point is found using a simple iteration or a Newton-Broyden iteration. The Newton-Broyden method is slower than the simple iteration, but it yields solutions at amplitudes that are significant compared to the 'dynamic aperture'. Invariant tori are found for the dynamics of a charged particle in a storage ring with sextupole magnets.

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[‡] Present address is SLAC Bln 26, P. O. Box 4349, Stanford, CA 94309.

ities, i.e. $V = 0$, the action I would be a constant of the motion. In accelerator physics, I_1 and I_2 are essentially the Courant-Snyder invariants.

The Hamilton-Jacobi Equation

The solution of the Hamilton-Jacobi equation, if it exists, is the canonical transformation to the action-angle variables of the nonlinear Hamiltonian. Eqn. (2). It yields a new Hamiltonian that is independent of the new angles. Hence, the new actions are constants of the motion. The canonical transformation $(\Phi, I) \rightarrow (\Psi, J)$ is generated by $F_2(\Phi, J, s) = \Phi \cdot J + G_1(\Phi, J, s)$ with the transformation equations

$$I = J + G_1(\Phi, J, s) \quad (3)$$

$$\Psi = \Phi + G_2(\Phi, J, s) \quad (4)$$

$$H'(\Psi, J, s) = \Omega \cdot (J + G_2) + V(\Phi, J + G_2, s) \quad (5)$$

Subscripts on G indicate differentiation: $G_\Phi = (\partial G / \partial \Phi_1, \partial G / \partial \Phi_2)$. Requiring the new Hamiltonian H' to be independent of the new angles yields the Hamilton-Jacobi equation. It can be projected onto a Fourier basis in Φ :

$$H'(\mathbf{J}, s) = \Omega(s) \cdot \mathbf{J} + \partial_s g(\mathbf{0}, \mathbf{J}, s) + v(\mathbf{0}, \mathbf{J}, s; g), \quad \mathbf{m} = \mathbf{0}, \quad (6)$$

$$0 = \partial_s g(\mathbf{m}, \mathbf{J}, s) + i\mathbf{m} \cdot \Omega(s)g(\mathbf{m}, \mathbf{J}, s) + v(\mathbf{m}, \mathbf{J}, s; g), \quad \mathbf{m} \neq \mathbf{0}, \quad (7)$$

$$v(\mathbf{m}, \mathbf{J}, s; g) = \int_0^{2\pi} \frac{d\Phi}{(2\pi)^N} e^{-i\mathbf{m} \cdot \Phi} V(\Phi, \mathbf{J} + G_\Phi, s), \quad (8)$$

with $G_\Phi(\Phi, \mathbf{J}, s) = \sum_{\mathbf{m} \in \mathcal{B}} i\mathbf{m} g(\mathbf{m}, \mathbf{J}, s) \exp(i\mathbf{m} \cdot \Phi)$.

For a numerical calculation, the set $\mathcal{B}$ includes all modes $\mathbf{m} = \{m_i\}$ with $0 \leq |m_i| \leq M_i$, except for the vector $\mathbf{m} = \mathbf{0}$. The number of independent modes in $\mathcal{B}$, for $N = 2$, is $2M_1M_2 + M_1 + M_2$, as one sees by applying the reality condition $g(\mathbf{m}, \mathbf{J}, s) = g^*(-\mathbf{m}, \mathbf{J}, s)$. For $M_1 = M_2 = 15$, typical values, there are 480 independent, complex modes. Let $\mathcal{M}$ denote the subset of $\mathcal{B}$ corresponding to the independent modes.

The coefficients with $\mathbf{m} \neq \mathbf{0}$ can be found from Eqn. (7) and the condition that the coefficients must be periodic with period C . This boundary condition yields solutions of the differential equation that give the invariant torus. The linear term in Eqn. (7) can be simplified using the following integrating factor, $\exp(i\mathbf{m} \cdot \mathbf{X}(s))$, where $\mathbf{X}(s) = \int_0^s \Omega(\sigma) d\sigma$ is the linear phase advance. This

defines a new coefficient $h(\mathbf{m}, \mathbf{J}, s) = g(\mathbf{m}, \mathbf{J}, s) \exp(i\mathbf{m} \cdot \mathbf{X}(s))$ that satisfies the differential equation

$$\partial_s h(\mathbf{m}, \mathbf{J}, s) = -e^{i\mathbf{m} \cdot \mathbf{X}(s)} v(\mathbf{m}, \mathbf{J}, s; g(h)), \quad \mathbf{m} \in \mathcal{M}, \quad (9)$$

$$v(\mathbf{m}, \mathbf{J}, s; g(h)) = \int_0^{2\pi} \frac{d\Phi}{(2\pi)^N} e^{-i\mathbf{m} \cdot \Phi} V(\Phi, \mathbf{J} + G_\Phi, s), \quad (10)$$

$$G_\Phi = -2 \operatorname{Im} \sum_{\mathbf{m} \in \mathcal{M}} \mathbf{m} h(\mathbf{m}, \mathbf{J}, s) e^{i\mathbf{m} \cdot (\Phi - \mathbf{X}(s))}. \quad (11)$$

Eqn. (9) is to be solved under the boundary condition

$$h(\mathbf{m}, \mathbf{J}, C) = e^{2i\mathbf{m} \cdot \nu} h(\mathbf{m}, \mathbf{J}, 0). \quad (12)$$

Notice that the boundary condition of h differs from the periodicity of g by the total phase advance, $\mathbf{X}(C) = 2\pi\nu$, of one revolution around the ring.

The solution to Eqns. (9)-(12) can be formulated as a fixed point of a nonlinear map.^{5,6} Writing $h(\mathbf{m}, \mathbf{J}, s)$ as $h(s)$, the (numerical) integration of Eqn. (9) from some initial condition $h(0)$ can be considered a map $U: h(0) \mapsto h(C) \mapsto h(0)$. Substituting Eqn. (12) for $h(C)$ gives a relation that the initial condition meets when it satisfies Eqns. (9)-(12):

$$h(\mathbf{m}, 0) = \frac{1}{e^{2i\mathbf{m} \cdot \nu} - 1} U(\mathbf{m}, h(0)) = A(\mathbf{m}, h(0)). \quad (13)$$

The fixed point, $h(0)$, of $A(h(0))$ yields the coefficients that solve the above differential equation and boundary condition.

We do not use a Fourier series to represent the s -dependence of h , since that would give poor accuracy when the perturbations are step functions in s , as is the case in storage rings.

Numerical Techniques

The problem of finding the solution of the Hamilton-Jacobi equation is now the problem of finding the fixed point of a nonlinear map. In this section, we discuss two techniques we use to find the fixed point: the simple iteration scheme and the Newton-Broyden iteration scheme.

The simple iteration, which works well for weak effective nonlinearities, is just the iteration $h^{i+1}(0) = A(h^i(0))$, where i labels the iterate. For a sufficiently weak A (proportional to V) this converges to a unique fixed point. The initial guess for the solution, $h^0(0)$, is found using the lowest order perturbative approximation for the solution. This amounts to solving Eqns. (9)-(12) with G_Φ in Eqn. (10) set to zero.

The Newton-Broyden iteration is a method of the Newton type for determination of roots of

$$F(\mathbf{m}, h(0)) = A(\mathbf{m}, h(0)) - h(0) = 0 \quad (14)$$

using a Broyden update^{7,8} procedure to approximate the new Jacobian of F at each iteration. The Jacobian must be fully calculated only for the first iteration. For large numbers of independent modes, this saves considerable computing time.

The complex derivative of F is not well defined; F is not an analytic function of $h(0)$. If Eqn. (14) is written as two real equations, then the derivative of F with respect to $\text{Re } h$ and $\text{Im } h$ is defined. A compact notation is used for the real equations, $\tilde{F} = (\text{Re } F, \text{Im } F)^T$. Equation (14) has the form $\tilde{F}(\mathbf{m}, \tilde{h}(0)) = 0$, where $\tilde{h}(0) = (\text{Re } h(0), \text{Im } h(0))^T$. The first-order expansion of this equation about the previous iterate is

$$\tilde{F}(\mathbf{m}, \tilde{h}^i(0)) + \sum_{\mathbf{n} \in \mathcal{M}} \tilde{D}(\mathbf{m}, \mathbf{n}, \tilde{h}^i(0)) \cdot [\tilde{h}^{i+1}(\mathbf{n}, 0) - \tilde{h}^i(\mathbf{n}, 0)] = 0 \quad (15)$$

with

$$\tilde{D}(\mathbf{m}, \mathbf{n}, \tilde{h}(0)) = \begin{pmatrix} \partial \text{Re } F(\mathbf{m}) / \partial \text{Re } h(\mathbf{n}, 0) & \partial \text{Re } F(\mathbf{m}) / \partial \text{Im } h(\mathbf{n}, 0) \\ \partial \text{Im } F(\mathbf{m}) / \partial \text{Re } h(\mathbf{n}, 0) & \partial \text{Im } F(\mathbf{m}) / \partial \text{Im } h(\mathbf{n}, 0) \end{pmatrix} \quad (16)$$

This system of equations is solved for the new iterate, $\tilde{h}^{i+1}$.

The Jacobian $\tilde{D}$ of the map is numerically calculated for the first iterate using divided differences. The components of $\tilde{h}(0)$ are perturbed separately and the map $\tilde{F}$ is evaluated. The elements of the derivative matrix are just the differences of the map at the nominal and perturbed coefficient divided by the magnitude of the appropriate perturbation.

If $\tilde{D}^i$ is the Jacobian at the i -th iteration, the Broyden update giving the Jacobian at the $(i+1)$ -th iteration is^{7,8}

$$\tilde{D}^{i+1} = \tilde{D}^i + \frac{[\tilde{F}(\tilde{h}^{i+1}) - \tilde{F}(\tilde{h}^i) - \tilde{D}^i(\tilde{h}^{i+1} - \tilde{h}^i)](\tilde{h}^{i+1} - \tilde{h}^i)^T}{(\tilde{h}^{i+1} - \tilde{h}^i)^T(\tilde{h}^{i+1} - \tilde{h}^i)} \quad (17)$$

Notice that this requires only one full calculation of the Jacobian at the start of the iteration and the subsequent updates need only one evaluation of the map $\tilde{F}$.

The map U is numerically evaluated using a fourth-order Runge-Kutta algorithm for the s integration and a fast Fourier transform for the Φ integration. We specify N_{RK} integration steps per nonlinear element. For strong effective nonlinearity, the number of integration steps must be increased to maintain the

accuracy of the solution. The perturbation estimate for $h^0(0)$ is not generally a good enough guess to ensure convergence for large N_{RK} , so we calculate the solution at a smaller, intermediate value for N_{RK} and use it as the starting point. We take the number of discrete steps in the Φ integration to be at least twice, and usually four times, the M_1 and M_2 mode numbers.

To save computation time, especially in the $2 \frac{1}{2}$ degree of freedom case, only numerically significant modes from the mode set are selected. At an early stage in the iteration, after performing one simple iteration, numerically significant modes (wherever in s) are kept for the calculation. The other modes are ignored. In practice, for the $2 \frac{1}{2}$ degree of freedom case, no more than 130-200 modes are kept for the longest calculations.

Once the fixed point, $h(\cdot)$, is reached, the s -dependence of the coefficients can be determined by integrating the differential equation, Eqn. (9).

Numerical Results

We present results of approximate invariant tori for a single cell of the Berkeley Advance Light Source (ALS).⁹ This storage ring lattice has very strong sextupoles and very good periodicity (the ring is made up of 12 identical cells). The parameters for the single cell of the ALS are given in Table 1.

TABLE 1. Lattice parameters for the single cell of the Advanced Light Source. Parameters are given at the beginning of the element. The cell length is 16.4 m and each sextupole is 0.20 m long. The tunes are $\nu_1=1.18973$ and $\nu_2=0.68158$.

name	position m	parameters				phase	
		β_1 , m	β_2 , m	α_1	α_2	X_1 , rads	X_2 , rads
SD	5.775	1.472	10.696	-1.779	8.401	2.480	0.866
SF	6.875	3.984	1.580	2.272	0.417	2.819	1.222
SF	9.325	3.137	1.443	-1.963	-0.268	4.600	2.928
SD	10.425	2.297	7.603	2.345	-7.062	4.886	3.395
Strengths are $S_{SD} = -88.090 \text{ m}^{-3}$, and $S_{SF} = 115.615 \text{ m}^{-3}$.							
At $s = 0$, $\beta_1 = 11.0 \text{ m}$, $\beta_2 = 4.0 \text{ m}$, and $\alpha_1 = \alpha_2 = 0$.							

In Fig. 1, we show some one-dimensional invariant surfaces (top) and results of symplectic integration¹⁰ (tracking) of the equations of motion (bottom). They are shown in the transverse phase space at $s = 0$. The phase coordinates, (x_{N1}, p_{N1}) , are normalized so that linear motion with $J_1 = 2.2 \cdot 10^{-5} \text{ m}$ would be represented

by a circle of unit radius. The tracking results are obtained by symplectically integrating the equations of motion using many initial conditions, some of which lie on the calculated invariant surfaces. (Corresponding graphs of surface points and tracking points are indistinguishable (see below for a discussion of the errors in the agreement of the torus and tracking). We also notice that the solution at the largest amplitude is very close to a region of large scale instability, and is outside of an island chain. This curve defines the approximate dynamic aperture for the one-dimensional motion.

The accuracy of the invariant surface can be estimated by comparing the surface with trajectories that begin on it and are symplectically integrated for N_r revolutions. We use 16 initial conditions uniformly distributed in Φ_1 (or Φ in two-dimensions) on the calculated invariant surface at $s = 0$. Each trajectory is tracked for $N_r = 1000$ revolutions and is compared to the invariant surface at each $s = 0$ crossing. Two parameters are defined that measure the degree to which the actions, I_1^T , of the trajectories differ from those of the solution:

$$\delta_1 = \max_{\{\Phi_1(0)\}} \frac{\sum_{k=1}^{N_r} |I_1(\Phi_{1k}^T) - I_{1k}^T|}{\sum_{k=1}^{N_r} |I_1(\Phi_{1k}^T) - J_1|}, \quad (18)$$

$$\delta_{J_1} = \max_{\{\Phi_1(0)\}} \frac{\sum_{k=1}^{N_r} |I_1(\Phi_{1k}^T) - I_{1k}^T|/(N_r J_1)}{\sum_{k=1}^{N_r} |I_1(\Phi_{1k}^T) - J_1|}. \quad (19)$$

Similar parameters are defined for each surface in the two-dimensional case. The two parameters differ only in the normalization of the deviation of the actions.

TABLE 2. Summary of parameters for the one-dimensional invariant surfaces.

J_1 (10^{-5} m)	M_1	N_{RK}	tracking comparison δ_1	smear	tune shift	cpu time
2.2	63	14/28	$4.10 \cdot 10^{-2}$	$\pm 35.8\%$	$-1.39 \cdot 10^{-2}$	8m20s
2.0	63	10/20	$3.67 \cdot 10^{-3}$	$\pm 31.6\%$	$-1.22 \cdot 10^{-2}$	5m13s
1.0	63	10/20	$5.41 \cdot 10^{-4}$	$\pm 20.5\%$	$-5.53 \cdot 10^{-3}$	7m50s
0.1	31	6/12	$5.05 \cdot 10^{-5}$	$\pm 6.4\%$	$-5.35 \cdot 10^{-4}$	0m55s
0.01	7	2	$1.78 \cdot 10^{-4}$	$\pm 2.0\%$	$-5.34 \cdot 10^{-5}$	0m05s

Several parameters for the calculated invariant tori shown in Fig. 1 are given in Table 2. It shows that the surfaces agree very well with short term symplectic tracking. We define the smear as $\pm [(I_1)_{\max} - (I_1)_{\min}] / [(I_1)_{\max} + (I_1)_{\min}]$. It

measures the degree of distortion of the surface from linearity. The tune shift is the change in the tune, ν_1 , from its linear value that results from the nonlinearity. The total 'CPU' time is the computation time for each calculation with diagnostics on the SLAC IBM 3090 computer.

For 2 1/2 degrees of freedom, the calculation of accurate surfaces for large amplitudes can be very time consuming. Several cases for the ALS cell are shown in Table 3. Notice the large mode sizes and generally long computation times. The maximum mode numbers for each angle are chosen to be equal, $M_1 = M_2$. The selected and total number of independent modes are shown, as well as the intermediate and the final number of integration steps. The comparison with short term tracking is given for each surface.

TABLE 3. Summary of parameters for the two-dimensional solutions.

J_1, J_2 (10^{-6} m)	M_i	modes	N_{RK}	tracking comparison δ_1	comparison δ_2	cpu time IBM 3090
0.2, 0.2	15	50/480	2/10	$2.66 \cdot 10^{-4}$	$4.37 \cdot 10^{-4}$	14m57s
4, 4	31	180/1984	7/16	$3.88 \cdot 10^{-2}$	$1.62 \cdot 10^{-2}$	3h13m10s
1, 3	31	125/1984	5/16	$5.64 \cdot 10^{-3}$	$5.16 \cdot 10^{-3}$	1h37m28s
3, 1	31	180/1984	4/16	$5.73 \cdot 10^{-4}$	$9.59 \cdot 10^{-4}$	2h00m45s

Typical two-dimensional invariant surfaces are shown in Figs. 2 and 3. We plot the invariant torus at $s = 0$ as $I_1(\Phi, J, s = 0)/J_1$ and $I_2(\Phi, J, s = 0)/J_2$. Fig. 2 is for a smaller constant action and shows less distortion than Fig. 3. Both surfaces agree well with short term tracking.

Summary

We have discussed a technique to find approximate invariant tori for nonlinear Hamiltonians with arbitrary nonlinearity. In particular, we gave results for a storage ring lattice with sextupole magnets as the perturbation. We showed that accurate solutions can be found for large amplitudes, significant when compared to the dynamic aperture, even in the case with 2 1/2 degrees of freedom.

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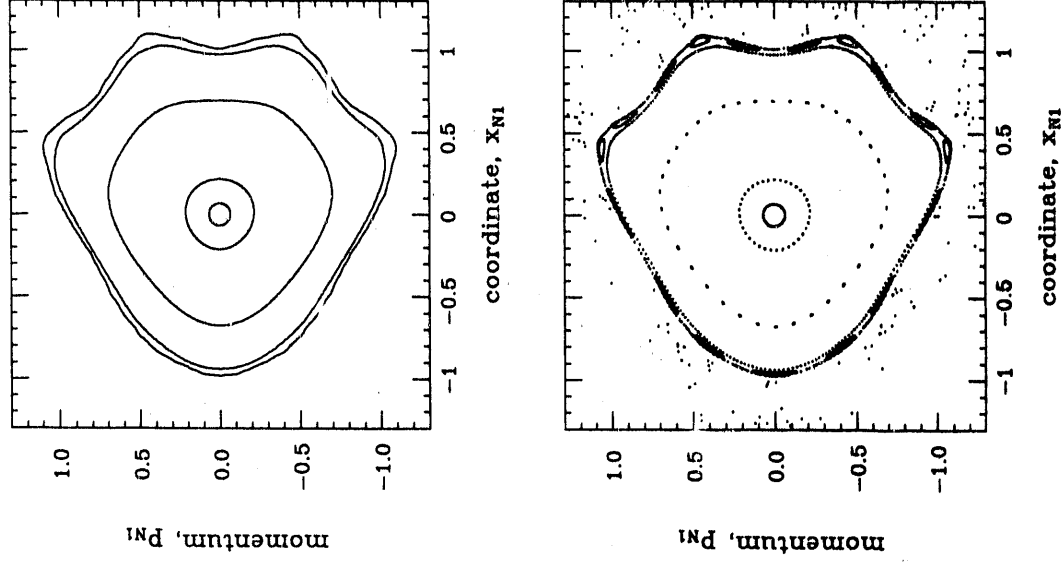


FIG. 1. Phase portrait for the ALS cell from the solution of the Hamilton-Jacobi equation (top) and symplectic tracking (bottom). The coordinates are normalized so that linear motion with $J_1 = 2.2 \cdot 10^{-5}$ m gives a circle of unit radius. The Hamilton-Jacobi solutions correspond to those in Table 2. The smallest, inner circle, is the invariant curve at $J_1 = 10^{-7}$ m; the outermost curve has $J_1 = 2.2 \cdot 10^{-5}$ m.

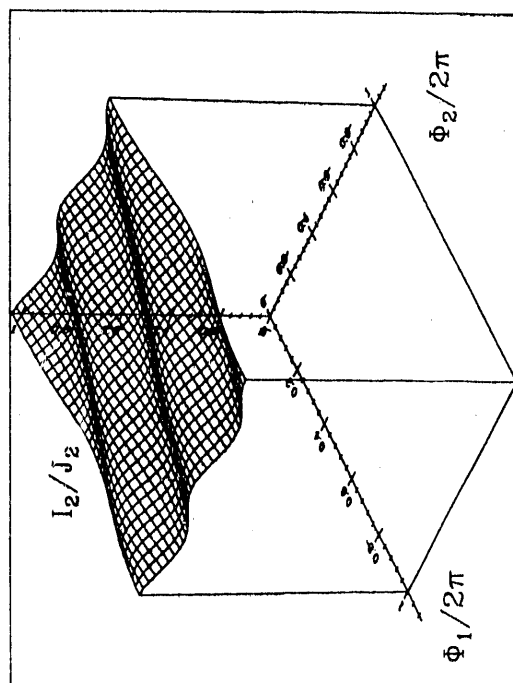
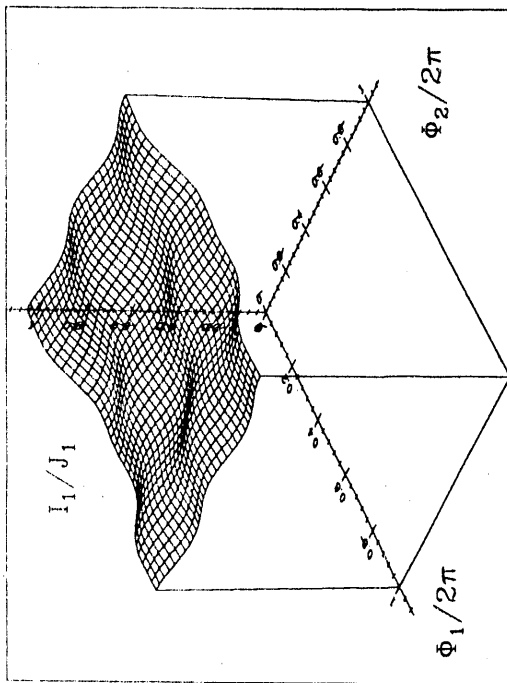


FIG. 2. The $I_1(\Phi)/J_1$ surface (top) and the $I_2(\Phi)/J_2$ surface (bottom) of the approximate invariant torus for $J_1 = J_2 = 2 \cdot 10^{-7}$ m at $s = 0$. Details are given in the text and in Table 3.

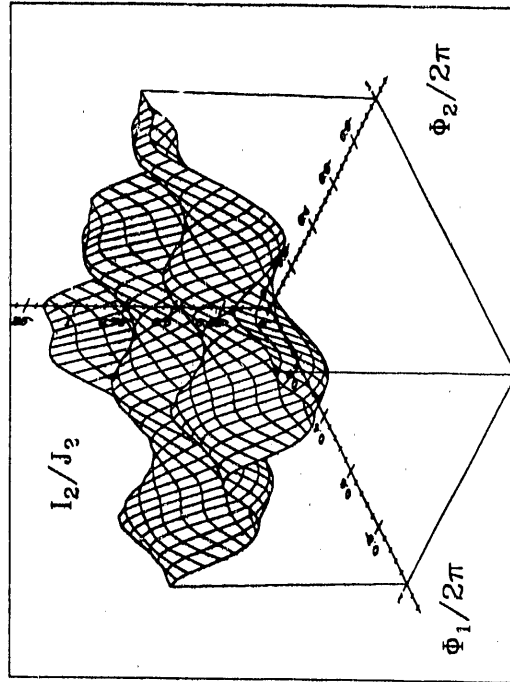
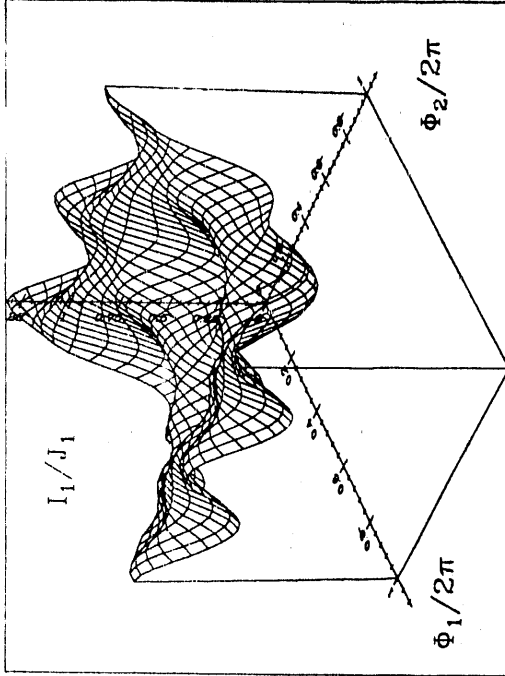


FIG. 3. The $I_1(\Phi)/J_1$ surface (top) and the $I_2(\Phi)/J_2$ surface (bottom) of the approximate invariant torus for $J_1 = J_2 = 4 \cdot 10^{-6}$ m at $s = 0$. Details are given in the text and in Table 3.

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