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J. Manickam, R. C. Grimm, and J. DeLucia

Plasma Physics Laboratory, Princeton University

P.O. Box 451, Princeton, New Jersey 08544

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ABSTRACT

A reduced set of ideal MHD equations is derived for large aspect ratio, low β tokamaks that adequately describes the linear and nonlinear evolution of ideal internal kink modes in tokamaks.

DISCLAIMER

I. INTRODUCTION

Recently the PDX experiment at PPPL has observed large MHD types of oscillations during high power neutral beam injection.[1] The MHD activity evident on both the soft X-ray detectors and the Mirnov loops has been dubbed "fishbones." From the nature of these oscillations, it appears that the ideal internal kink may be their cause.[2] Evidence from linear eigenvalue codes indicates that $\epsilon\beta_p$ is becoming large enough to cause the internal kink to cross the th hold of stability with growth rates larger than the resistive kink growth rate. The nonlinear ideal bouncing of the internal kink could then account for the periodic nature of the oscillation and for the fact that reconnection, as occurs during a sawtooth, does not take place during fishbones.

To simulate the nonlinear evolution of the internal kink mode in tokamaks where $\beta \sim \epsilon^2$ ($\epsilon \equiv$ the inverse aspect ratio) with a reduced set of MHD equations means that we must first carry out the expansion to high order in ϵ. The growth rate, γ_k, typical of free boundary kink modes is the time scale of the lowest order reduced equations. The internal kink is marginally stable in this order. Pressure and toroidal curvature are introduced in the next order. Instead of $\gamma^2 \sim \epsilon\gamma_k^2$ as one might expect, it is found that $\gamma^2 \sim \epsilon^2\gamma_k^2$ because the bad and good curvature average to zero in leading order. We must, therefore, go to higher order in ϵ to find the growth rate of the internal kink accurately, i.e., $\gamma_{Ik}^2 \sim \epsilon^2\gamma_k^2$.

II. HIGH ORDER REDUCED EQUATIONS

The equations of ideal MHD are

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p \quad ,$$

$$\vec{J} = \nabla \times \vec{B} \quad ,$$

$$\frac{dp}{dt} = -\Gamma p \nabla \cdot \vec{V} \quad ,$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{V} \quad ,$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) \quad .$$

We use a cylindrical coordinate system R, Z, ζ. The major radius of the device is denoted by R_o, the minor radius by a, and the ratio of specific heats by Γ.

The starting point of the procedure is to note that with $\beta \sim \varepsilon^2 (\varepsilon = a/R_o)$ the energy principle shows that for modes to be unstable their growth rates must be of the order of or smaller than $\gamma_k = R/V_A$ (V_A is the Alfvén velocity and time dependence is $e^{\gamma t}$.) . The momentum equation then shows that the variation of the toroidal field from 1/R must be of the order of $\varepsilon^2 B_o$. To lowest order in ε this unknown variation of the toroidal field can be eliminated from the problem by taking the curl of the momentum equation. The resulting equations are the standard low β tokamak reduced equations that describe free boundary kink modes.[3]

$$R_o^2 \frac{d\nabla^2 u}{dt} = \vec{B} \cdot \nabla (\nabla_\perp^2 \psi) \quad ,$$

$$\frac{\partial \psi}{\partial t} = R_o^2 \vec{B} \cdot \nabla u \quad ,$$

$$\vec{B} = \nabla \psi \times \nabla \zeta + I_o \nabla \zeta \quad ,$$

$$\vec{V} = R_o^2 \nabla u \times \nabla\zeta \quad ,$$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2} \quad .$$

Here $I_o = B_o R_o$ and $\nabla\zeta = \hat{\zeta}/R_o$.

In this order the internal kink mode is stable and we need to go to higher order in inverse aspect ratio. One might assume at this point that to make progress it is necessary to go back to the momentum equation to find the variation of the toroidal field. But this is not the case. Instead, by first multiplying the momentum equation by R^2 and then taking the curl, the unknown variation of toroidal field can be eliminated to the next order as well. The resulting equations are

$$R_o^2 \frac{d\nabla^2 u}{dt} = \vec{B}\cdot\nabla(\Delta^* \psi) + 2\ \nabla R \times \nabla_\perp p \cdot \nabla\zeta \quad ,$$

$$\frac{\partial\psi}{\partial t} = R^2\ \vec{B}\cdot\nabla u \quad ,$$

$$\frac{dp}{dt} = 0 \quad ,$$

$$\vec{B} = \nabla\psi \times \nabla\zeta + I_o \nabla\zeta \quad ,$$

$$\vec{V} = R^2 \nabla u \times \nabla\zeta \quad ,$$

where $\nabla\zeta = \hat{\zeta}/R$ and we have taken $\rho \propto 1/R^2$ to satisfy the continuity equation trivially. With the inclusion of resistivity these equations have been successful in describing the effects of finite aspect ratio on $\beta = 0$ tearing modes.[4] However, as mentioned in the introduction these next order corrections lead to a growth or oscillation with $|\gamma|^2 \sim \varepsilon^2\gamma_k^2$. This means we must go still higher order in ε to find the correct growth rate or oscillation frequency of the internal kink mode. To do this we must now solve for the variation in the toroidal field. This is accomplished by going back to the momentum equation and making use of the fact that the growth rate is small to eliminate the leading order inertia terms. To proceed we use the following forms for $\vec{v}$ and $\vec{B}$.

$$\vec{V} = R^2\nabla u \times \nabla\zeta + \nabla_\perp\chi + V_\zeta R \nabla\zeta \quad ,$$

$$\vec{B} = \vec{H} \times \nabla\zeta + I\nabla\zeta \quad .$$

These are the most convenient forms, since they reduce to the results for the lower order equations. The expressions for both $\vec{v}$ and $\vec{B}$ are also completely general. The magnetic field components $\vec{H}$ and I are related to the magnetic vector potential $\vec{A}$ by

$$\vec{H} = \nabla\psi - \frac{\partial\vec{A}_\perp}{\partial\zeta} \quad ,$$

$$I = I_o + R^2\nabla_\perp \times \vec{A}_\perp \cdot \nabla\zeta \quad ,$$

where

$$\vec{B} = \nabla \times \vec{A} + I_o\nabla\zeta \quad ,$$

$$\psi = RA_\zeta \quad .$$

Although it is possible to justify a priori the ordering of the variables, we will for convenience set out the consistent ordering. With

$$B_o = 1 \ , \ \rho = 1 \ , \ a = 1$$

we take

$$p \sim \varepsilon^2 \ , \quad t \sim \frac{1}{\varepsilon} \ , \ \psi \sim 1 \ , \ v_\zeta \sim \varepsilon^2$$

$$v_\perp \sim \varepsilon \ , \ u \sim \varepsilon \ , \ I \sim \frac{1}{\varepsilon} \ , \ \nabla I \sim \varepsilon$$

$$A_R \sim \varepsilon^2 \ , \ A_z \sim \varepsilon^2 \ , \ B_\perp \sim \varepsilon \quad .$$

This ordering differs from that of Strauss[5] because we assume $\beta \sim \varepsilon^2$ rather than ε. However, our resulting equations uniformly recover the equations in Ref. 5 when $\beta \sim \varepsilon$.

We start the derivation by taking the curl of R^2 times the momentum equation, but now we keep all terms on the right-hand side and find,

$$R_o^2 \frac{R d \nabla_\perp^2 u}{dt} = -R\vec{B}\cdot\nabla(RJ_\zeta) + \frac{1}{R}\frac{\partial \vec{H}}{\partial \zeta}\cdot\nabla I - \frac{H_R}{R^2}\frac{\partial I}{\partial \zeta} + J_\zeta \frac{\partial I}{\partial \zeta} + 2R\,\nabla R \times \nabla p \cdot \nabla\zeta$$

where,

$$J_\zeta = -R\nabla\cdot\left(\frac{\vec{H}}{R^2}\right) \quad .$$

We can see from this equation that we only need I correct to leading order. Ignoring inertia, the perpendicular component of the momentum equation then gives

$$I_o \nabla_\perp I = \vec{k} \quad ,$$

where

$$\vec{k} = -R^2 \nabla_\perp p - \Delta^* \psi \nabla \psi - I_o \frac{\partial \vec{B}_\perp}{\partial \zeta} \quad .$$

At equilibrium this gives the Grad-Shafranov equation

$$I_o I' + R^2 p' = -\Delta^* \psi \quad .$$

We can solve for I by integrating the above, but it is easier if one takes the divergence of the equation for I and uses the line integral only for the boundary condition. Doing so we find

$$I_o \nabla_\perp^2 I = \nabla \cdot \vec{k} \quad ,$$

$$I_o I_b = \int \vec{k} \cdot d\vec{\ell} + C(t) \quad .$$

We previously used the $\hat{\zeta}$ component of Faraday's law to show that the fluid was incompressible to leading order. We will now use this equation to evolve χ. We find

$$\frac{dI}{dt} = -I_o \left(\nabla_\perp^2 \chi + \frac{1}{R} \frac{\partial v_\zeta}{\partial \zeta} \right) + R(\vec{B} \cdot \nabla) v_\zeta \quad .$$

Next we proceed to find the magnetic field variables from Faraday's law

$$\frac{\partial \vec{A}}{\partial t} = \vec{V} \times \vec{B} + I_o \nabla u + \nabla \phi$$

where we have taken the potential $\overline{\Phi} = I_o u + \phi$ for convenience $(\phi \sim \varepsilon^3)$. This yields

$$\frac{\partial \psi}{\partial t} = R^2 \nabla u \times \vec{H} \cdot \nabla \zeta + I_o \frac{\partial u}{\partial \zeta} - \vec{H} \cdot \nabla \chi + \frac{\partial \phi}{\partial \zeta} \quad .$$

All that remains is to choose the gauge to determine the evolution of the potential ϕ. A convenient choice is

$$\nabla_{\perp} \cdot \vec{A}_{\perp} = 0$$

with

$$\vec{A}_{\perp} = R \nabla F \times \nabla \zeta \quad .$$

Faraday's law then gives

$$\nabla_{\perp} \cdot \left(\frac{\nabla_{\zeta} \vec{H}}{R}\right) - \nabla_{\perp} \cdot \left[(I - I_o)\nabla_{\perp} u\right] + \nabla_{\perp}^2 \phi = 0 \quad ,$$

and from the definition of I we have

$$\nabla_{\perp}^2 F = - \frac{(I-I_o)}{R} \quad .$$

Collecting the results, and keeping terms only to the necessary order:

$$R_o^2 \frac{d\nabla_\perp^2 u}{dt} = \vec{B}\cdot\nabla(\Delta^*\psi) + 2\nabla R \times \nabla p\cdot\nabla\zeta - \frac{\Delta^*\psi}{R^2}\frac{\partial I}{\partial\zeta} + \frac{1}{R}\nabla\left(\frac{\partial\psi}{\partial\zeta}\right)\cdot\nabla I + \nabla\frac{\partial F}{\partial\zeta}\cdot\nabla\Delta^*\psi \quad ,$$

$$\frac{\partial\psi}{\partial t} = R^2\vec{B}\cdot\nabla u - \nabla\psi\cdot\nabla\chi + R\nabla u\cdot\nabla\frac{\partial F}{\partial\zeta} + \frac{\partial\phi}{\partial\zeta} + \left(I_o - I\right)\frac{\partial u}{\partial\zeta} \quad ,$$

$$\frac{dp}{dt} = -\Gamma p\nabla\cdot\vec{v} \quad ,$$

$$\rho\frac{\partial v_\zeta}{\partial t} - \rho R\nabla u \times \nabla v_\zeta \cdot \hat{\zeta} = -\frac{1}{R}\frac{\partial p}{\partial\zeta} - \frac{1}{R^3}\nabla\left(\frac{\partial\psi}{\partial\zeta}\right)\cdot\nabla\psi - \frac{1}{R}\nabla I \times \nabla\zeta \cdot \nabla\psi \quad ,$$

$$I_o\nabla_\perp^2 I = \nabla\cdot\vec{k} \quad ,$$

$$\vec{k} = -R^2\nabla_\perp p - \Delta^*\psi\nabla\psi - I_o\frac{\partial\vec{B}_\perp}{\partial\zeta} \quad ,$$

$$\nabla_\perp^2\chi = -\frac{1}{I_o}\frac{dI}{dt} + \frac{R_o}{I_o}(\vec{B}\cdot\nabla)v_\zeta - \frac{1}{R_o}\frac{\partial v_\zeta}{\partial\zeta} \quad ,$$

$$\nabla_\perp^2\phi = \nabla_\perp\cdot\left[\left(I - I_o\right)\nabla_\perp u\right] - \nabla_\perp\cdot\left(\frac{v_\zeta\nabla\psi}{R}\right) \quad ,$$

$$\nabla_\perp^2 F = -\frac{(I-I_o)}{R} \quad ,$$

$$\vec{v} = R^2\nabla u \times \nabla\zeta + \nabla\chi + v_\zeta R\nabla\zeta \quad ,$$

$$\vec{B} = \nabla\psi \times \nabla\zeta + I\nabla\zeta \quad .$$

The boundary conditions for these equations are a) $\vec{B}\cdot\hat{n}\big|_b = 0$, b) $v_r\big|_b = 0$, and c) $\vec{E} \times \hat{n}\big|_b = 0$. These imply that on b

a) $\psi = 0$, $\frac{\partial f}{\partial r} = h(t)$

b) $u = 0 \;,\; \frac{\partial \chi}{\partial r} = 0$

c) $\Phi = 0$

where h(t) is chosen to satisfy Gauss' theorem. There is also the boundary condition for I

$$I_o I_b = \int \vec{k} \cdot d\vec{\ell} + C(t)$$

where C(t) is chosen to keep

$$\int \frac{I}{R}\, dA = \text{constant}$$

which follows from the conservation of toroidal flux with $E \times n\big|_b = 0$.

III. RESULTS

The above set of reduced equations has been coded using a finite difference scheme in the radial direction and spectral analysis in the poloidal and toroidal direction. From the work of Bussac et al.[6] we know that the toroidal case does not reduce to the cylindrical case for poloidal and toroidal mode numbers m = 1 and n = 1. Therefore a flag has been put into the code to null the toroidal terms and allow a check with a 1-D cylindrical solution obtained by quadrature. For the cylinder, the product of the parallel wave number and the minor radius, ka, plays the same role as the inverse aspect ratio, ϵ, for a torus. The results of this test are shown in the first figure. The results are good even for aspect ratios of 5. The toroidal version with the flag turned off can be checked against the cylindrical case for $m = 1$, $n >> 1$ as the results of Bussac et al. show. (However, $\epsilon\beta_p$ must go like $1/n$ and nq must remain fixed.) The results of this test are also shown in Fig. 1.

To check the toroidal terms themselves we have carried out a comparison with the PEST[7] code for a case with an aspect ratio of 10. The results are displayed in Fig. 2. The two codes are expected to agree within ϵ. Indeed in regions where γ is not a strong function of q(o) the agreement is quite good. In addition, the computed marginal points agree rather well. Both codes indicate that the $m = 1$ ideal, internal kink is linearly unstable for a particular window in q(o). This window will depend upon the equilibrium tested.

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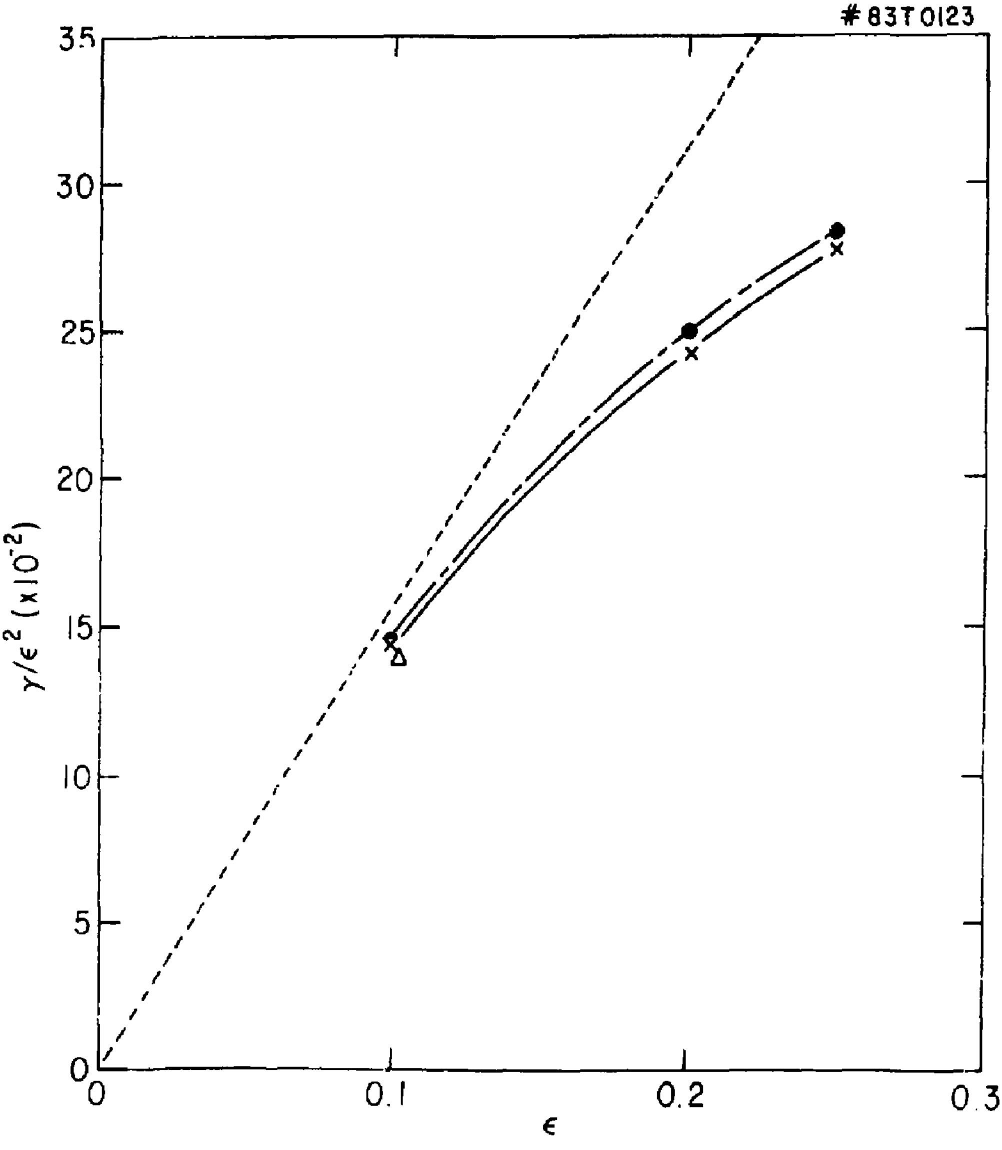
FIGURE CAPTIONS

FIG. 1. γ/ε^2 vs ε for $J_z = J_o\,(1 - r^2/a^2)^2$ $\rho = 1$, $B_z = 1$, $a = 1$. Dashed curve is from Rosenbluth, Dagazian, Rutherford, Phys. Fluids 16, 1894 (1973). • are from quadrature. X are from reduced equations. All the above are for $m = 1$, $n = 1$ cylinder. Δ is from reduced equations for torus and $m = 1$, $n = 5$.

FIG. 2. Linear growth rate of $n = 1$ ideal mode vs $q(0)$ for $\varepsilon\beta_p = 0.55$ and $q(a)/q(0) = 2.5$. Dashed curve is from reduced equations and solid curve is from PEST.



#83T0123
35
30
25
20
15
10
5
0
$\gamma/\epsilon^2 (\times 10^{-2})$
0
0.1
0.2
0.3
ϵ

Fig. 1

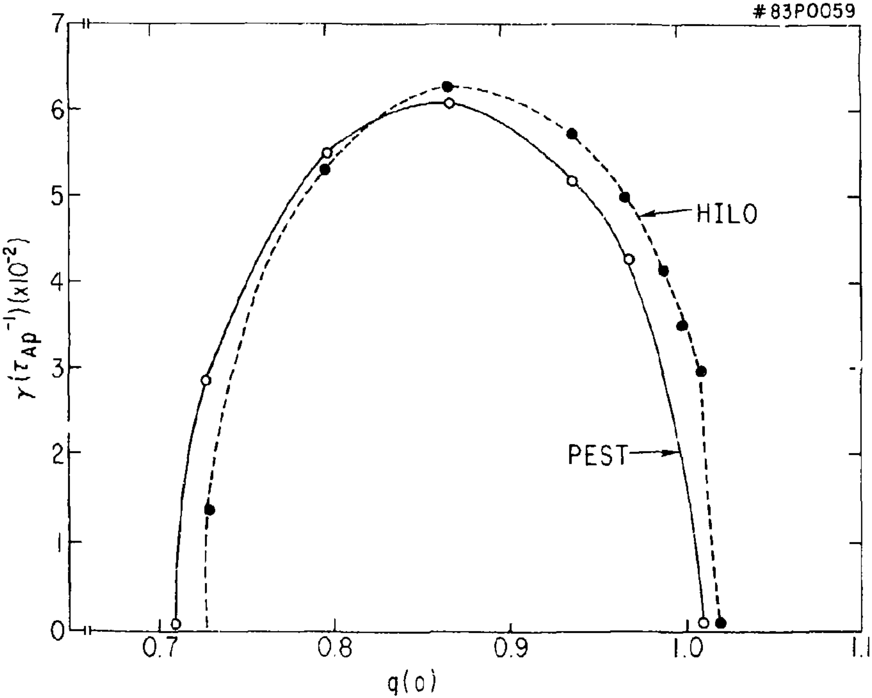


Fig. 2

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