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TRANSVERSE CORRELATIONS IN START-UP OF A FREE ELECTRON LASER FROM NOISE*

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ABSTRACT

Linearized Vlasov-Maxwell equations are used to derive a partial differential equation determining the 3-dimensional slowly varying envelope function of the radiated electric field. The equation is solved analytically. From the correlation function $\langle E(z,\vec{r},t)\, E^*(z',\vec{r}',t)\rangle$ of the electric field averaged over the stochastic ensemble describing the initial shot noise in the beam, we compute the longitudinal and transverse correlation lengths $\sigma_\parallel$ and $\sigma_\perp$. The radiated power S per unit cross-sectional area of the electron beam is

$$S = \frac{\rho S_e}{9 n_o V_c} \exp(\sqrt{3}\, 4\pi N_w \rho),$$

where $V_c = (2\pi)^{3/2}\, \sigma_\parallel \sigma_\perp^2$ is the coherence volume, n_o the electron density, $S_e = (\gamma_0 mc^2)\, n_0 c$ the power per unit area in the electron beam, N_w the number of wiggler periods and ρ the Pierce parameter. The angular distribution of the radiation is characterized by the Gaussian factor $\exp(-\theta^2/2\sigma_\theta^2)$, where $2\pi\sigma_\theta\sigma_\perp = \lambda$ (radiated wavelength). Our analysis is applicable for wiggler length $L = N_w\lambda_w$ long enough for the exponential regime to be reached, but short enough so that $L\, \sigma_\theta \lesssim a$, the electron beam radius.

INTRODUCTION

There is great interest in using a free electron laser (FEL) operating in the high-gain regime for the generation of high intensity coherent radiation at wavelengths below 1000 Å. Amplification in a long wiggler magnet of the initial spontaneous radiation emitted by individual electrons has the attractive feature that the use of an optical resonator is avoided. The process of self-amplified spontaneous emission is still not well understood. Three key issues which require further elucidation to facilitate the design of a single pass FEL are the start-up[1-3] of the laser from the shot noise in the electron beam, the guiding[4,5] (or self-focusing) of the radiation by the electron beam, and the saturation of the exponential growth of the radiation field due to nonlinear effects.

MASTER

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EAB

In this paper we present the results of an analysis of the start-up of an FEL from shot-noise. Linearized Vlasov-Maxwell equations have been used to derive a partial differential equation determining the three-dimensional slowly varying envelope function of the emitted radiation, extending an earlier one-dimensional treatment[2,3] to include transverse variations. The problem with the one-dimensional model is that individual electrons are treated as two-dimensional charge sheets, hence the angular distribution of the radiation cannot be properly described, and the total radiated power cannot be correctly determined. In the three-dimensional calculation which we shall present, individual electrons are described as point charges (Fig. 1), allowing us to determine the angular distribution of the emitted radiation and the build-up of transverse correlations.

Initially each electron radiates independently of all others, and the angular distribution of the radiation is that of the spontaneous radiation from a point charge. As the electron beam proceeds down the wiggler magnet different electrons communicate via their emitted radiation and correlations build up. As the transverse correlation length increases, the angular distribution of the radiation narrows. The description of the development of transverse correlations and the narrowing of the angular distribution are the key subjects of this paper.

ENVELOPE EQUATION

Suppose a highly relativistic electron beam is moving in the positive z-direction through a periodic helical wiggler with vector potential $\vec{A}_w = A_w(\hat{e}_- e^{ik_w z} + c.c.)/\sqrt{2}$, where $\hat{e}_\pm = (\hat{e}_1 \pm i\hat{e}_2)/\sqrt{2}$ and $\hat{e}_1$ and $\hat{e}_2$ are orthogonal unit vectors transverse to $\hat{z}$. The transverse electron velocity is approximated by $\vec{v}_\perp \simeq -e\vec{A}_w/m\gamma$ and the longitudinal velocity by $v_\parallel \simeq c(1 - \frac{1+K^2}{2\gamma^2})$, where γ is the electron energy in units of its rest mass and $K = eA_w/mc$ is the wiggler strength parameter. The electron beam is assumed to be initially monoenergetic with all electrons having energy γ_o and longitudinal velocity $v_\parallel(\gamma_o) = v_o$. The spontaneous radiation emitted by the electrons in the forward direction is left circularly polarized with wave number k_o and frequency $\omega_o = k_o c$. The combined action of the static wiggler field and the radiation field produces a ponderomotive potential, which has the dependence $e^{ik_o z - i\omega_o t}\, e^{ik_w z}$. Because the electron beam moves with velocity v_o, the modulation should be of the form $e^{ik_r(z - v_o t)}$. To be in resonance, these two exponential expressions should be the same, hence we have

$$k_r = k_o + k_w \quad (1)$$

and

$$k_r v_o = k_o c = \omega_o . \quad (2)$$

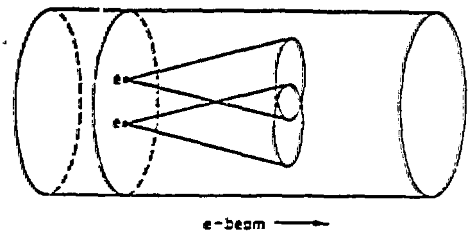

Fig. 1. In the three-dimensional calculation, individual electrons are described as point charges allowing the proper determination of angular distribution and transverse correlations of the emitted radiation.

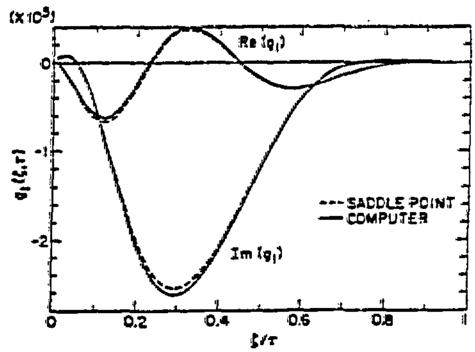

Fig. 2. Numerical evaluation of the one-dimensional Green's function $g_1(\zeta,\tau)$ introduced in Eq. (21). Note the saddle point approximation is very accurate. The case shown corresponds to $\rho = 3\times10^{-3}$, $N_w = 300$, $2\rho\tau = 3.6\pi$.

It follows that $k_o/k_w = v_o/(c-v_o) = 2\gamma^2/(1+K^2)$ and $k_r = \omega_o/v_o = \omega_w/(c-v_o)$, where $\omega_w = k_w c$.

The radiated electric field $\vec{\varepsilon}$ satisfies the wave equation, in mks units,

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\vec{\varepsilon} = \mu_o \frac{\partial \vec{j}}{\partial t} \tag{3}$$

The current density $\vec{j}$ is given by

$$\vec{j} = en_o \int \vec{v}_\perp\, f d\gamma \tag{4}$$

with n_o being the peak density of the electron beam and $n_o f(z,\vec{r},\gamma,t)\, dz d^2r d\gamma$ being the number of electrons in element $dz d^2 r d\gamma$. (Transverse coordinates denoted by $\vec{r}$.) Writing the distribution as $f = f_0 + f_1$, the linearized Vlasov equation is

$$\frac{\partial f}{\partial t} + v_\parallel(\gamma)\frac{\partial f}{\partial z} + \dot{\gamma}\frac{\partial f_0}{\partial \gamma} = 0 \tag{5}$$

where

$$\dot{\gamma} = \frac{e}{mc^2}\,\vec{v}\cdot\vec{\varepsilon}\,. \tag{6}$$

It is convenient to introduce dimensionless variables measuring spatial and temporal variations:

$$\tau = \omega_w t\ ,\ \zeta = k_r(z-v_o t),$$

$$\vec{x} = \sqrt{2k_o k_w}\ \vec{r}\ , \qquad \nabla_\perp^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\ .$$

The unperturbed equilibrium distribution is taken to be

$$f_o = u(\zeta,\vec{x})\ \delta(\gamma-\gamma_o), \tag{7}$$

the smooth function $u(\zeta,\vec{x})$ describes the average properties of the initial electron beam, in the absence of the high-frequency shot noise. The distribution f is determined from Eq. (5) subject to the initial condition at $t = 0$,

$$f(t=0) = \frac{1}{n_o}\sum_i\ \delta(z-z_i)\ \delta(\vec{r}-\vec{r}_i)\ \delta(\gamma-\gamma_o)\ . \tag{8}$$

The shot noise is taken into account by treating the initial coordinates $z_i,\vec{r}_i$ of the i th electron as stochastic variables and determining physical quantities as averages over the ensemble of possible $z_i,\vec{r}_i$.

Introducing the slowly varying envelope function E by writing $\varepsilon = E\exp(ik_0z-i\omega_0t)$, and using the paraxial approximation, the coupled Vlasov-Maxwell equations can be shown to take the form:

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta} - i\nabla_\perp^2\right) E = J \quad , \tag{9}$$

$$\frac{\partial^2 J}{\partial\tau^2} = \alpha \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta} + i\right)(uE) \ , \tag{10}$$

$$J \equiv \frac{n_o\mu_o e^2c^2A_w}{2m\omega_w} e^{-i\zeta} \int \frac{d\gamma}{\gamma} f \ . \tag{11}$$

The constant α in Eq. (10) is related to the Pierce parameter ρ of Bonifacio et al.[6] by

$$\alpha = (2\rho)^3 = \frac{n_o\mu_o e^4A_w^2}{2m^3\gamma_o^3\omega_w^2} \tag{12}$$

Eqs. (9) and (10) immediately lead to the envelope equation:

$$\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta} - i\nabla_\perp^2\right) E = \alpha\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta} + i\right)(uE) \ . \tag{13}$$

GREEN'S FUNCTION

For an initially uniform electron beam, $u(\zeta,\vec{x}) = 1$, we introduce the Green's function $g(\zeta,\vec{x},\tau)$ via

$$\left[\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta} - i\nabla_\perp^2\right) - \alpha\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta} + i\right)\right]g = \delta(\zeta)\ \delta(\vec{x})\ \delta(\tau) \ . \tag{14}$$

Solving by Fourier-Lapace transform yields

$$g(\zeta,\vec{x},\tau) = \int_{-\infty+is}^{\infty+is} \frac{d\Omega}{2\pi i} \int \frac{dq_\parallel d^2q_\perp}{(2\pi)^3} \frac{\exp(iq_\parallel\zeta+i\vec{q}_\perp\cdot\vec{x}-i\Omega\tau)}{D(\Omega,q_\parallel,\vec{q}_\perp)} \ , \tag{15}$$

with

$$D(\Omega,q_\parallel,\vec{q}_\perp) = \Omega^3 - (q_\parallel + q_\perp^2)\ \Omega^2 + \alpha\Omega - \alpha(1 + q_\parallel) \ . \tag{16}$$

It follows that

$$g(\zeta,\vec{x},\tau) = \int dq_\parallel d^2q_\perp\ G_q(\tau)e^{iq_\parallel\zeta+i\vec{q}_\perp\cdot\vec{x}} \tag{17}$$

where

$$G_q(\tau) = \frac{-1}{(2\pi^3)} \left[\frac{e^{-i\Omega_1\tau}}{(\Omega_1-\Omega_2)(\Omega_1-\Omega_3)} + \frac{e^{-i\Omega_2\tau}}{(\Omega_2-\Omega_1)(\Omega_2-\Omega_3)} + \frac{e^{-i\Omega_3\tau}}{(\Omega_3-\Omega_1)(\Omega_3-\Omega_2)} \right] \quad (18)$$

and $\Omega_1, \Omega_2, \Omega_3$ are the three solutions of $D(\Omega, q_\parallel, \vec{q}_\perp) = 0$.

Another useful representation of the Green's function can be obtained from Eq. (15) by employing the identity

$$\frac{i}{D} = \int_0^{\infty+i\varepsilon} d\nu \, e^{i\nu D} \quad (19)$$

which allows the integrals over $q_\parallel$ and $\vec{q}_\perp$ to be performed, one obtains

$$g(\zeta,\vec{x},\tau) = \frac{i\theta(\zeta)\theta(\tau-\zeta)}{8\pi^2\zeta} e^{ix^2/4\zeta} \int_{-\infty+is}^{\infty+is} \frac{d\Omega}{\Omega^2} \exp[-i\Omega(\tau-\zeta) - \frac{i\alpha\zeta}{\Omega^2+\alpha} + \frac{i\alpha x^2}{4\Omega^2\zeta}] \quad (20)$$

We chose $s > \sqrt{\alpha}$ in Eq. (20), so the integral vanishes[7] for $\tau - \zeta < 0$.

In the regime of exponential growth, $\tau \to \infty$, τ/ζ finite, the integral in Eq. (18) is dominated by a saddle point[8] at $\Omega_o = [2\alpha\zeta/(\tau-\zeta)]^{1/3} e^{2\pi i/3}$. We consider $\rho \ll 1$ so $\alpha \ll |\Omega_o^2|$. Then we found the saddle point method is valid if $\rho\tau \gg 1$, and

$$g(\zeta,\vec{x},\tau) \simeq g_E(\zeta,\vec{x},\tau) \, g_1(\zeta,\tau) \, R(\zeta,\vec{x},\tau) , \quad (21)$$

where

$$g_E(\zeta,x,\tau) = - \frac{i\theta(\zeta)\theta(\tau-\zeta)}{4\pi\zeta} e^{ix^2/4\zeta} , \quad (22)$$

$$g_1(\zeta,\tau) = \frac{1}{4\pi\rho} \sqrt{\frac{\pi}{6\rho\zeta}} \exp \{- \frac{3}{2} e^{2\pi i/3} [2\zeta(\tau-\zeta)^2]^{1/3} 2\rho i - \frac{\pi}{4} i\}, \quad (23)$$

$$R(\zeta,\vec{x},\tau) = \exp \{\frac{1}{8} i \, e^{2\pi i/3} [2\zeta(\tau-\zeta)^2]^{1/3} 2\rho \frac{x^2}{\zeta^2}\} \quad (24)$$

Here, g_E is the Green's function (within the paraxial approximation) for the low density limit $\alpha = 0$; g_1 is the Green's function for the one-dimensional model[3] [see Fig. 2]; R describes the transverse fall-off of the radiation. From the derivation of these results, it can be seen that the transverse fall-off is a consequence of the term $\alpha\partial g/\partial\zeta$ in Eq. (14) and correspondingly of the term $\alpha q_\parallel$ in Eq. (16). This term is negligible in the one-dimensional treatment, but is of primary importance in three-dimensions, since zero detuning corresponds to $q_\parallel + q_\perp^2 = 0$ as shown later [see Eq. (36)], so $q_\parallel \approx -q_\perp^2$ and the term $\alpha q_\parallel$ determines the divergence angle of the radiation, and the transverse size of the radiation cone.

The maximum of $g_1(\zeta,\tau)$ is at $\zeta = \tau/3$. From Eq. (24), we find

$$\left|R\ (\zeta = \tau/3,\ \vec{x},\ \tau)\right|^2 = e^{-x^2/2\sigma_x^2} \tag{25}$$

with

$$\sigma_x^2 = \frac{\tau}{3\sqrt{3}\,\rho}. \tag{26}$$

FEL START-UP FROM SHOT NOISE

We wish to solve the envelope equation (13) subject to initial conditions specified at $t = 0$. In particular, we specify $E(\zeta,\vec{x},o) = E_o(\zeta,\vec{x})$, $J(\zeta,\vec{x},o) = J_o(\zeta,\vec{x})$ and $\dot{J}(\zeta,\vec{x},o) = \dot{J}_o(\zeta,\vec{x})$, where the dot denotes $\partial/\partial\tau$ and the current J was introduced in Eqs. (9-11). The envelope function is then determined by

$$E(\zeta,\vec{x},\tau) = \int d\zeta' d^2x'\ [E_o(\zeta',\vec{x}')\ \ddot{g}(\zeta-\zeta',\ \vec{x}-\vec{x}',\ \tau) + J_o(\zeta',\vec{x}')\ \dot{g}(\zeta-\zeta',\vec{x}-\vec{x}',\tau) + \dot{J}_o(\zeta',\vec{x}')\ g(\zeta-\zeta',\vec{x}-\vec{x}',\ \tau)] \tag{27}$$

where $g(\zeta,\vec{x},t)$ is the Green's function defined in Eq. (14). Here, E_o represents an initial electric field possibly due to an external laser; J_o describes the initial spatial bunching of the electron beam and $\dot{J}_o$ corresponds to an initial energy modulation of the electron beam.

We assume the absence of an external radiation field, $E_o = 0$, and describe the shot noise by

$$J_o = \frac{n_o \mu_o e^2 c^2 A_w}{2m\omega_w} e^{-i\zeta} \int \frac{d\gamma}{\gamma}\ f \tag{28}$$

$$f = \frac{1}{n_o} \sum_i \delta(z-z_i)\ \delta(\vec{r}-\vec{r}_i)\ \delta(\gamma-\gamma_o)\ , \tag{29}$$

where the coordinates z_i, $\vec{r}_i$ of the i th electron are treated as independent random variables. For the purposes of the present discussion we ignore the spread in energies of the electrons, hence $\dot{J}_o = 0$. Although $\langle E\rangle = 0$, averages of quantities quadratic in E do not vanish. In particular, the correlation function of the electric field at two different spatial points is found to be expressed in terms of the Fourier transform of the Green's function $G_q(\tau)$ [see Eqs. (17-18)] via

$$C(z,\vec{r},\tau) = \frac{1}{Z_o} \langle E^*(z,\vec{r},\tau)\, E(o,o,\tau)\rangle$$

$$= 4\pi^3 \alpha c(\gamma_o mc^2) \int dk_\parallel d^2k_\perp \left|\dot{G}(q_\parallel,\vec{q}_\perp,\tau)\right|^2 e^{ik_\parallel z + i\vec{k}_\perp \cdot \vec{r}} , \quad (30)$$

where $k_\parallel = k_r q_\parallel$, $\vec{k}_\perp = \sqrt{2k_o k_w}\,\vec{q}_\perp$, and $Z_o = \mu_o c$ is the impedance of free space. Denoting the radiated power per unit area by S and the radiated power per unit area, per unit solid angle, per unit frequency $dP/dAd\Omega d\omega$, we see that

$$S = C(o,o,\tau) = \int d\omega d\Omega \; \frac{dP}{dAd\Omega d\omega} \; . \quad (31)$$

Using $dk_\parallel d^2k_\perp = k^2 dk d\Omega$ with $k = \omega/c$, Eqs. (30) and (31) show

$$\frac{dP}{dAd\Omega d\omega} = 4\pi^3 \alpha m \gamma_o \omega^2 \left|\dot{G}(q_\parallel,q_\perp^2,\tau)\right|^2 . \quad (32)$$

In the high gain regime, keeping only the dominant term,

$$\dot{G}_q(\tau) \simeq \frac{i}{(2\pi)^3} \frac{\Omega_1 e^{-i\Omega_1 \tau}}{(\Omega_1-\Omega_2)(\Omega_1-\Omega_3)} , \quad (33)$$

where $D(\Omega_1,q_\parallel,\vec{q}_\perp) = 0$ [Eq. (16)] and $\mathrm{Im}\Omega_1 > 0$. We consider $\rho \ll 1$ and $2\rho\tau > 1$. Introducing μ by

$$\Omega = 2\rho(1 + q_\parallel)^{1/3} \mu , \quad (34)$$

to good approximation μ is determined from

$$\mu^3 - \Delta\mu^2 - 1 = 0 , \quad (35)$$

$$\Delta = \frac{q_\parallel + q_\perp^2}{2\rho(1+q_\parallel)^{1/3}} \simeq \frac{\omega - \omega_1(\theta)}{2\rho\omega_1(\theta)} , \quad (36)$$

where[9] $\omega_1(\theta) = 2\gamma^2\omega_w/(1+K^2+\gamma^2\theta^2)$. Note that $\mathrm{Im}\mu_1$ is maximum at $\Delta = 0$, so $q_\parallel = -q_\perp^2$, and

$$\left|\dot{G}_q(\tau)\right|^2 \simeq \frac{1}{(2\pi)^6} \frac{1}{9(2\rho)^2} \exp\left[\sqrt{3}\, 2\rho\tau\left(1 - \frac{\Delta^2}{9} - \frac{q_\perp^2}{3}\right)\right] . \quad (37)$$

From Eq. (32) we find

$$\frac{dP}{dAd\Omega d\omega} = \frac{1}{9(2\pi)^3} \rho(\gamma_o mc^2)k_o^2 \, e^{\sqrt{3}\, 2\rho\tau} \, e^{-(\omega-\omega_1(\theta))^2/2\sigma_\omega^2} \, e^{-\theta^2/2\sigma_\theta^2} , \quad (38)$$

with

$$2\sigma_\omega^2 = \frac{9(2\rho\omega_o)^2}{\sqrt{3}\ 2\rho\tau} \tag{39}$$

$$2\sigma_\theta^2 = \frac{\sqrt{3}\ (1+K^2)}{\gamma^2 2\rho\tau} \tag{40}$$

Integrating Eq. (38) over frequency and solid angle we obtain the radiated power per unit cross-sectional area of the electron beam

$$S = \frac{\rho S_e}{9(2\pi)^{3/2}} \frac{k_o^2\sigma_\theta^2}{n_o\sigma_\parallel} e^{\sqrt{3}\ 2\rho\tau} , \tag{41}$$

where

$$S_e = (\gamma_o mc^2)n_o c \tag{42}$$

is the power per unit area in the electron beam and

$$\sigma_\parallel = c/\sigma_\omega \tag{43}$$

is the longitudinal correlation length.

In a similar fashion, the correlation function of Eq. (30) can be evaluated, yielding

$$C(z,\vec{r},\tau) = \frac{\rho S_e e^{\sqrt{3}\ 2\rho\tau}}{9n_o V_c w(z)} \exp\left(-\frac{z^2}{2\sigma_\parallel^2}\right) \exp\left(-\frac{r^2}{2\sigma_\perp^2 w(z)}\right) , \tag{44}$$

where the transverse correlation length for $z = 0$ is

$$\sigma_\perp = 1/k_o\sigma_\theta , \tag{45}$$

the coherence volume V_c is

$$V_c = (2\pi)^{3/2}\ \sigma_\parallel\sigma_\perp^2 \tag{46}$$

and

$$w(z) = 1 + \frac{i\sqrt{3}\ k_r z}{2\rho\tau} . \tag{47}$$

The total power radiated per unit area given in Eq. (41) can now be rewritten as

$$S = C(o,o,\tau) = \frac{1}{9}\ \rho S_e e^{\sqrt{3}\ 2\rho\tau} \frac{1}{n_o V_c} , \tag{48}$$

and is seen to be inversely proportional to the number of electrons in a coherence volume.

In our analysis the transverse variation of the electron density has been neglected, i.e. u appearing in Eq. (13) has been taken to be unity. This is a reasonable approximation for times short enough that radiation emitted by the electrons in the bulk of the beam has not reached the edge, i.e. (ct) $\sigma_\theta \lesssim a$, where a is the electron beam radius. Once the radiation reaches the edge of the beam, the transverse fall-off of u becomes important and may lead to self-focusing or guiding.[4,5]

One-dimensional analysis gives the total radiated power[2,3]

$$P_{1D} = \frac{1}{9}\,\rho P_e\, e^{\sqrt{3}\,2\rho\tau}\,\frac{1}{N_c} \qquad (49)$$

where N_c is the number of electrons within a coherence length, rather than a coherence volume. If one thinks of keeping the electron density constant and increasing the electron beam cross-section, P_{1D} would remain constant, because $P_e = S_e A$ and N_c are both proportional to the beam cross-section. This puzzling result is removed by the 3-dimensional treatment leading to Eq. (48), since N_c is replaced by $n_0 V_c$, the number of electrons in a coherence volume. Hence the transverse coherence length $\sigma_\perp$ is important in the start-up process.

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7. The paraxial approximation has resulted in a plane wavefront $\tau - \zeta = 0$. An improved paraxial approximation (IPA) leads to $D_{IPA}(\Omega, q_\parallel, \vec{q}_\perp) = (1+q_\parallel)\,[\Omega^3 - (q_\parallel + q_\perp^2/(1 + q_\parallel))\,\Omega^2 + \alpha\Omega - \alpha(1 + q_\parallel)]$, and a curved wavefront $\tau - \zeta - x^2/4\zeta = 0$.

8. For the one-dimensional Green's function this has been noted in the Appendix of ref. 4.

9. The approximate equality in Eq. (36) holds over a larger angular range in the improved paraxial approximation mentioned in ref. 7. For $\gamma >> 1$, $\theta << 1$, one sees that $q_\parallel + q_\perp^2/(1 + q_\parallel) = (\omega - \omega_1(\theta))/\omega_1(\theta)$.